Hierarchical Anderson Model

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ABSTRACT

In this thesis, we study the spectral properties of the hierarchical Anderson model. This model is an approximation of the Anderson tight-binding model on $\mathbb{Z}^{\mathrm{d}}$, with the usual discrete Laplacian replaced by a hierarchical long-range interaction operator. In the hierarchical Anderson model, we are given a countable set $\mathbb{X}$ endowed with a hierarchical structure. The free hierarchical Laplacian is a self-adjoint operator Δ acting on the Hilbert space $l^2(\mathbb{X})$. The spectrum of Δ consists of isolated infinitely degenerate eigenvalues. We look at small random perturbations of the operator Δ. The disorder is modeled by a random potential V_ω, $(V_\omega\psi)(x) = \omega(x)\psi(x)$ for $\psi \in l^2(\mathbb{X})$. The numbers $\omega(x)$ are identically distributed independent random variables with a bounded density. The hierarchical Anderson model is the random self-adjoint operator $H_\omega = \Delta + V_\omega$. We prove the following two results. If the model has a spectral dimension $\mathrm{d_{sp}} \leq 4$ then, almost surely, the spectrum of H_ω is dense pure-point. The second result is on eigenvalue statistics. For $\mathrm{d_{sp}} < 1$, the energy levels for H_ω are asymptotically a Poisson point process in the thermodynamic limit, after a proper rescaling.

ABRÉGÉ

Dans ce mémoire, nous étudions les propriétés spectrales du modèle hiérarchique d'Anderson. Ce modèle est une approximation du modèle d'Anderson sur $\mathbb{Z}^{\mathrm{d}}$, avec le Laplacien discret remplacé par un opérateur d'interaction hiérarchique de longue portée. Dans le modèle hiérarchique d'Anderson, nous considérons un ensemle dénombrable $\mathbb{X}$, muni d'une structure hiérarchique. Le Laplacien hiérarchique libre est un opérateur auto-adjoint Δ qui agit sur l'espace d'Hilbert $l^2(\mathbb{X})$. Le spectre de Δ consiste en valeurs propres isolées et infiniment dégénérées. Nous considérons des faibles perturbations aléatoires de l'opérateur Δ. Le désordre est modélisé par un potentiel aléatoire V_ω, $(V_\omega\psi)(x) = \omega(x)\psi(x)$ pour $\psi \in l^2(\mathbb{X})$. Les nombres $\omega(x)$ sont des variables aléatoires identiquement distribuées, avec une densité bornée. Le modèle hiérarchique d'Anderson est l'opérateur aléatoire auto-adjoint $H_\omega = \Delta + V_\omega$. Nous démontrons les deux résultats suivants. Si le modèle a une dimension spectrale $\mathrm{d_{sp}} \leq 4$ alors, presque sûrement, le spectre de H_ω is dense pur-point. Le second résultat concerne la statistique des valeurs propres. Pour $\mathrm{d_{sp}} < 1$, les niveaux d'énergie de H_ω sont asymptotiquement un processus de Poisson dans la limite thermodynamique, après un changement d'échelle approprorié.

TABLE OF CONTENTS

Introduction

The mathematical discipline of random Schrödinger operators has its origins in solid state physics as an attempt to describe localization and transport in randomly disordered quantum systems. The notion of a random discrete Schrödinger operator can be formulated in the following general framework. Let $\mathbb{X}$ be a countable set and let H_0 be a self-adjoint operator acting on the Hilbert space $l^2(\mathbb{X})$. H_0 is the energy operator of the unperturbed quantum system, and is usually taken to be a Laplacian. Let $\omega = \{\omega(x)\}_{x\in\mathbb{X}}$ be a family of independent identically distributed (i.i.d.) random variables. For the purpose of this introduction, let us assume that $\omega(x)$ are i.i.d. random variables uniformly distributed on the interval $[-1, 1]$. The disorder is modeled by a random potential V_ω acting diagonally on $l^2(\mathbb{X})$:

$$(V_\omega\psi)(x) = \omega(x)\psi(x), \qquad \psi \in l^2(\mathbb{X}), x \in \mathbb{X}.$$

The disordered quantum system is then described by the random self-adjoint operator

$$H_\omega = H_0 + cV_\omega,$$

where $c > 0$ is a coupling constant measuring the strength of the disorder. The generic spectral properties of H_ω reflect the physical properties of the disordered system. Localization corresponds to a pure-point spectrum, whereas transport corresponds to an absolutely continuous (a.c.) spectrum.

Since the pioneering work of Anderson [A], the most famous example of a random discrete Schrödinger operator is the Anderson tight-binding model on the lattice

$\mathbb{X} = \mathbb{Z}^{\mathrm{d}}$. In the Anderson model, H_0 is equal to the discrete Laplacian $\Delta_{\mathbb{Z}^{\mathrm{d}}}$,

$$(\Delta_{\mathbb{Z}^{\mathrm{d}}}\psi)(x) = \sum_{|y-x|=1} \psi(y), \qquad \psi \in l^2(\mathbb{Z}^{\mathrm{d}}), x \in \mathbb{Z}^{\mathrm{d}}, \tag{1}$$

where $x = (x_1, \cdots, x_{\mathrm{d}}), y = (y_1, \cdots, y_{\mathrm{d}}) \in \mathbb{Z}^{\mathrm{d}}$ and $|x-y| = \sum_{j=1}^{\mathrm{d}} |x_j - y_j|$. A remarkable feature of the Anderson model is that many spectral properties hold with probability one, and thus are generic. There exist deterministic sets Σ, Σ_{ac}, Σ_{pp} such that with probability one, the spectrum of H_ω is equal to Σ, the essential support of the a.c. spectrum of H_ω is equal to Σ_{ac} (up to a set of zero Lebesgue measure), and the pure-point spectrum of H_ω is equal to Σ_{pp}. We refer the reader to [PF, CKFS, CL] for the proofs of these facts and for a general introduction to the field of random Schrödinger operators. The Anderson conjecture is that in dimension d $= 3$, there is a critical strength of coupling c_0 separating the following regimes.

(1) For $c > c_0$, H_ω is in the *localized regime:* with probability one, the spectrum of H_ω is pure-point and the corresponding eigenfunctions decrease exponentially.

(2) For $c < c_0$, H_ω is in the *delocalized regime:* with probability one, H_ω has some a.c. spectrum.

The large desorder regime $c >> 1$ is by now well understood [FS, AM]: with probability one, H_ω is in the localized regime (1). The methods of proof are based on the idea that for large enough c, H_0 can be considered as a small perturbation term added to cV_ω, and that perturbation can be controlled by probabilistic estimates. The fractional moments method [AM, ASFH] is a robust technique allowing to prove localization at large disorder for a wide class of random discrete Schrödinger operators. The analysis of the weak disorder regime $c << 1$ is a much more difficult problem. A fundamental open question is to prove the existence of some a.c. spectrum for the Anderson model in dimension d = 3. At the present moment, there does not exist a satisfactory mathematical technique to tackle the weak disorder regime. Nevertheless, very interesting special methods have been developed to analyze simpler models. For the one-dimensional Anderson model, there is no phase transition. For every $c > 0$, with probability one, the spectrum of H_ω is pure-point and the corresponding eigenfunctions decrease exponentially [KuSo]. Localization was first proven for the continuous one-dimensional model in [GMP]. One can also consider more general Anderson models on graphs. In this case $\mathbb{X}$ is the set of vertices of the graph and the operator H_0 is taken to be the graph Laplacian $(H_0\psi)(x) = \sum \psi(y), \psi \in l^2(\mathbb{X}), x \in \mathbb{X}$, where the sum is taken over the vertices y adjacent to x. The Anderson tight-biding model on $\mathbb{Z}^{\mathrm{d}}$ is a special case in this framework when $\mathbb{Z}^{\mathrm{d}}$ is viewed as a graph with the vertices x and y adjacent iff $|x - y| = 1$. After the Anderson model on $\mathbb{Z}^{\mathrm{d}}$, the next most studied graph model is the Bethe lattice, for which $\mathbb{X}$ is a regular tree with branching $K \geq 2$ (the case $K = 1$ is identical to the one-dimensional Anderson model). For the Bethe lattice, a result very close to the Anderson phase transition has been proven in [Kl, ASW, FHS]. For small enough c, $c < c_1$ H_ω has purely a.c. spectrum with probability one and for large enough c, $c > c_2$, H_ω has pure-point spectrum with probability one. The situation on $[c_1, c_2]$ is not resolved yet.

A possible approach to study the Anderson model is via a hierarchical toy model. The idea of using hierarchical interactions goes back to Dyson's [D] in the context of statistical physics. The density of states for hierarchical Anderson model was studied by Bovier [Bo] in the supersymmetric formalism. Later, Molchanov [M2, M3] considered the spectral localization problem for the hierarchical Anderson model and proved localization in the case of a Cauchy random potential. The hierarchical Anderson model turns out to be technically the simplest nontrivial random discrete Schrödinger operator for which localization holds at arbitrary coupling c. The main guiding principle for the hierarchical Anderson model is the same as in proof of localization in the large disorder regime for the Anderson model on $\mathbb{Z}^{\mathrm{d}}$. The hierarchical Laplacian $\Delta = H_0$ is considered as a perturbation term added to V_ω. We will see that the very special hierarchical structure of Δ allows to understand this perturbation via simple recursive resolvent formulas. The parameter quantifying the importance of the perturbation will be the spectral dimension $\mathrm{d_{sp}}$ of Δ. A small spectral dimension will mean that Δ very weakly couples distant regions of space and thus is not a too strong perturbation term. As a result, localization and eigenvalue statistics are easier to analyze for smaller spectral dimensions.

This thesis is based on the author's papers [K1, K2, K3]. Although the main results can be extracted from [K1, K2, K3], in this work we have adapted a uniform notation, simplified certain arguments of [K1, K2, K3] and fixed minor mathematical mistakes. In Chapter 1, we give the definition of the free hierarchical Laplacaian Δ and we study its basic spectral properties. The material of this chapter is elementary and is essentially present in [M3] and [K1]. Chapter 2 is based on [K1, K2] and is devoted to the study of the generic spectral properties of H_ω. We prove localization for the hierarchical Anderson model with spectral dimension $d_{sp} \leq 4$ and a generalization of Molchanov's theorem for mixed Cauchy distributions. The techniques of proof have their origin in [M3] and this chapter can be viewed as a natural extension of Molchanov's work. Chapter 3 is devoted to the study of the fine eigenvalue statistics. We prove Poisson statistics of eigenvalues for the hierarchical Anderson model with spectral dimension $d_{sp} < 1$. This chapter is based on the author's work [K3] and exploits the method of Minami [Mi].

CHAPTER 1
The Free Hierarchical Laplacian

1.1 Hierarchical structures

Let $\mathbb{X}$ be an infinite countable set and $\mathtt{d} : \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{N}$ a metric on $\mathbb{X}$. For $x \in \mathbb{X}$ and $r \in \mathbb{N}$, we denote by $B(x,r)$ the (closed) ball with center x and radius r,

$$B(x,r) = \{y \in \mathbb{X} : \mathtt{d}(x,y) \leq r\}.$$

We shall call $\mathtt{d}$ a hierarchical distance on $\mathbb{X}$ if the following three conditions are satisfied:

(1) Two balls of the same radius are either disjoint or identical, i.e. given $x_1, x_2 \in \mathbb{X}$ and $r \in \mathbb{N}$, we have either $B(x_1,r) \cap B(x_2,r) = \emptyset$ or $B(x_1,r) = B(x_2,r)$.

(2) Every ball of radius $r \geq 1$ is a disjoint union of balls of radius $r-1$, i.e. given $x \in \mathbb{X}$ and $r \geq 1$, there exist $\mathtt{n}_r(x) \in \{1, 2, \cdots\}$ and $y_1(x), \cdots, y_{\mathtt{n}_r(x)}(x) \in \mathbb{X}$ such that

$$B(x,r) = \bigcup_{j=1}^{\mathtt{n}_r(x)} B(y_j(x), r-1).$$

(3) For every $x \in \mathbb{X}$, $\mathbb{X} = \bigcup_{r=0}^{\infty} B(x,r)$.

If $\mathtt{d}$ is a hierarchical distance on $\mathbb{X}$, we call the pair $(\mathbb{X}, \mathtt{d})$ a *hierarchical structure.* By convention, we set $\mathtt{n}_0(x) = \mathtt{n}_0 = 1$. We denote by $|B(x,r)|$ the cardinality of $B(x,r)$.

A hierarchical structure can be equivalently described in terms of a system of embedded partitions. A partition $\mathcal{P}$ of $\mathbb{X}$ is a collection of finite disjoint subsets of $\mathbb{X}$ whose union is equal to $\mathbb{X}$, i.e. $\mathcal{P} = \{B_i\}_{i \in I}$, $X = \bigcup_{i \in I} B_i$, and $B_i \cap B_j = \emptyset$ for $i \neq j$. Given two partitions $\mathcal{P}$ and $\mathcal{P}'$, we write $\mathcal{P} > \mathcal{P}'$ if every $B \in \mathcal{P}$ is a subset of some $B' \in \mathcal{P}'$. A hierarchical structure $(X, \mathtt{d})$ naturally induces a system of embedded partitions of $\mathbb{X}$. By the property (1), for every $r \geq 0$, the balls of radius r form a partition $\mathcal{P}_r$ of $\mathbb{X}$. By the property (2), $\mathcal{P}_r$ is finer than $\mathcal{P}_{r+1}$ and thus we have a system of embedded partitions

$$\mathcal{P}_0 > \mathcal{P}_1 > \mathcal{P}_2 > \cdots .$$

Conversely, let $\mathcal{P}_0 = \{\{x\}\}_{x \in \mathbb{X}}$ be the singleton partition suppose that $\mathcal{P}_1 > \mathcal{P}_2 > \cdots$ is a given system of embedded partitions on $\mathbb{X}$ such that for every $x, y \in \mathbb{X}$, there exist $r \geq 0$ and $B \in \mathcal{P}_r$ containing both x and y. If we define $\mathtt{d}(x, y)$ to be the such smallest r, then $(\mathbb{X}, \mathtt{d})$ defines a hierarchical distance. These two constructions establish a one to one correspondence between hierarchical structures and systems of embedded partitions.

If for every $r \geq 1$, the number $\mathtt{n}_r(x)$ is independent of x, $\mathtt{n}_r(x) = \mathtt{n}_r$, we say that $(\mathbb{X}, \mathtt{d})$ is a *regular hierarchical structure.* In this case, we have

$$|B(x, r)| = \mathtt{N}_r = \prod_{s=0}^{r} \mathtt{n}_s,$$

for every $x \in \mathbb{X}$. If moreover $\mathtt{n}_r = \mathtt{n}$ for $r \geq 1$, we say that $(\mathbb{X}, \mathtt{d})$ is *a homogeneous hierarchical structure of degree* $\mathtt{n}$. The structure is nontrivial for $\mathtt{n} \geq 2$. For a homogeneous hierarchical structure of degree $\mathtt{n}$, we have $\mathtt{N}_r = \mathtt{n}^r$.

Example 1.1.1. *Let $\mathbb{X}$ be the lattice $\mathbb{Z}^{\mathrm{d}}$. The elements of $\mathbb{X}$ are pairs $x = (x_1, x_2, \cdots x_{\mathrm{d}})$, with $x_1, x_2, \cdots x_{\mathrm{d}} \in \mathbb{Z}$. Let $m \geq 2$ be a given integer. For $r \geq 0$ and $x \in \mathbb{Z}^{\mathrm{d}}$, consider the cube*

$$Q_{x,r} = \left\{ y \in \mathbb{Z}^{\mathrm{d}} : 0 \leq y_k - x_k < m^r \text{ for } 1 \leq k \leq \mathrm{d} \right\}.$$

Let $\mathcal{P}_r$ be the partition

$$\mathcal{P}_r = \{Q_{x,r}\}_{x \in (m^r \mathbb{Z})^{\mathrm{d}}},$$

of $\mathbb{Z}^{\mathrm{d}}$. Then the system of partitions $(\mathcal{P}_r)_{r \geq 0}$ defines a homogeneous hierarchical structure of degree $\mathrm{n} = m^{\mathrm{d}}$.

1.2 The hierarchical Laplacian

Let $(\mathbb{X}, \mathrm{d})$ be a hierarchical structure. We consider the Hilbert space $l^2(\mathbb{X})$ consisting of square summable functions $\psi : \mathbb{X} \to \mathbb{C}$,

$$\sum_{x \in \mathbb{X}} |\psi(x)|^2 < \infty.$$

The inner product on $l^2(\mathbb{X})$ is given by

$$\langle \psi | \phi \rangle = \sum_{x \in \mathbb{X}} \overline{\psi(x)} \phi(x).$$

For $r \geq 0$, we define the averaging operator $E_r : l^2(\mathbb{X}) \to l^2(\mathbb{X})$ by

$$(E_r \psi)(x) = |B(x,r)|^{-1} \sum_{y \in B(x,r)} \psi(y).$$

Thus E_r is the orthogonal projection onto the closed subspace $\mathcal{H}_r \subset l^2(X)$ consisting of functions that are constant on every ball of radius r. Let $(\mathrm{p}_r)_{r \geq 1}$ be a sequence of real numbers such that $\mathrm{p}_r > 0$ and $\sum_{r=1}^{\infty} \mathrm{p}_r = 1$. We set $\mathrm{p}_0 = 0$ and

$$\lambda_r = \sum_{s=0}^{r} \mathrm{p}_s, \qquad r = 0, 1, 2, \cdots, \infty.$$

The hierarchical Laplacian Δ is defined by

$$\Delta = \sum_{r=0}^{\infty} \mathrm{p}_r E_r.$$

The triple $(\mathbb{X}, \mathsf{d}, \Delta)$ is called a *hierarchical model.* If $(\mathbb{X}, \mathsf{d})$ is regular (resp. homogeneous) than we say that $(\mathbb{X}, \mathsf{d}, \Delta)$ is a regular (resp. homogeneous). It is easy to see that Δ is a bounded self-adjoint operator and $0 \leq \Delta \leq 1$.

Proposition 1.2.1. *We have the following diagonalization of Δ:*

(1) *The spectrum of Δ is given by*

$$\mathrm{sp}(\Delta) = \{\lambda_r : r = 0, \cdots, \infty\}. \tag{1.1}$$

Each λ_r, $r < \infty$, is an eigenvalue of Δ of infinite multiplicity. The point $\lambda_\infty = 1$ is not an eigenvalue.

(2) *$E_r - E_{r+1}$ is the orthogonal projection onto the eigenspace of λ_r and*

$$\Delta = \sum_{r=0}^{\infty} \lambda_r (E_r - E_{r+1}).$$

Proof. Note that

$$l^2(X) = \mathcal{H}_0 \supset \mathcal{H}_1 \supset \mathcal{H}_2 \supset \mathcal{H}_3 \supset \ldots,$$

and that $\bigcap_{r=0}^{\infty} \mathcal{H}_r = \{0\}$ since a nonzero function constant on every ball would be identically constant on $\mathbb{X}$ and hence would have infinite l^2 norm. These observations yield that

$$l^2(X) = \bigoplus_{r=0}^{\infty} L_r,$$

where L_r is the orthogonal complement of $\mathcal{H}_{r+1}$ in $\mathcal{H}_r$. Note that L_r is the infinite dimensional subspace of functions ψ s.t. $E_s\psi = \psi$ for $0 \leq s \leq r$ and $E_s\psi = 0$ for $s > r$. Hence for every $\psi \in L_r$, $\Delta\psi = \lambda_r\psi$, and this proves (1) and (2). □

For $x \in \mathbb{X}$, we denote by δ_x the Kronecker delta function at x: $\delta_x(y) = 1$ for $y = x$ and $\delta_x(y) = 0$ for $y \neq x$. The spectral measure for δ_x and Δ is the probability measure μ_x on $\mathbb{R}$ such that

$$\langle \delta_x | f(\Delta)\delta_x \rangle = \int f d\mu_x,$$

for every bounded Borel measurable function $f : \mathbb{R} \to \mathbb{C}$. Proposition 1.2.1 allows to compute μ_x explicitly.

Proposition 1.2.2. *We have*

$$\mu_x = \sum_{r=0}^{\infty} \left(|B(x,r)|^{-1} - |B(x,r+1)|^{-1}\right) \delta(\lambda_r), \tag{1.2}$$

where $\delta(\lambda_r)$ denotes the Dirac delta mass at λ_r.

Proof. Using the spectral decomposition of Δ given by Proposition 1.2.1, we have

$$\begin{aligned}
\langle \delta_x | f(\Delta)\delta_x \rangle &= \sum_{r=0}^{\infty} f(\lambda_r)\langle \delta_x | (E_r - E_{r+1})\delta_x \rangle \\
&= \sum_{r=0}^{\infty} f(\lambda_r)\langle \delta_x |\, |B(x,r)|^{-1} \mathbf{1}_{B(x,r)} - |B(x,r+1)|^{-1} \mathbf{1}_{B(x,r+1)} \rangle \\
&= \sum_{r=0}^{\infty} f(\lambda_r) \left(|B(x,r)|^{-1} - |B(x,r+1)|^{-1}\right),
\end{aligned}$$

where $\mathbf{1}_{B(x,r)}(y) = 1$ if $y \in B(x,r)$ and 0 otherwise. $\square$

For a regular hierarchical structure, μ_x is independent of x and

$$\mu_x = \mu = \sum_{r=0}^{\infty} \left(\frac{1}{\mathrm{N}_r} - \frac{1}{\mathrm{N}_{r+1}} \right) \delta(\lambda_r). \tag{1.3}$$

In this case, we will call the measure μ *the spectral measure for* Δ.

1.3 The spectral dimension

In this section, we discuss the important notion of the spectral dimension of a hierarchical model. For sake of simplicity, we only consider regular hierarchical models, but the discussion can be generalized to arbitrary hierarchical models. Informally, the spectral dimension is a measure of coupling of distant regions of $\mathbb{X}$. A small spectral dimensions means weak coupling and a large spectral dimension means strong coupling. Hence faster decay of p_r or rapid growth of N_r should imply smaller spectral dimension. The precise definition of the spectral dimension is motivated by the analogy with the edge asymptotics of the spectral measure of the discrete Laplacian $\Delta_{\mathbb{Z}^{\mathrm{d}}}$ on $\mathbb{Z}^{\mathrm{d}}$, for which the spectral and the spacial dimensions coincide.

Definition 1.3.1. *The spectral dimension of a hierarchical model is the number* d_{sp} *given by*

$$\lim_{t \downarrow 0} \frac{\log \mu([1-t, 1])}{\log t} = \mathrm{d}_{\mathrm{sp}}/2,$$

provided the limit exists.

Since

$$\sum_{y \in \mathbb{X}} \langle \delta_x | \Delta \delta_y \rangle = 1,$$

for all $x \in \mathbb{X}$, Δ generates a random walk on $\mathbb{X}$ with transition probability $p(x,y) = \langle \delta_x | \Delta \delta_y \rangle$. Starting at a fixed point $x_0 \in \mathbb{X}$, the random walk is a path $(x_0, x_1, x_2, \cdots)$ in $\in \mathbb{X}$, where the points x_k, $k \geq 1$ are chosen according to the following procedure. Given x_k, $k \geq 0$, we choose a random radius r with probability p_r. After that, we choose a random point x_{k+1} in $B(x_k, r)$ according to the uniform distribution. All the choices are made independently of each other. The random walk is called recurrent if with probability one, $x_k = x_0$ for infinitely many k's and otherwise, the random walk is called transient. We recall without proof the following well-known general criterion for recurrence. Let

$$R = \sum_{k=0}^{\infty} \langle \delta_{x_0} | \Delta^k \delta_{x_0} \rangle = \langle \delta_{x_0} | (1 - \Delta)^{-1} \delta_{x_0} \rangle.$$

The random walk starting at x_0 is recurrent for $R = \infty$ and transient for $R < \infty$. We also recall Polya's result saying that that the random walk on $\mathbb{Z}^{\mathrm{d}}$ generated by the discrete Laplacian $\Delta_{\mathbb{Z}^{\mathrm{d}}}$ is recurrent if $\mathrm{d} = 1, 2$ and transient if $\mathrm{d} > 2$. The corresponding result for the hierarchical Laplacian is:

Proposition 1.3.2. *Consider a homogeneous hierarchical model of degree* $\mathrm{n} \geq 2$. *Suppose that there exist constants* $C_1 > 0, C_2 > 0$ *and* $\rho > 1$ *such that*

$$C_1 \rho^{-r} \leq \mathrm{p}_r \leq C_2 \rho^{-r},$$

for r *big enough. Then:*

(1) *The spectral dimension is*

$$\mathrm{d_{sp}}(\mathrm{n}, \rho) = 2 \frac{\log \mathrm{n}}{\log \rho}. \tag{1.4}$$

Hence $0 < \mathrm{d_{sp}}(\mathrm{n}, \rho) \leq 2$ *iff* $\mathrm{n} \leq \rho$.

(2) *The random walk generated by* Δ *is recurrent if* $0 < \mathrm{d_{sp}}(\mathrm{n}, \rho) \leq 2$ *and transient if* $\mathrm{d_{sp}}(\mathrm{n}, \rho) > 2$.

Proof of Proposition 1.3.2. Note that $\mu([1-t, 1])$ is a piecewise constant function of t with jump discontinuities at the points $1 - \lambda_r$. Since

$$C_1(\rho - 1)^{-1}\rho^{-r} \leq 1 - \lambda_r = \sum_{s=r+1}^{\infty} \mathrm{p}_s \leq C_2(\rho - 1)^{-1}\rho^{-r},$$

and $\mu([1 - \lambda_r, 1]) = 1/\mathrm{N}_r = \mathrm{n}^{-r}$, we have that

$$\lim_{t \downarrow 0} \frac{\log \mu([1 - t, 1])}{\log t} = \frac{\log \mathrm{n}}{\log \rho},$$

which proves (1). Part (2) of Proposition 1.2.1 allows to compute R explicitly:

$$R = \langle \delta_{x_0} | (1 - \Delta)^{-1} \delta_{x_0} \rangle = \int \frac{d\mu(\xi)}{1 - \xi} = \sum_{r=0}^{\infty} \frac{\mathrm{N}_r^{-1} - \mathrm{N}_{r+1}^{-1}}{1 - \lambda_r}.$$

The bounds

$$C_2^{-1}(\rho - 1)(1 - 1/\mathrm{n}) \sum_{r=0}^{\infty} (\rho/\mathrm{n})^r \leq R \leq C_1^{-1}(\rho - 1)(1 - 1/\mathrm{n}) \sum_{r=0}^{\infty} (\rho/\mathrm{n})^r$$

show that $R < \infty$ for $\rho < \mathrm{n}$ and $R = \infty$ for $\rho \geq \mathrm{n}$, and part (2) follows. □

1.4 The density of states

In this section, $(\mathbb{X}, \mathrm{d})$ is a regular hierarchical structure given by the parameters n_r and Δ is a hierarchical Laplacian specified by the parameters p_r. We demonstrate below that the spectral measure μ is equal to the density of states measure for Δ. The density of states is a probability measure that describes the asymptotic distribution of eigenvalues of a large finite volume approximation to Δ. There are many possible finite volume approximations of Δ. We will consider two such approximations and we will show that they yield the same limiting density of states, equal to μ.

Let us fix a point $x_0 \in \mathbb{X}$ and consider the increasing sequence of balls

$$B_k = B(x_0, k) \qquad k \geq 0.$$

Each B_k has size $|B_k| = \mathtt{N}_k$. For $k \geq 0$, we define Δ_k to be the truncated hierarchical Laplcacian

$$\Delta_k = \sum_{s=0}^{k} \mathtt{p}_s E_s \tag{1.5}$$

Note that the subspace

$$l^2(B_k) = \left\{ \psi \in l^2(\mathbb{X}) : \psi(x) = 0 \text{ for } x \notin B_k \right\}, \tag{1.6}$$

is invariant for Δ_k. Let $e_j^k, k = 1, \cdots, \mathtt{N}_r$ denote the eigenvalues of the restricted operator $\Delta_k \restriction l^2(B_k)$ and let μ_k be the corresponding normalized eigenvalue counting measure,

$$\mu_k = \frac{1}{\mathtt{N}_r} \sum_{j=1}^{\mathtt{N}_r} \delta(e_j^k). \tag{1.7}$$

We denote by $C_0(\mathbb{R})$ the space of continuous functions $f : \mathbb{R} \to \mathbb{C}$ vanishing at infinity, i.e. $\lim_{|t| \to \infty} |f(t)| = 0$. If $(\nu_k)_{k \geq 1}$ and ν are Borel probability measures on $\mathbb{R}$, we say that ν_k converges to ν in the weak-* topology if for every $f \in C_0(\mathbb{R})$,

$$\lim_{k \to \infty} \int f(t) d\nu_k(t) = \int f(t) d\nu(t).$$

Proposition 1.4.1. *The sequence μ_k converges to the spectral measure μ in the weak-* topology as $k \to \infty$.*

Proof. As in Proposition 1.2.1, one can explicitly diagonalize Δ_k. Summing by parts, we have the spectral decomposition

$$\begin{aligned}\Delta_k &= \mathrm{p}_0E_0 + \mathrm{p}_1E_1 + \mathrm{p}_2E_2 + \cdots + \mathrm{p}_kE_k \\ &= \mathrm{p}_0(E_0 - E_1) + (\mathrm{p}_0 + \mathrm{p}_1)(E_1 - E_2) + \cdots \\ &+ (\mathrm{p}_0 + \mathrm{p}_1 + \cdots + \mathrm{p}_{k-1})(E_{k-1} - E_k) \\ &+ (\mathrm{p}_0 + \mathrm{p}_1 + \cdots + \mathrm{p}_{k-1} + \mathrm{p}_k)E_k.\end{aligned}$$

It follows that

$$\mu_k = \frac{1}{N_k}\delta(\lambda_k) + \sum_{r=0}^{k-1}\left(\frac{1}{N_r} - \frac{1}{N_{r+1}}\right)\delta(\lambda_r), \tag{1.8}$$

which yields the statement after taking $k \to \infty$. □

Suppose that $\widetilde{\Delta_k}$ is another sequence of bounded self-adjoint operators approximating to Δ and that $l^2(B_k)$ is an invariant subspace for $\widetilde{\Delta_k}$. If $\widetilde{\mu_k}$ is the normalized eigenvalue counting measure corresponding to $\widetilde{\Delta_k}$, then we have

$$\int f d\widetilde{\mu_k} = \frac{1}{\mathrm{N}_k}\sum_{x\in B_k}\langle\delta_x|f(\widetilde{\Delta_k})\delta_x\rangle,$$

for every bounded Borel function $f : \mathbb{R} \to \mathbb{C}$.

Proposition 1.4.2. *Suppose that* $\left\|\Delta_k - \widetilde{\Delta_k}\right\| \to 0$ *as* $k \to \infty$. *Then* $\widetilde{\mu_k}$ *converges to* μ *in the weak-* topology.*

Proof. Since μ_k converges to μ in the weak-* topology, it suffices to show that for every $\operatorname{Im} z \neq 0$, the difference

$$D_k(z) = \int (t-z)^{-1}d\mu_k(t) - \int (t-z)^{-1}d\widetilde{\mu_k}(t),$$

converges to 0 as $k \to \infty$. The resolvent identity yields

$$\left\|(\Delta_k - z)^{-1} - (\widetilde{\Delta_k} - z)^{-1}\right\| \leq \frac{\left\|\Delta_k - \widetilde{\Delta_k}\right\|}{|\mathrm{Im}\, z|^2},$$

and therefore

$$\begin{aligned} |D_k(z)| &= \left|\frac{1}{\mathrm{N}_r} \sum_{x \in B_k} \langle \delta_x | \left((\Delta_k - z)^{-1} - (\widetilde{\Delta_k} - z)^{-1}\right) \delta_x \rangle\right| \\ &\leq \frac{\left\|\Delta_k - \widetilde{\Delta_k}\right\|}{|\mathrm{Im}\, z|^2}, \end{aligned}$$

which converges to 0 as $k \to \infty$. □

Let us consider the following simple special case. We denote by P_k the operator of orthogonal projection onto $l^2(B_k)$ and we set

$$\widetilde{\Delta_k} = P_k \Delta P_k.$$

Then $\left\|\Delta_k - \widetilde{\Delta_k}\right\| \leq \sum_{r=k+1}^{\infty} \mathrm{p}_r$ and the sequence $\widetilde{\Delta_k}$ verifies the hypothesis of the previous proposition. Therefore the corresponding limiting normalized eigenvalue counting measure is equal to μ. Among the different possible finite-volume approximations to Δ, the truncated Laplacian Δ_k is the most convenient and will be often used in the following chapters. Hence, the spectral measure μ can be naturally interpreted as the density of states of Δ. An analogous statement is well known for the discrete Laplacian on $\mathbb{Z}^{\mathrm{d}}$.

CHAPTER 2
Spectral Localization in the Hierarchial Anderson Model

2.1 Definition of the model and its basic properties

Let $(\mathbb{X}, \mathsf{d}, \Delta)$ be a regular hierarchical model. The associated hierarchical Anderson model is defined as follows. Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\Omega = \mathbb{R}^{\mathbb{X}}$, $\mathcal{F}$ is the product Borel σ-algebra in Ω, and $\mathbb{P}$ is a given probability measure on $(\Omega, \mathcal{F})$. For $\omega \in \Omega$, we set

$$V_\omega = \sum_{x \in X} \omega(x) \langle \delta_x | \cdot \rangle \delta_x, \tag{2.1}$$

and

$$H_\omega = \Delta + V_\omega. \tag{2.2}$$

If the set $\{\omega(x) : x \in \mathbb{X}\}$ is unbounded, then V_ω and H_ω are unbounded self-adjoint operators with the domain

$$\mathcal{D}_\omega = \left\{ \psi : \sum_{x \in \mathbb{X}} |\psi(x)|^2 \left(1 + |\omega(x)|^2\right) < \infty \right\}.$$

The family of self-adjoint operators $\{H_\omega\}_{\omega \in \Omega}$ indexed by the random parameter ω of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called the *hierarchical Anderson model.* Our goal in this chapter will be to understand the generic spectral properties of H_ω. By *generic* we mean a property that holds with probability one. More precisely, there must be a set $\widetilde{\Omega} \in \mathcal{F}$ with $\mathbb{P}(\widetilde{\Omega}) = 1$ and such that for all $\omega \in \widetilde{\Omega}$, H_ω has the desired property. In this case we shall write: for $\mathbb{P}$-a.a. $\omega \in \Omega$, H_ω has the desired property. The first natural question is to describe the spectrum of H_ω, denoted $\mathrm{sp}(H_\omega)$, as a set. Note that, unlike the spectrum of the discrete Laplacian on $\mathbb{Z}^{\mathrm{d}}$, the spectrum of the hierarchical Laplacian, $\mathrm{sp}(\Delta)$, is a disconnected set.

Theorem 2.1.1. *Suppose that the random variables $\{\omega(x)\}_{x\in\mathbb{X}}$ are i.i.d. with a distribution μ^V, i.e. $\mathbb{P} = \times_{x\in\mathbb{X}}\mu^V$. Let S^V be the support of μ^V and I^V the smallest (possibly unbounded) closed interval containing S^V. Then, for $\mathbb{P}$-a.a. $\omega \in \Omega$, we have*

$$\mathrm{sp}(\Delta) + S^V \subset \mathrm{sp}(H_\omega) \subset (\mathrm{sp}(\Delta) + I^V) \cap ([0,1] + S^V).$$

In particular, if S^V is connected, then $S^V = I^V$ and

$$\mathrm{sp}(H_\omega) = \mathrm{sp}(\Delta) + S^V,$$

for $\mathbb{P}$-a.a. $\omega \in \Omega$.

Proof. Let $S_\omega = \mathrm{sp}(H_\omega)$. Then S_ω is a random closed subset of $\mathbb{R}$. For $\mathbb{P}$-a.a. $\omega \in \Omega$, we have $\omega(x) \in S^V$ and therefore $\mathrm{sp}(V_\omega) \subset S^V$. For such ω, the proof of the inclusion

$$S_\omega \subset (\mathrm{sp}(\Delta) + I^V) \cap ([0,1] + S^V),$$

is an immediate consequence of the following general fact:

Lemma 2.1.2. *Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$. If B is a bounded self-adjoint operators on $\mathcal{H}$, $b_1 = \inf \mathrm{sp}(B)$ and $b_2 = \sup \mathrm{sp}(B)$, then*

$$\mathrm{sp}(A + B) \subset \mathrm{sp}(A) + [b_1, b_2].$$

If A is bounded and B is bounded from below, then

$$\mathrm{sp}(A+B) \subset [\inf \mathrm{sp}(A) + \inf \mathrm{sp}(B), \infty).$$

If A is bounded and B is bounded from above, then

$$\mathrm{sp}(A+B) \subset (-\infty, \sup \mathrm{sp}(A) + \sup \mathrm{sp}(B)].$$

Proof. Suppose that B is bounded. By adding a constant to B, we can assume without of generality that $b_1 = -\|B\|$ and $b_2 = \|B\|$. If $z \notin \mathrm{sp}(A) + [-\|B\|, \|B\|]$, then $dist(z, \mathrm{sp}(A)) > \|B\|$ and hence $\|(z-A)^{-1}\| < \|B\|^{-1}$. Hence

$$\left\|(z-A)^{-1}B\right\| \leq \left\|(z-A)^{-1}\right\| \|B\| < 1,$$

and the operator

$$z - A - B = (z-A)(1-(z-A)^{-1}B),$$

has a bounded inverse. Suppose now B is bounded from below, say $\inf \mathrm{sp}(B) = b \in \mathbb{R}$. Let $a = \inf \mathrm{sp}(A)$. Then for any unit vector ψ in the domain of B,

$$\langle (A+B)\psi | \psi \rangle \geq a + b,$$

and hence $\mathrm{sp}(A+B) \subset [a+b, \infty)$. The case when B is bounded from above is proven similarly. $\square$

The proof of the inclusion

$$\mathrm{sp}(\Delta) + S^V \subset S_\omega,$$

for $\mathbb{P}$-a.a. $\omega \in \Omega$ is trickier. Let $S = \mathrm{sp}(\Delta) + S^V$. For rational numbers $r_1 < r_2$, consider the event

$$\mathcal{Q}_{r_1,r_2} = \{S_\omega \cap (r_1, r_2) \neq \emptyset\}.$$

Proposition 2.1.3. *If* $(r_1, r_2) \cap S \neq \emptyset$ *then* $\mathbb{P}(\mathcal{Q}_{r_1,r_2}) = 1$.

Before proving Proposition 2.1.3, let us see how it implies that $S \subset S_\omega$, for $\mathbb{P}$-a.a. $\omega \in \Omega$. Consider the event

$$\mathcal{Q} = \bigcap_{r_1<r_2:(r_1,r_2)\cap S\neq\emptyset} \mathcal{Q}_{r_1,r_2}.$$

Then $\mathbb{P}(\mathcal{Q}) = 1$. We claim that

$$\omega \in \mathcal{Q} \Rightarrow S \subset S_\omega.$$

Indeed, let $\omega \in \mathcal{Q}$. Suppose $u \in S$. For every rationals $r_1 < u < r_2$, we then have have $(r_1, r_2) \cap S \neq \emptyset$ and hence $\omega \in \mathcal{Q}_{r_1,r_2}$. Hence $S_\omega \cap (r_1, r_2) \neq \emptyset$. Since S_ω is a closed set, we conclude, after letting $r_1 \uparrow u$ and $r_2 \downarrow u$, that $u \in S_\omega$.

Proof of Proposition 2.1.3:

Proof. Suppose $(r_1, r_2)\cap S \neq \emptyset$ and let $u \in (r_1, r_2)\cap S$. Then we can write $u = \lambda_s + e$ where $e \in S^V$ and $s \in \{1, 2, \cdots, \infty\}$. For each $\varepsilon > 0$ and a ball B_r or radius r, consider the event

$$\mathcal{E}(B_r, \varepsilon) := \left\{\omega : \max_{x\in B_r} |\omega(x) - e| < \varepsilon\right\}.$$

Since $\mu^{\mathrm{V}}((e-\varepsilon, e+\varepsilon)) > 0$ and the random variables $\{\omega(x) : x \in B_r\}$ are independent, we have that

$$\mathbb{P}\left(\mathcal{E}(B_r, \varepsilon)\right) = \left(\mu^{\mathrm{V}}((e - \varepsilon, e + \varepsilon))\right)^{\mathrm{N}_r} > 0.$$

According to the hierarchical structure, $\mathbb{X}$ is a countable union of disjoint balls of radius r: $\mathbb{X} = \bigcup_{j=1}^{\infty} B_{r,j}$. Let

$$\mathcal{E}_{r,\varepsilon} = \limsup_{j} \mathcal{E}(B_{r,j}, \varepsilon).$$

Since the events $\{\mathcal{E}(B_{r,j}, \varepsilon), j = 1, \ldots, \infty\}$ are independent and $\sum_{j=1}^{\infty} \mathbb{P}\left(\mathcal{E}(B_{r,j}, \varepsilon)\right) = \infty$, the Borel-Cantelli Lemma yields that $\mathbb{P}(\mathcal{E}_{r,\varepsilon}) = 1$. Then the event

$$\mathcal{E} = \bigcap_{r \geq 0} \mathcal{E}_{r,1/(r+1)},$$

also has $\mathbb{P}(\mathcal{E}) = 1$. By construction, for $\omega \in \mathcal{E}$, there is a sequence $(B_{r,j_r})_{r \geq 0}$ of balls of increasing radius such that

$$(*) \qquad \max_{x \in B_{r,j_r}} |\omega(x) - e| < \frac{1}{r+1},$$

for all $r \geq 0$.

For $r \geq 1$, we let $q_r = \min(r-1, s)$. It is easy to see that for each $r \geq 1$, there exists a unit eigenfunction ψ_r of Δ corresponding to the eigenvalue λ_{q_r}, such that $\psi_r(x) = 0$ for $x \notin B_{r,j_r}$. Indeed, we can take a normalized function ψ_r constant on every radius-q_r sub-ball of B_{r,j_r}, zero outside B_{r,j_r} and such that $E_{q_r+1}\psi_r = 0$. Then

$$\lim_{r \to \infty} \|\Delta \psi_r - \lambda_s \psi_r\| = 0,$$

and, by $(*)$,

$$\lim_{r \to \infty} \|H_\omega \psi_r - (\lambda_s + e)\psi_r\| = 0,$$

for $\omega \in \mathcal{E}$. Hence, ψ_r is a Weyl sequence for the operator H_ω and $u = \lambda_s + e \in \mathrm{sp}(H_\omega)$. Hence $u \in (r_1, r_2) \cap S_\omega$ and

$$(r_1, r_2) \cap S_\omega \neq \emptyset,$$

i.e. $\omega \in \mathcal{Q}_{r_1,r_2}$. We have shown that $\mathcal{E} \subset \mathcal{Q}_{r_1,r_2}$. Since $\mathbb{P}(\mathcal{E}) = 1$, we conclude that $\mathbb{P}(\mathcal{Q}_{r_1,r_2}) = 1$. □

This completes the proof of Theorem 2.1.1. □

Remark: The corresponding statement is well known for the discrete Laplacian on $\mathbb{Z}^{\mathrm{d}}$, see for example [CL, CKFS, PF]. The proof given above corrects a mistake made in [K2].

2.2 Statement of spectral localization theorems

We denote by $\mathrm{sp}_{\mathrm{ac}}(H_\omega)$ the a.c.part of the spectrum of H_ω and by $\mathrm{sp}_{\mathrm{cont}}(H_\omega)$ the continuous part. In [M2], Molchanov proved the following localization result:

Theorem 2.2.1. *Let* $(\mathbb{X}, \mathsf{d}, \Delta)$ *be a regular hierarchical model. Assume that*

$$\sum_{r=1}^{\infty} \mathrm{p}_r r^{1+\varepsilon} < \infty, \tag{2.3}$$

for some $\varepsilon > 0$. *If the random variables* $\{\omega(x) : x \in \mathbb{X}\}$ *are i.i.d. with a Cauchy distribution,*

$$k_{a,h}(e)de = \frac{1}{\pi} \frac{h}{(e-a)^2 + h^2} de, \tag{2.4}$$

for some parameters $a \in \mathbb{R}$ *and* $h > 0$, *then* $\mathrm{sp}_{\mathrm{cont}}(H_\omega) = \emptyset$ *for* $\mathbb{P}$*-a.a.* ω.

Molchanov's theorem is remarkable for several reasons. First of all, it is a result valid for any disorder. Given a coupling constant $c > 0$, the random operator $\Delta + cV_\omega$ can be rewritten as $\Delta + \sum_{x \in \mathbb{X}} c\omega(x) \langle \delta_x | \cdot \rangle \delta_x$. If $\omega(x)$ has a Cauchy distribution $k_{a,h}$, then $c\omega(x)$ has a Cauchy distribution $k_{ca,ch}$. Hence, under the hypotheses of the theorem, $\Delta + cV_\omega$ has pure point spectrum for $\mathbb{P}$-a.a. ω. The next observation is that the condition (2.3) is very weak, it particular it does not at all depend on n_r. This mens that Theorem 2.2.1 holds in arbitrary spectral dimension. Indeed, let $\mathrm{n} \geq 2$ be given and let $\mathrm{d_{sp}}$ be a positive real number. Then there is a unique $\rho > 1$ with $\mathrm{d_{sp}} = \mathrm{d_{sp}}(\mathrm{n}, \rho)$. Set $\mathrm{p}_r = C\rho^{-r}$, where C is a normalization constant making $\sum_{r=1}^{\infty} \mathrm{p}_r = 1$. Then p_r satisfies (2.3). Therefore, we can construct homogeneous hierarchical Anderson models of arbitrary spectral dimension for which Theorem 2.2.1 holds.

Cauchy random variables play a very special role in the theory of random discrete Schrödinger operators and it is natural to ask whether one can extend Theorem 2.2.1 to distributions other than Cauchy. A partial answer to this question is that one can prove localization for very general distributions of $\omega(x)$, at any disorder, provided that one imposes stronger decay conditions on p_r than (2.3). This restriction will in turn impose an upper bound on the spectral dimension.

Concerning the probability measure $\mathbb{P}$, we will make a technical assumption having to do with the notion of conditional density. We denote by $\mathcal{L}$ the Lebesgue measure on $\mathbb{R}$. For any $x \in \mathbb{X}$, Ω can be decomposed along the x'th coordinate as $\Omega = \mathbb{R} \times \widetilde{\Omega}$, $\widetilde{\Omega} = \mathbb{R}^{\mathbb{X}\setminus\{x\}}$. Let $\widetilde{\mathbb{P}}_x$ be the corresponding marginal of $\mathbb{P}$ defined by $\widetilde{\mathbb{P}}_x(\widetilde{\mathcal{B}}) = \mathbb{P}(\mathbb{R} \times \widetilde{\mathcal{B}})$, where $\widetilde{\mathcal{B}} \subset \widetilde{\Omega}$ is a Borel set. Then for $\widetilde{\mathbb{P}}_x$-a.a. $\widetilde{\omega} \in \widetilde{\Omega}$, there is a probability measure $\mathbb{P}_x^{\widetilde{\omega}}$ on $\mathbb{R}$ s.t. the conditional Fubini theorem holds: for all

$f \in L^1(\Omega, \mathbb{P})$ we have

$$\int_\Omega f(\omega) d\mathbb{P}(\omega) = \int_{\widetilde{\Omega}} \left(\int_{\mathbb{R}} f(\xi, \widetilde{\omega}) d\mathbb{P}_x^{\widetilde{\omega}}(\xi) \right) d\widetilde{\mathbb{P}}_x(\widetilde{\omega}).$$

If for $\widetilde{\mathbb{P}}_x$-a.a. $\widetilde{\omega} \in \widetilde{\Omega}$, $\mathbb{P}_x^{\widetilde{\omega}}$ is a.c. with respect to $\mathcal{L}$, then we say that $\mathbb{P}$ has a conditional density along the x'th coordinate. $\mathbb{P}$ is called *conditionally a.c.* if for every $x \in \mathbb{X}$, $\mathbb{P}$ has a conditional density along the x'th coordinate. An important special case of a conditionally a.c. probability measure is the product measure $\mathbb{P} = \otimes_{x \in \mathbb{X}} \mathbb{P}_x$, where each $\mathbb{P}_x$ is a probability measure on $\mathbb{R}$ a.c. with respect to $\mathcal{L}$.

Our main result on spectral localization is.

Theorem 2.2.2. *Let* $(\mathbb{X}, \mathtt{d}, \Delta)$ *be a regular hierarchical model. Assume that there exists a sequence* $u_r > 0$ *with* $\sum_{r=1}^\infty u_r^{-1} < \infty$ *and*

$$\sum_{r=1}^{\infty} \mathtt{p}_r u_r \sqrt{\mathtt{N}_r} < \infty. \tag{2.5}$$

Then:

(1) *For all* $\omega \in \Omega$, $\mathrm{sp}_{\mathrm{ac}}(H_\omega) = \emptyset$.

(2) *If* $\mathbb{P}$ *is conditionally a.c. then* $\mathrm{sp}_{\mathrm{cont}}(H_\omega) = \emptyset$ *for* $\mathbb{P}$*-a.a.* ω.

Remarks on Theorem 2.2.2

Remark 1. The condition (2.5) is more demanding than (2.3). The required decay of $\mathtt{p}_r$ imposes an upper bound on the spectral dimension of Δ. Theorem 2.2.2 and Proposition 1.3.2 allow us to construct hierarchical models with spectral dimension $\mathrm{d_{sp}} \leq 4$ that exhibit localization at arbitrary disorder. If $(\mathbb{X}, \mathtt{d})$ is a homogeneous hierarchical structure of degree $\mathtt{n} \geq 2$ and $\mathtt{p}_r = C\rho^{-r}$ with $\rho > \sqrt{\mathtt{n}}$, then the hypothesis (2.5) is fulfilled for $u_r = r^{1+\epsilon}$. Given $0 < \mathrm{d_{sp}} < 4$ one can adjust $\rho > \sqrt{\mathtt{n}}$

to make $d_{sp}(n, \rho) = d_{sp}$. If $p_r = Cr^{-2-\varepsilon} n^{-r/2}$, then the model has spectral dimension $d_{sp} = 4$ and (2.5) is verified for $u_r = r^{1+\varepsilon/2}$. One can also construct trivial models with $d_{sp} = 0$ by taking p_r to decrease faster than ρ^{-r} for any ρ. We emphasize that homogeneity of the hierarchical structure is not required for Theorem 2.2.2.

Remark 2. The fractional moments method of Aizenman and Molchanov [AM] allows to prove localization for $\Delta + cV_\omega$ for large disorder c or for large energies. One needs an extra regularity hyphothesis on the random variables $\omega(x)$ and the condition on Δ that

$$M := \sup_x \sum_{y \in X} |\langle \delta_x | \Delta \delta_y \rangle|^s < \infty \tag{2.6}$$

for some $0 < s < 1$. Simple estimates show that

$$\sum_{r=1}^{\infty} p_r N_r^{1-s} \leq M \leq \sum_{r=1}^{\infty} p_r^s N_r^{1-s}.$$

The requirement (2.6) on the decay of p_r is comparable to the hypothesis (2.5), while Theorem 2.2.2 is valid at arbitrary disorder or energy.

Remark 3. By a general result of [JL], if $\omega(x)$ are i.i.d. random variables with a density, then the eigenvalues of H_ω are simple with probability one.

Remark 4. Part (2) of Theorem 2.2.2 does not hold for all ω. Our method of proof combined with the general results of [DMS], [G] yields that H_ω will have singular continuous spectrum for some ω's.

2.3 A criterion for spectral localization

In this section we formulate and prove a sufficient condition for H_ω to have $\mathrm{sp}_{ac}(H_\omega) = \emptyset$ for all $\omega \in \Omega$, and a sufficient condition for H_ω to have $\mathrm{sp}_{cont}(H_\omega) = \emptyset$

for $\mathbb{P}$-a.a. ω. Both theorems 2.2.1 and 2.2.2 will follow from this localization criterion and will be proven in the next section.

Consider the truncated operators

$$H_{\omega,r} = \sum_{s=0}^{r} \mathrm{p}_s E_s + V_\omega, \qquad r \geq 0. \tag{2.7}$$

Fix $\omega \in \Omega$. For any ball B_r of radius r, the subspace $l^2(B_r)$ is invariant for $H_{\omega,r}$. This is the main reason for working with the truncated hierarchical Laplacian. Let $\sigma(\omega, B_r)$ be the set of the eigenvalues of the restricted operator $H_{\omega,r} \restriction l^2(B_r)$ and $\sigma_\omega = \bigcup \sigma(\omega, B_r)$ where the union is over all balls in $\mathbb{X}$ with all possible radii. Clearly, σ_ω is a countable subset of $\mathbb{R}$, and hence of zero Lebesgue measure. For $z \in \mathbb{C} \backslash \sigma_\omega$, $r \geq 0$, and $x, y \in \mathbb{X}$, we set

$$G_{\omega,r}(x, y; z) = \langle \delta_x | (H_{\omega,r} - z)^{-1} \delta_y \rangle.$$

For $z \in \mathbb{C} \backslash \sigma_\omega$, $r \geq 0$ and $t \in \mathbb{X}$, let $g_{\omega,r}(t; z)$ be the average of $G_{\omega,r}(\cdot, t; z)$ over the ball $B(t, r)$, i.e.

$$g_{\omega,r}(t; z) = \frac{1}{\mathrm{N}_r} \sum_{\mathrm{d}(t',t) \leq r} G_{\omega,r}(t', t; z).$$

Since the joint spectral measure for δ_t, $\delta_{t'}$ and $H_{\omega,r}$ is real, $G_{\omega,r}(t', t; z) = G_{\omega,r}(t, t'; z)$ and

$$g_{\omega,r}(t; z) = \frac{1}{\mathrm{N}_r} \sum_{\mathrm{d}(t',t) \leq r} G_{\omega,r}(t, t'; z) = \frac{1}{\mathrm{N}_r} \langle \delta_t | (H_{\omega,r} - z)^{-1} \mathbf{1}_{B(t,r)} \rangle. \tag{2.8}$$

For $B_r \in \mathcal{P}_r$, we set

$$\gamma_\omega(B_r; z) = \frac{1}{\mathrm{N}_r} \langle \mathbf{1}_{B_r} | (H_{\omega,r} - z)^{-1} \mathbf{1}_{B_r} \rangle.$$

For $x \in \mathbb{X}$ and $\omega \in \Omega$, we denote by μ_x^ω the spectral measure for H_ω and δ_x, by $\mu_{x,\mathrm{cont}}^\omega$ the continuous part of μ_x^ω and by $\mu_{x,\mathrm{ac}}^\omega$ the a.c. part. The main spectral localization criterion for the hierarchical Anderson model is the following

Theorem 2.3.1. *Assume that there exists a sequence* $u_r > 0$ *with* $\sum_{r=1}^\infty u_r^{-1} < \infty$ *and*

$$\sum_{r=1}^{\infty} \mathrm{p}_r u_r < \infty. \tag{2.9}$$

Let $x \in \mathbb{X}$ *and a Borel set* $\mathcal{B} \subset \mathbb{R}$ *be given. Then:*

(1) *If for a fixed* $\omega \in \Omega$,

$$\sum_{s=1}^{\infty} \mathrm{p}_s \left|\gamma_\omega(B(x,s);e)\right| < \infty, \tag{2.10}$$

for $\mathcal{L}$*-a.e.* $e \in \mathcal{B}$, *then* $\mu_{x,\mathrm{ac}}^\omega(\mathcal{B}) = 0$.

(2) *If* $\mathbb{P}$ *has a conditional density along the* x*'th coordinate, and* (2.10) *holds for* $\mathbb{P} \otimes \mathcal{L}$*-a.a.* $(\omega, e) \in \Omega \times \mathcal{B}$, *then* $\mu_{x,\mathrm{cont}}^\omega(\mathcal{B}) = 0$ *for* $\mathbb{P}$*-a.a.* $\omega \in \Omega$.

(3) *Assume that for* $\mathbb{P} \otimes \mathcal{L}$*-a.a.* $(\omega, e) \in \Omega \times \mathcal{B}$, *there is a finite constant* $C_{\omega,e,x}$ *such that*

$$|\gamma_\omega(B(x,r);e)| \leq C_{\omega,e,x} u_r, \tag{2.11}$$

for all $r \geq 0$. *If* $\mathbb{P}$ *has a conditional density along the* x*'th coordinate, then* $\mu_{x,\mathrm{cont}}^\omega(\mathcal{B}) = 0$ *for* $\mathbb{P}$*-a.a.* $\omega \in \Omega$.

To facilitate the exposition, we will first prove three simple propositions (2.3.2, 2.3.3 and 2.3.4). Then we will review the Simon-Wolff localization criterion (Theorem 2.3.5) and give the proof of Theorem 2.3.1.

Proposition 2.3.2. *Let $\omega \in \Omega$, $x, y \in \mathbb{X}$, $z \in \mathbb{C}\backslash\sigma_\omega$ and $r \geq 0$ be given. Then*

$$G_{\omega,r}(x,y;z) = G_{\omega,0}(x,y;z) - \sum_{s=\mathtt{d}(x,y)}^{r} \mathtt{p}_s \mathtt{N}_{s-1} g_{\omega,s-1}(x;z) g_{\omega,s}(y;z), \tag{2.12}$$

and

$$g_{\omega,r}(x;z) = \frac{1}{\mathtt{N}_r} G_{\omega,0}(x,x;z) \prod_{s=1}^{r} (1 - \mathtt{p}_s \gamma_\omega(B(x,s);z)). \tag{2.13}$$

Proof. The formula (2.12) holds for $r = 0$ since $\mathtt{p}_0 = 0$. For $s \geq 1$, the resolvent identity yields

$$(H_{\omega,s} - z)^{-1}\delta_y - (H_{\omega,s-1} - z)^{-1}\delta_y = -(H_{\omega,s-1} - z)^{-1} \mathtt{p}_s E_s (H_{\omega,s} - z)^{-1}\delta_y.$$

Observe that $E_s(H_{\omega,s} - z)^{-1}\delta_y = g_{\omega,s}(y;z)\mathbf{1}_{B(y,s)}$. Taking $\langle \delta_x | \cdot \rangle$ in the above equation yields

$$G_{\omega,s}(x,y;z) - G_{\omega,s-1}(x,y;z) = -\mathtt{p}_s g_{\omega,s}(y;z) \langle \delta_x | (H_{\omega,s-1} - z)^{-1} \mathbf{1}_{B(y,s)} \rangle. \tag{2.14}$$

Note that by (2.8),

$$\langle \delta_x | (H_{\omega,s-1} - z)^{-1} \mathbf{1}_{B(y,s)} \rangle = \begin{cases} \mathtt{N}_{s-1} g_{\omega,s-1}(x;z), & \text{if } \mathtt{d}(x,y) \leq s, \\ 0, & \text{if } \mathtt{d}(x,y) > s. \end{cases}$$

The formula (2.12) follows after adding (2.14) for $s = 1, 2, \cdots, r$.

The proof of (2.13) is similar. The resolvent identity yields

$$\langle \delta_x | (H_{\omega,r} - z)^{-1} \mathbf{1}_{B(x,r)} \rangle = (1 - \mathtt{p}_r \gamma_\omega(B(x,r);z)) \langle \delta_x | (H_{\omega,r-1} - z)^{-1} \mathbf{1}_{B(x,r-1)} \rangle,$$

and (2.13) follows after iterating the above formula. $\square$

Proposition 2.3.3. *Let $u_r > 0$ be a sequence with $\sum_{r=1}^{\infty} u_r^{-1} < \infty$. Let $\omega \in \Omega$ and $x \in \mathbb{X}$ be fixed. Then for $\mathcal{L}$-a.e. $e \in \mathbb{R}$, there exists a finite constant C_e such that*

$$\left(\sum_{\mathrm{d}(x,y) \le s} |g_{\omega,s}(y;e)|^2 \right)^{1/2} \le C_e u_s, \tag{2.15}$$

for all $s \ge 0$.

Proof. Since $l^2(B(x,r))$ is an $\mathtt{N}_r$-dimensional invariant subspace for $H_{\omega,r}$ and since $\left\|\mathbf{1}_{B(x,r)}\right\|_2 = \sqrt{\mathtt{N}_r}$, we have by Lemma 3.4.7(see the appendix) that for $M_r > 0$,

$$\mathcal{L}\left(\left\{e \in \mathbb{R}\backslash\sigma_\omega : \left\|(H_{\omega,r} - e)^{-1}\mathbf{1}_{B(x,r)}\right\|_2^2 \ge M_r\right\}\right) \le \frac{4\mathtt{N}_r}{\sqrt{M_r}}.$$

Let $M_r = (u_r \mathtt{N}_r)^2$. Then $\sum_{r=1}^{\infty} \mathtt{N}_r M_r^{-1/2} < \infty$. By the Borel-Cantelli lemma, for $\mathcal{L}$-a.a. $e \in \mathbb{R}\backslash\sigma_\omega$, there exists a finite constant C_e such that

$$\left\|(H_{\omega,r} - e)^{-1}\mathbf{1}_{B(x,r)}\right\|_2 < C_e\sqrt{M_r}, \tag{2.16}$$

for all $r \ge 0$. Observe that

$$\begin{aligned}
\left(\sum_{\mathrm{d}(x,y) \le s} |g_{\omega,s}(y;e)|^2 \right)^{1/2} &= \left(\sum_{\mathrm{d}(x,y) \le s} \left| \frac{1}{\mathtt{N}_s} \langle \delta_y | (H_{\omega,s} - e)^{-1}\mathbf{1}_{B(y,s)} \rangle \right|^2 \right)^{1/2} \\
&= \frac{1}{\mathtt{N}_s} \left(\sum_{\mathrm{d}(x,y) \le s} \left| \langle \delta_y | (H_{\omega,s} - e)^{-1}\mathbf{1}_{B(x,s)} \rangle \right|^2 \right)^{1/2} \\
&= \frac{1}{\mathtt{N}_s} \left\|(H_{\omega,s} - e)^{-1}\mathbf{1}_{B(x,s)}\right\|_2.
\end{aligned}$$

Inequality (2.16) yields (2.15). □

The key step in proving Theorem 2.3.1 is:

Proposition 2.3.4. *Assume that there exists a sequence $u_r > 0$ with $\sum_{r=1}^{\infty} u_r^{-1} < \infty$ and $\sum_{r=1}^{\infty} \mathrm{p}_r u_r < \infty$. Let $\mathcal{B} \subset \mathbb{R}$ be a Borel set and let $\omega \in \Omega$, $x \in \mathbb{X}$ be fixed. Suppose that*

$$\sum_{s=1}^{\infty} \mathrm{p}_s \left|\gamma_\omega(B(x,s);e)\right| < \infty, \tag{2.17}$$

for $\mathcal{L}$-a.a. $e \in \mathcal{B}$. Then, for $\mathcal{L}$-a.a. $e \in \mathcal{B}$,

$$\sup_{r\geq 0} \sum_{y\in\mathbb{X}} \left|G_{\omega,r}(x,y;e)\right|^2 < \infty. \tag{2.18}$$

Proof. For all $e \in \mathbb{C}\backslash\sigma_\omega$, the representation formula (2.12) and Cauchy-Schwarz inequality yield

$$\begin{aligned}\left(\sum_{y\in\mathbb{X}} \left|G_{\omega,r}(x,y;e)\right|^2\right)^{1/2} &\leq \left|G_{\omega,0}(x,x;e)\right| \\ &+ \sum_{s=1}^{r} \mathrm{p}_s \mathrm{N}_{s-1} \left|g_{\omega,s-1}(x;e)\right| \left(\sum_{\mathrm{d}(x,y)\leq s} \left|g_{\omega,s}(y;e)\right|^2\right)^{1/2}.\end{aligned}$$

Hence, by Proposition 2.3.3, for $\mathcal{L}$-a.a. $e \in \mathbb{R}$,

$$\left(\sum_{y\in\mathbb{X}} \left|G_{\omega,r}(x,y;e)\right|^2\right)^{1/2} \leq \left|G_{\omega,0}(x,x;e)\right| + C_e \sum_{s=1}^{r} \mathrm{p}_s u_s \mathrm{N}_{s-1} \left|g_{\omega,s-1}(x;e)\right|. \tag{2.19}$$

Hypothesis (2.17) implies that for $\mathcal{L}$-a.a. $e \in \mathcal{B}$, the product $\prod_{s=1}^{\infty}(1-\mathrm{p}_s\gamma_\omega(B(x,s);e))$ converges. For such e, it follows from (2.13) that there is a finite constant $C_e^{'}$ such that

$$\mathrm{N}_s \left|g_{\omega,s}(x;e)\right| \leq C_e^{'}, \tag{2.20}$$

for all $s \geq 0$.

Combination of (2.19) with (2.20) yields the estimate

$$\left(\sum_{y\in X}|G_{\omega,r}(x,y;e)|^2\right)^{1/2} \leq |G_{\omega,0}(x,x;e)| + C_e C_e' \sum_{s=1}^{r} \mathrm{p}_s u_s,$$

and the result follows. □

Before proving Theorem 2.3.1, let us recall the Simon-Wolff localization criterion. Define the function $\mathcal{G}_{\omega,x} : \mathbb{R} \to [0,+\infty]$ by

$$\mathcal{G}_{\omega,x}(e) := \int_{\mathbb{R}} \frac{d\mu_x^\omega(\lambda)}{(e-\lambda)^2} = \lim_{\epsilon\downarrow 0} \left\|(\Delta + V_\omega - e - i\epsilon)^{-1}\delta_x\right\|^2.$$

By the Theorem of de la Vallé Poussin,

$$d\mu_{x,\mathrm{ac}}^\omega(e) = \pi^{-1}\left(\lim_{\epsilon\downarrow 0} \varepsilon \left\|(\Delta + V_\omega - e - i\epsilon)^{-1}\delta_x\right\|^2\right) de.$$

Hence, if for a fixed $\omega \in \Omega$ we have that $\mathcal{G}_{\omega,x}(e) < \infty$ for $\mathcal{L}$-a.a. $e \in \mathbb{R}$, then $\mu_{x,\mathrm{ac}}^\omega = 0$.

The Simon-Wolff localization criterion is summarized in:

Theorem 2.3.5. *Assume that $\mathbb{P}$ has a conditional density along the x'th coordinate. Let $\mathcal{B} \subset \mathbb{R}$ be a Borel set such that $\mathcal{G}_{\omega,x}(e) < \infty$ for $\mathbb{P}\otimes\mathcal{L}$-a.a. $(\omega, e) \in \Omega \times \mathcal{B}$. Then $\mu_{x,\mathrm{cont}}^\omega(\mathcal{B}) = 0$ for $\mathbb{P}$-a.a. $\omega \in \Omega$.*

Theorem 2.3.5 is a well known consequence of the rank-1 Simon-Wolff theorem [SW] (see also the lecture notes [J]) and the conditional Fubini's theorem.

Now we can prove the main localization criterion.

Proof of Theorem 2.3.1. Fix $\omega \in \Omega$ and fix $e \in \mathbb{R}\backslash\sigma_\omega$ for which (2.9) holds. Proposition 2.3.4 yields that then we also have the bound (2.18). By monotone convergence

$$\int_{\mathbb{R}} \frac{d\mu_x^\omega(\lambda)}{(e-\lambda)^2} = \lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}} \frac{d\mu_x^\omega(\lambda)}{(e-\lambda)^2+\varepsilon^2} = \sup_{\varepsilon>0} \int_{\mathbb{R}} \frac{d\mu_x^\omega(\lambda)}{(e-\lambda)^2+\varepsilon^2}.$$

Since for any $z \in \mathbb{C} \backslash \mathbb{R}$,

$$\lim_{r \to \infty} \left\| (H_{\omega,r} - z)^{-1} - (H_\omega - z)^{-1} \right\| = 0,$$

we have that the weak-* limit $\lim_{r \to \infty} \mu^\omega_{x,r}$ equals μ^ω_x, where $\mu^\omega_{x,r}$ is the spectral measure for $H_{\omega,r}$ and δ_x. Therefore

$$\begin{aligned}
\int_{\mathbb{R}} \frac{d\mu^\omega_x(\lambda)}{(e-\lambda)^2} = \sup_{\varepsilon>0} \lim_{r\to\infty} \int_{\mathbb{R}} \frac{d\mu^{\omega,r}_x(\lambda)}{(e-\lambda)^2+\varepsilon^2} &\leq \sup_{\varepsilon>0, r\geq 1} \int_{\mathbb{R}} \frac{d\mu^{\omega,r}_x(\lambda)}{(e-\lambda)^2+\varepsilon^2} \\
&= \sup_{r\geq 1} \int_{\mathbb{R}} \frac{d\mu^{\omega,r}_x(\lambda)}{(e-\lambda)^2} \\
&= \sup_{r\geq 1} \left\| (H_{\omega,r} - e)^{-1} \delta_x \right\|^2 \\
&= \sup_{r\geq 1} \sum_{y\in\mathbb{X}} \left| G_{\omega,r}(x,y;e) \right|^2 < \infty.
\end{aligned}$$

In the final equality we used the fact that $\{\delta_y : y \in \mathbb{X}\}$ is an orthonormal basis for $l^2(\mathbb{X})$. Since $\mathcal{L}(\sigma_\omega) = 0$ and since the bound (2.18) holds for $\mathcal{L}$-a.a. $e \in \mathbb{R} \backslash \sigma_\omega$, we have that for every fixed $\omega \in \Omega$, $\mathcal{G}_{\omega,x}(e) < \infty$ for $\mathcal{L}$-a.a. $e \in \mathbb{R}$. This proves part (1). Part (2) follows from the fact that $\mathcal{G}_{\omega,x}(e) < \infty$ for $\mathbb{P} \otimes \mathcal{L}$-a.a. $(\omega, e) \in \Omega \times \mathbb{R}$ and the Simon-Wolff criterion. Part (3) is an immediate consequence of part (2) and assumption (2.9). $\square$

2.4 Proofs of spectral localization theorems

In this section we show how to apply the localization criterion (Theorem 2.3.1) to prove theorems 2.2.2 and 2.2.1.

Proof of Theorem 2.2.2. Since $\gamma_\omega(B(x,s);e) = \sum_{\mathsf{d}(x,y)\leq s} g_{\omega,s}(y;e)$, Cauchy-Schwarz inequality and Proposition 2.3.3 yield that for $\mathcal{L}$-a.e. $e \in \mathbb{R}$,

$$|\gamma_\omega(B(x,s);e)| \leq C_e u_s \sqrt{\mathtt{N}_s}. \tag{2.21}$$

Hence $\sum_{s\geq 0} \mathrm{p}_s \left|\gamma_\omega(B(x,s);e)\right| < \infty$ and the result follows from parts (1) and (2) of Theorem 2.3.1. $\square$.

The proof of Molchanov's theorem 2.2.1 requires more work. Let us introduce some notation. It is convenient to form the complex number $v = a + ih \in \mathbb{C}_+$ (in the upper half-plane) and to denote by k_v the Cauchy distribution with density $k_{a,h}$. If ν is a Borel probability measure on the Riemann sphere $\mathbb{S} = \mathbb{C} \cup \{\infty\}$, and $\tau : \mathbb{S} \to \mathbb{S}$ is a Borel measurable map, then $\tau\nu$ will denote the induced Borel probability measure on $\mathbb{S}$:

$$(\tau\nu)(\mathcal{B}) := \nu(\tau^{-1}(\mathcal{B})),$$

for Borel sets $\mathcal{B} \subset \mathbb{S}$. If $n \geq 2$ is a integer, and ν is a Borel probability measure on $\mathbb{S}$, we let

$$A_n\nu = \tau(\underbrace{\nu \star \cdots \star \nu}_{n \text{ times}}),$$

where $\tau(z) = z/n$, and $\star$ is the usual convolution of measures. Hence, if $Y_1, \ldots Y_n$ are i.i.d. random variables on $\mathbb{S}$ with distribution ν, then $A_n\nu$ is the distribution of $(Y_1 + \cdots + Y_n)/n$. The following proposition summarizes some of the basic facts about Cauchy distributions.

Proposition 2.4.1. *Cauchy distributions have the following properties:*

1. *If $v_1, v_1 \in \mathbb{C}_+$, then*

$$k_{v_1} \star k_{v_2} = k_{v_1+v_2}.$$

2. *If $\tau(z) = -\overline{z}$ or if $\tau(z) = (az+b)/(cz+d)$, where $a,b,c,d \in \mathbb{R}$, $ad - bc > 0$, then*

$$\tau k_v = k_{\tau v},$$

for all $v \in \mathbb{C}_+$.

3. *For all $v \in \mathbb{C}_+$ and integers $n \geq 2$,*

$$A_n k_v = k_v.$$

Proof. The proof uses basic harmonic function theory. For bounded continuous functions $h : \mathbb{R} \to \mathbb{R}$, let $U_h : \mathbb{C}_+ \to \mathbb{R}$ be defined by $U_h(v) = \int_{\mathbb{R}} h(t)dk_v(t)$. Then $U_h(v)$ is the unique bounded harmonic function in $\mathbb{C}_+$ which extends continuously to $\mathbb{R}$ with $\lim_{y \downarrow 0} U_h(x+iy) = h(x)$. If X is a random variable with a Cauchy distribution k_v, $X_1 \sim k_v$, then $\mathbb{E}h(X) = U_h(v)$. Let $X_1 \sim k_{v_1}$ and $X_1 \sim k_{v_1}$. For fixed $t \in \mathbb{R}$, we let $h_t(s) = h(s+t)$. Then

$$\begin{aligned}
\mathbb{E}h(X_1 + X_2) &= \int \int h(s+t) dk_{v_1}(s) dk_{v_2}(t) \\
&= \int \left\{ \int h_t(s) dk_{v_1}(s) \right\} dk_{v_2}(t) \\
&= \int \{U_{h_t}(v_1)(s)\} \, dk_{v_2}(t) \\
&= \int \{U_h(t + v_1)(s)\} \, dk_{v_2}(t) \\
&= U_h(v_1 + v_2).
\end{aligned}$$

Since h is arbitrary, we conclude that $X_1 + X_2 \sim k_{v_1+v_2}$. This proves 1. For the proof of 2., consider the case $\tau(z) = (az+b)/(cz+d)$. The mapping τ is an analytic bijection $\mathbb{C} \setminus \{-d/c\} \to \mathbb{C} \setminus \{a/c\}$ with an analytic inverse. We have

$$\operatorname{Im} \tau(z) = \frac{(ad - bc)\operatorname{Im} z}{|cz + d|^2}.$$

Hence τ maps $\mathbb{C}_+ \to \mathbb{C}_+$, $\mathbb{C}_- \to \mathbb{C}_-$, $\mathbb{R}\backslash\{-d/c\} \to \mathbb{R}\backslash\{a/c\}$. Let $X \sim k_v$, let $h : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function whose support does not contain a/c and let $f(s) = h(\tau(s))$. Then

$$U_f(v) = U_h(\tau(v)), \qquad \operatorname{Im} v > 0.$$

Hence

$$\int h dk_{\tau(v)} = U_h(\tau(v)) = U_f(v) = \mathbb{E}f(X) = \mathbb{E}h(\tau(X)).$$

Since h is arbitrary, this implies that $\tau(X) \sim k_{\tau(v)}$. The case $\tau(z) = -\overline{z}$ is proved along the same lines using the fact that τ preserves harmonicity. 3. follows from 1. and 2.. $\square$

We will use fractional linear transformations of the special form

$$\tau_p(z) = \frac{z}{1+pz},$$

where $p \geq 0$. Note that $\tau_{p+p'} = \tau_p \circ \tau_{p'}$. The transformations τ_p are important in the hierarchical Anderson model because of the following recursive relation, proven by Molchanov in [M2].

Proposition 2.4.2. *Let $\omega \in \Omega$ and $z \in \mathbb{C}\backslash\sigma_\omega$ be given. Then for all $x \in \mathbb{X}$,*

$$\gamma_\omega(B(x,0); z) = \frac{1}{\omega(x) - z}. \tag{2.22}$$

For all $r \geq 1$ and $B_r \in \mathcal{P}_r$, we have the recursive formula

$$\gamma_\omega(B_r; z) = \tau_{\mathrm{p}_r}\left(\frac{1}{\mathrm{n}_r}\sum_{B_{r-1,j}\subset B_r} \gamma_\omega(B_{r-1,j}; z)\right), \tag{2.23}$$

where the sum contains n_r *terms, corresponding to the* n_r *balls* $B_{r-1,j}$ *of radius* $r-1$ *contained in* B_r.

Proof. The formula (2.22) is clear since $G_{\omega,0}(x,y;z) = (\omega(x)-z)^{-1}\langle\delta_x|\delta_y\rangle$. For $r \geq 1$, the resolvent identity yields

$$(H_{\omega,r} - z)^{-1}\mathbf{1}_{B_r} - (H_{\omega,r-1} - z)^{-1}\mathbf{1}_{B_r} = -\mathrm{p}_r\gamma_\omega(B_r;z)(H_{\omega,r-1} - z)^{-1}\mathbf{1}_{B_r}.$$

Taking $\langle \mathbf{1}_{B_r}|\cdot\rangle$ in the above equation yields

$$\mathtt{N}_r\gamma_\omega(B_r;z) - \sum_{B_{r-1,j}\subset B_r} \mathtt{N}_{r-1}\gamma_\omega(B_{r-1,j};z) = -\mathrm{p}_r\gamma_\omega(B_r;z) \sum_{B^j_{r-1}\subset B_r} \mathtt{N}_{r-1}\gamma_\omega(B_{r-1,j};z).$$

The formula (2.23) follows after dividing by $\mathtt{N}_r$ in the above equation and then solving for $\gamma_\omega(B_r;z)$. □

Proof of Theorem 2.2.1.

Since for every fixed ω, $\mathcal{L}(\sigma_\omega) = 0$, we have $e \notin \sigma_w$ for $\mathbb{P}\times\mathcal{L}$-a.a. $(\omega, e) \in \Omega\times\mathbb{R}$. Hence, there is a Borel mesurable set $E \subset \mathbb{R}$ with the properties

1. $\mathcal{L}(\mathbb{R}\backslash E) = 0$, and
2. for every fixed $e \in E$, there is a set $\Omega_e \in \mathcal{F}$ with $\mathbb{P}(\Omega_e) = 1$ and such that $e \notin \sigma_w$ for all $\omega \in \Omega_e$.

Let us fix $e \in E$. Then for all $\omega \in \Omega_e$, and hence for $\mathbb{P}$-a.a. $\omega \in \Omega$, all the representation formulas in propositions 2.3.2 and 2.4.2 are valid. It follows from (2.22) and from Proposition 2.4.1 that $\{\gamma_\omega(B(x,0);e)\}_{x\in\mathbb{X}}$ are i.i.d. Cauchy random variables with distribution k_{v_0}, where $v_0(e) = -(e-a+ih)^{-1}$. Moreover, (2.23) and Proposition 2.4.1 yield that for $r \geq 1$, $\{\gamma_\omega(B_r;e)\}_{B_r\in\mathcal{P}_r}$ are i.i.d. Cauchy random variables with distribution k_{v_r}, where $v_r(e) = \tau_{\lambda_r} v_0(e)$. Since $\lambda_r \longrightarrow \lambda_\infty = 1$ as $r \longrightarrow \infty$, the closure of the orbit

$$V(e) = \bigcup_{r \geq 0} \{v_r(e)\},$$

is equal to $V(e) \cup \{\tau_1 v_0(e)\}$, a compact set in $\mathbb{C}_+$. Since $\sup_{r\geq 0} |v_r(e)| < \infty$, there is a constant $K(e) < \infty$, such that

$$\mathbb{P}\left(|\gamma_\omega(B_r; e)| > u\right) \leq \frac{K(e)}{u},$$

for all real $u > 0$, integer $r \geq 0$ and $B_r \in \mathcal{P}_r$. Let us now fix $x \in \mathbb{X}$ and take $u_r = r^{1+\varepsilon}$. By the Borel-Cantelli lemma, for $\mathbb{P}$-a.a. $\omega \in \Omega_e$, there exists a finite constant $L_\omega(e)$ such that

$$|\gamma_\omega(B(x,r); e)| \leq L_\omega(e) u_r, \tag{2.24}$$

for all $r \geq 0$. Part (3) of Theorem 2.3.1 yields that $\mu^\omega_{x,\mathrm{cont}} = 0$ for $\mathbb{P}$-a.a. $\omega \in \Omega$. Since $x \in \mathbb{X}$ is arbitrary, the result follows. $\square$.

Note that, in the proof given above, (2.24) is a significant improvement of (2.21) since the factor $\sqrt{\mathtt{N}_r}$ is no longer present.

2.5 A generalization of Molchanov's theorem

In this section we generalize Theorem 2.2.1 to convex combinations of Cauchy distributions. Let $\mathcal{M}$ denote the set of Borel probability measures on $\mathbb{C}_+$. Given $\nu \in \mathcal{M}$ and a Borel set $\mathcal{B} \subset \mathbb{R}$, we set

$$k_\nu(\mathcal{B}) = \int k_v(\mathcal{B}) d\nu(v).$$

Then k_ν is a Borel probability measure on $\mathbb{R}$ and for every bounded Borel function $f : \mathbb{R} \to \mathbb{C}$,

$$\int_{\mathbb{R}} f dk_\nu = \int \left(\int_{\mathbb{R}} f dk_v \right) d\nu(v).$$

We call k_ν a *mixed Cauchy distribution.* Note that the usual Cauchy distribution k_{a+ih} is a special case of the above general defintion: $k_{a+ih} = k_\nu$ for $\nu = \delta(a+ih)$. Let S be the set of mixed Cauchy distributions k_ν, such that the distance from $\mathbb{R}$ to the support of ν is strictly positive.

Theorem 2.5.1. *Assume that there exists a sequence $u_r > 0$ with $\sum_{r=1}^{\infty} u_r^{-1} < \infty$ and*

$$\sum_{r=1}^{\infty} \mathrm{p}_r u_r < \infty.$$

If the random variables $\{\omega(x)\}_{x \in \mathbb{X}}$ are i.i.d. with a mixed Cauchy distrubution $k_\nu \in S$, then $\mathrm{sp}_{\mathrm{cont}}(H_\omega) = \emptyset$ for $\mathbb{P}$-a.a. ω.

Remark 1. Theorem 2.5.1 is valid at arbitrary spectral dimension or disorder. Indeed, if $\omega(x)$ has a distribution $k_\nu \in S$, and $c > 0$, then $c\omega$ has again a distribution of the form $k_{\nu'} \in S$. Precisely, $\nu' = \tau\nu$, where $\tau(z) = cz$.

Remark 2. If f is a C_0 probability density on $\mathbb{R}$, and $\varepsilon > 0$, then there is a mixed Cauchy distribution $k_\nu \in S$ s.t.

$$\sup_{e \in \mathbb{R}} |f(e) - dk_\nu/de| < \varepsilon.$$

The proof of Theorem 2.5.1 parallels that of Molchanov's theorem 2.2.1. First, we need a generalization of Proposition 2.4.1 to the case of mixed Cauchy distributions.

Proposition 2.5.2. *Mixed Cauchy distributions have the following properties:*

1. *If $\nu_1, \nu_1 \in \mathcal{M}$, then*

$$k_{\nu_1} \star k_{\nu_2} = k_{\nu_1 \star \nu_2}.$$

2. *If $\tau(z) = -\overline{z}$ or if $\tau(z) = (az+b)/(cz+d)$, where $a, b, c, d \in \mathbb{R}$, $ad - bc > 0$, then*

$$\tau k_\nu = k_{\tau\nu},$$

for all $\nu \in \mathcal{M}$.

3. *For all $\nu \in \mathcal{M}$ and integers $n \geq 2$,*

$$A_n k_\nu = k_{A_n \nu}.$$

The proof is a straight forward consequence of the definitions together with Propostion 2.4.1 and Fubini's theorem.

Proof of Theorem 2.5.1. Let E be the set of full Lebesgue measure as in the beginning of the proof of Theorem 2.2.1, and let $e \in E$ be fixed. It follows from (2.22) and from Proposition 2.5.2 that $\{\gamma_\omega(B(x,0);e)\}_{x \in \mathbb{X}}$ are i.i.d. mixed Cauchy random variables with distribution k_{ν_0}, where $\nu_0 = \tau_0 \nu$, $\tau_0(z) = -(e - \overline{z})^{-1}$. Since the distance from the support of ν to $\mathbb{R}$ is strictly positive, there is a closed disk $D_0 \subset \mathbb{C}_+$ such that $\mathrm{supp}(\nu_0) \subset \mathrm{D}_0$. The relation (2.23) and Proposition 2.5.2 yield that for $r \geq 1$, $\{\gamma_\omega(B_r;e) : B_r \in \mathcal{P}_r\}$ are i.i.d. mixed Cauchy random variables with distribution k_{ν_r}, where

$$\nu_r = \tau_{\mathrm{p}_r} A_{\mathrm{n}_r} \nu_{r-1}.$$

For $r \geq 0$, let $D_r = \tau_{\lambda_r} D_0$. Since τ_{λ_r} is a fractional linear transformation taking $\mathbb{C}_+$ onto itself, D_r is again a closed disk in $\mathbb{C}_+$. We claim that for all $r \geq 0$,

$$(*) \qquad \mathrm{supp}(\nu_r) \subset \mathrm{D_r}.$$

The proof is by induction. By construction of D_0, $(*)$ holds for $r = 0$. Let $r \geq 1$ and suppose $(*)$ holds for $r - 1$. Since $\mathrm{supp}(\nu_{r-1}) \subset \mathrm{D_{r-1}}$, and D_{r-1} is convex, we have that $\mathrm{supp}(\mathrm{A}_{n_r}\nu_{r-1}) \subset \mathrm{D_{r-1}}$. Hence $\mathrm{supp}(\tau_{p_r}\mathrm{A}_{n_r}\nu_{r-1}) \subset \tau_{p_r}\mathrm{D_{r-1}} = \mathrm{D_r}$, and $(*)$ holds for r, proving the claim.

Let D be the closure of $\cup_{r\geq 0} D_r$. D is a compact subset of $\mathbb{C}_+$ and for all $r \geq 0$, $\mathrm{supp}(\nu_r) \subset \mathrm{D}$. It follows that there is a constant $K(e) < \infty$, such that

$$\mathbb{P}\left(|\gamma_\omega(B_r; e)| > u\right) \leq \frac{K(e)}{u},$$

for all real $u > 0$, integer $r \geq 0$ and $B_r \in \mathcal{P}_r$. The rest of the proof is the same as in Theorem 2.2.1, with the more general sequence u_r instead of $r^{1+\epsilon}$. □

CHAPTER 3
Eigenvalue Statistics

3.1 Introduction: the problem of the eigenvalue statistics

Let us describe the problem of eigenvalue statistics in a more general framework of random discrete Schrödinger operators. In this framework, recall that we are given an infinite countable set $\mathbb{X}$, a bounded self-adjoint operator H_0 acting on the Hilbert space $l^2(\mathbb{X})$ and a random potential V_ω acting diagonally on $l^2(\mathbb{X})$:

$$(V_\omega \psi)(x) = \omega(x)\psi(x), \qquad \psi \in l^2(\mathbb{X}), x \in \mathbb{X}.$$

For our study of the eigenvalue statistics, we will always make the assumption that $\{(\omega(x))\}_{x\in\mathbb{X}}$ are i.i.d. random variables with common distribution μ^{V}. Hence the random parameter ω is an element of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \mathbb{R}^{\mathbb{X}}$, $\mathcal{F}$ is the product Borel σ-algebra on Ω and $\mathbb{P}$ is the product probability measure is $\mathbb{P} = \times_{x\in\mathbb{X}}\mu^{\mathrm{V}}$. We consider the random discrete Schrödinger operator

$$H_\omega = H_0 + V_\omega.$$

Of course, the hierarchical Anderson model fits into the above framework if $\mathbb{X}$ is endowed with a hierarchical distance $\mathtt{d}$ and $H_0 = \Delta$ is the hierarchical Laplacian. The Anderson model on $\mathbb{Z}^{\mathrm{d}}$, the Bethe lattice and any other Anderson model on an infinite graph fit into the above framework as well.

The finite volume approximations to H_ω are in general given by an increasing sequence $(B_k)_{k\geq 1}$ of finite subsets of $\mathbb{X}$, $\bigcup_{k\geq 1} B_k = \mathbb{X}$, and a corresponding sequence of operators $(H_k^\omega)_{k\geq 1}$ approximating H_ω, such that the subspace $l^2(B_k)$ is invariant for H_k^ω. We are interested in the asymptotic behavior of the random eigenvalues

$$e_1^{\omega,k} \leq e_2^{\omega,k} \leq \cdots \leq e_{|B_k|}^{\omega,k},$$

of $H_k^\omega \restriction l^2(B_k)$ as $k \to \infty$. Usually, the first step is to prove the existence of the *density of states measure.* One establishes that there is a nonrandom probability measure μ^{av} on $\mathbb{R}$ such that, with probability one, the random normalized eigenvalue counting measure

$$\mu_k^\omega = |B_k|^{-1} \sum_{j=1}^{|B_k|} \delta(e_j^{\omega,k}), \tag{3.1}$$

converges to μ^{av} in the weak-* topology as $k \to \infty$. If the model has enough symmetry and regularity, then μ^{av} is equal to the averaged spectral measure for H_ω and any δ_x. The measure μ^{av} is called the density of states for H_ω. Typically, the proof of the existence of the density of states does not require any special regularity of μ^{V} and is based on general probabilistic arguments (Birkhoff's ergodic theorem in the case of the Anderson model on $\mathbb{Z}^{\mathrm{d}}$ and, as we will see, Kolmogorov's strong law of large numbers in the case of the hierarchical Anderson model). The interpretation of the existence of the density of states is that for large k, the number of eigenvalues in a small interval $(e-\varepsilon, e+\varepsilon)$ around a point $e \in \mathrm{supp}(\mu^{\mathrm{av}})$ is typically of the order of $|B_k|\, \mu^{av}((e+\varepsilon, e-\varepsilon))$. The fine eigenvalue statistics near e are then captured by the rescaled point measure

$$\xi_k^{\omega,e} = \sum_{i=1}^{|B_k|} \delta(|B_k|\,(e_i^{\omega,k} - e)). \tag{3.2}$$

Unlike μ_k^ω, the study of the asymptotic behavior of $\xi_k^{\omega,e}$ requires finer probabilistic and spectral theoretical tools. Minami's technique [Mi] is a method allowing to prove that, in appropriate situations, $\xi_k^{\omega,e}$ is asymptotically a Poisson point process as $k \to \infty$. This means that for disjoint Borel sets $A_1, A_2, \cdots, A_m \subset \mathbb{R}$, the corresponding numbers of rescaled eigenvalues in each of the sets,

$$\xi_k^{\omega,e}(A_1), \xi_k^{\omega,e}(A_2), \cdots, \xi_k^{\omega,e}(A_m),$$

are approximately independent Poisson random variables and hence the eigenvalues near e are uncorrelated. In Minami's technique, one makes the assumption that the distribution μ^{V} has a bounded density: $d\mu^{\mathrm{V}}(t) = \gamma(t)dt$ and $\|\gamma\|_\infty < \infty$.

Minami originally considered the Anderson tight-binding model on $\mathbb{Z}^{\mathrm{d}}$. He proved Poisson statistics of eigenvalues in the localized regime ([Mi, KN]). Minami's method has its origins in Molchanov's paper [M1], where the first rigorous proof of the absence of energy level repulsion is given for a continuous one-dimensional model. After Minami's paper [Mi], the technique and its variations have been used to prove Poisson statistics of eigenvalues for different models [AW, BHS, KN, KS, S]. Along with the Anderson conjecture, it is believed that there is a phase transition on the the level of eigenvalue fluctuations. A localized regime should always yields Poisson statistics of eigenvalues. In the delocalized regime, one should observe universal repulsion laws for rescaled eigenvalues, as in the random matrix ensembles. A phase transition for eigenvalue fluctuations has been recently rigorously analyzed [KS] in the context of CMV matrices. As for the problem of the existence of a.c. spectrum for the Anderson model on $\mathbb{Z}^{\mathrm{d}}$, there does not yet exist a general technique allowing to show eigenvalue repulsion for Anderson type models at weak disorder.

The probabilistic part of Minami's technique shared by most models is based on the theory of infinitely divisible point processes. As a result, one sometimes has to go though a substantial body of material also concerned with other questions e.g. [Ka, DV] in order to extract the necessary results. One of our goals is to give a self-contained elementary exposition of the probabilistic part, only assuming standard material taught in a first graduate course on probability. The spectral part of the technique is based on decoupling, i.e. on approximating H_ω^k by a direct sum of a large number of statistically independent infinitesimal components. The analysis is specific to each model and the decoupling is possible only in an appropriate regime. In this chapter, we prove Poisson statistics of eigenvalues for the hierarchical Anderson model with spectral dimension $\mathrm{d_{sp}} < 1$. In subsection 2, we discuss the necessary probabilistic preliminaries on Poisson point processes. In subsection 3, we study the density of states for the hierarchical Anderson model in arbitrary spectral dimension. In subsection 4, we provide a complete proof of Poisson statistics of eigenvalues in the hierarchical Anderson model. In the Appendix, we outline, within our framework, Minami's original proof of Poisson statistics of eigenvalues in the Anderson model on $\mathbb{Z}^{\mathrm{d}}$ in the localized regime.

3.2 Probabilistic preliminaries

3.2.1 Why the Poisson distribution

The Poisson distribution with parameter λ is the discrete probability measure $\mathbb{P}_\lambda$ on $\mathbb{N} = \{0, 1, 2, \cdots\}$ given by

$$\mathbb{P}_\lambda = e^{-\lambda} \sum_{r \in \mathbb{N}} \frac{\lambda^r}{r!} \delta(r).$$

The simplest example where the Poisson distribution appears naturally in connection with the rescaled measure $\xi_k^{\omega,e}$ is the trivial case of a random discrete Schrödinger operator: $\mathbb{X} = \{1, 2, \cdots\}$,$H_0 = 0$ and the finite volume approximations are $B_k = \{1, \cdots, k\}$, $H_k^\omega = H_\omega \restriction l^2(B_k)$. Then $H_k^\omega \restriction l^2(B_k)$ has statistically independent eigenvalues $\{\omega(x)\}_{x \in B_k}$ and it follows from Kolmogorov's strong law of large numbers that for every Borel set $A \subset \mathbb{R}$,

$$\lim_{k \to \infty} \mu_k^\omega(A) = \mu^{av}(A) = \int_A \gamma(t) dt,$$

for $\mathbb{P}$-a.a. $\omega \in \Omega$.

Let us assume that γ is continuous at a point $e \in \mathbb{R}$ and that $\gamma(e) > 0$. If $A_1, A_2, \cdots, A_m \subset \mathbb{R}$ are disjoint bounded Borel sets, then the random vector

$$[\xi_k^{\omega,e}(A_1), \xi_k^{\omega,e}(A_2), \cdots, \xi_k^{\omega,e}(A_m)],$$

has a multinomial distribution

$$\mathbb{P}\left\{\xi_k^{\omega,e}(A_1) = r_1, \xi_k^{\omega,e}(A_2) = r_2, \cdots, \xi_k^{\omega,e}(A_m) = r_m\right\}$$

$$= \frac{k!}{r_1! r_2! \cdots r_{m+1}!} q_{k,1}^{r_1} q_{k,2}^{r_2} \cdots q_{k,m+1}^{r_{m+1}}, \qquad r_s = 0, \cdots, k, \sum_{s=1}^{m+1} r_s = k,$$

where

$$q_{k,s} = \mathbb{P}\{k(\omega(1) - e) \in A_s\} = \int_{e+k^{-1}A_s} \gamma(t)dt, \qquad s = 1, \cdots, m+1,$$

and $A_{m+1} = \mathbb{R} \backslash (\bigcup_{s=1}^{m} A_s)$. Continuity of γ at e yields that

$$\lim_{k\to\infty} kq_{k,s} = \gamma(e)\mathcal{L}(A_s),$$

and hence

$$\lim_{k\to\infty} \mathbb{P}\{\xi_k^{\omega,e}(A_1) = r_1, \xi_k^{\omega,e}(A_2) = r_2, \cdots, \xi_k^{\omega,e}(A_m) = r_m\} = \prod_{s=1}^{m} \mathbb{P}_{\lambda_s}(\{r_s\}),$$

with $\lambda_s = \gamma(e)\mathcal{L}(A_s)$. Hence the random variables $\xi_k^{\omega,e}(A_s), s = 1, \cdots, m$ are asymptotically independent and have Poisson distributions $\mathbb{P}_{\lambda_s}$.

In nontrivial situations, the operator $H_0 \neq 0$ introduces statistical dependence between the eigenvalues of $H_k^\omega \restriction l^2(B_k)$ and therefore the analysis of the rescaled measure $\xi_k^{\omega,e}$ is more involved. However, if the dependence introduced by H_0 is not too big in a suitable sense, then Minami's method allows to show that $\xi_k^{\omega,e}(A_s), s = 1, \cdots, m$ are still asymptotically independent Poisson random variables. In the next subsection, we discuss a general limit theorem needed for Minami's method.

3.2.2 The Poisson point process and Grigelionis' limit theorem

Although $\xi_k^{\omega,e}$ as well as the other measures of interest to us are on $\mathbb{R}$, we discuss, for sake of clarity, the general situation of random point measures on a metric space S. We equip S with the Borel σ-algebra $\mathcal{B}_S$, i.e. the σ-algebra generated by open sets. We denote by $\mathcal{M}$ the set of all nonnegative Borel measures μ on $(S, \mathcal{B}_S)$ such that $\mu(A) < \infty$ for every bounded Borel set $A \subset S$. A measure $\mu \in \mathcal{M}$ is called a point measure if μ can be written in the form

$$\mu = \sum_{j \in J} \delta(x_j), \qquad x_j \in S,$$

where J is a countable index set. We denote by $\mathcal{M}_p$ the set of all point measures on $(S, \mathcal{B}_S)$. If $\mu \in \mathcal{M}_p$, then we must have $\mu(A) \in \mathbb{N}$ for every bounded Borel set $A \subset S$. A point process on S is map $\omega \to \mu^\omega$ from some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $\mathcal{M}_p$ such that for every bounded Borel set $A \subset S$, the map $\omega \to \mu^\omega(A)$ is measurable. If μ^ω is a point process, then the map

$$\nu(B) = \mathbb{E}\mu^\omega(B), \qquad B \in \mathcal{B}_S,$$

defines a measure on $(S, \mathcal{B}_S)$. The measure ν is called the intensity measure of the point process μ^ω.

Definition 3.2.1. *Let $\nu \in \mathcal{M}$. A Poisson point process on S with intensity ν is a point process ξ^ω with the following properties:*

1. *for every bounded Borel set $A \subset S$, the random variable $\xi^\omega(A)$ has a Poisson distribution with parameter $\nu(A)$.*
2. *given disjoint bounded Borel sets $A_1, A_2, \cdots, A_m$ in S, the random variables $\xi^\omega(A_1), \xi^\omega(A_2), \cdots, \xi^\omega(A_m)$ are independent.*

It can be shown [Ki] that given any $\nu \in \mathcal{M}$, there exists a Poisson process on S with intensity ν, constructed on a suitable probability space. The Poisson point process is an idealized model of noninteraction and the point process $\xi_k^{\omega,e}$ in the study of eigenvalue statistics never exactly verifies conditions (1) and (2) of definition 3.2.1.

Definition 3.2.2. *A sequence ξ_k^ω of point processes on S, defined on the same probability space, is said to converge to a Poisson point process on S with intensity $\nu \in \mathcal{M}$ if for any given disjoint bounded Borel sets $A_1, A_2, \cdots, A_m$ in S, we have*

$$\lim_{k\to\infty} \mathbb{P}\left\{\xi_k^\omega(A_1) = r_1, \xi_k^\omega(A_2) = r_2, \cdots, \xi_k^\omega(A_m) = r_m\right\} = \prod_{s=1}^{m} \mathbb{P}_{\nu(A_s)}(\{r_s\}), \tag{3.3}$$

for all $r_1, r_2, \cdots, r_m \in \mathbb{N}$.

Hence, in the previous subsection, the sequence of point processes $\xi_k^{\omega,e}$ on $\mathbb{R}$ converges to a Poisson process on $\mathbb{R}$ with intensity $\gamma(e)\mathcal{L}$. In general, it can be difficult to verify the condition (3.3) directly and it is more convenient to verify an equivalent condition in terms of the characteristic functions, namely

$$\lim_{k\to\infty} \mathbb{E}e^{i\sum_{s=1}^m t_s \xi_k^\omega(A_s)} = \prod_{s=1}^{m} \exp\left(\nu(A_s)(e^{it_s} - 1)\right), \tag{3.4}$$

for all $t_1, t_2, \cdots, t_m \in \mathbb{R}$. Both (3.3) and (3.4) are equivalent to the usual definition of convergence in law for random vectors in $\mathbb{N}^m$.

The basic limit theorem guaranteeing the convergence of a sequence of point processes to a Poisson point processes is due to Griegelionis [G]. Originally formulated for step processes on $\mathbb{R}$, Grigelionis' theorem remains valid in more general settings and in our case it translates to:

Theorem 3.2.3. *(Grigelionis, 1963) Let $(n_k)_{k\geq 1}$ be a natural subsequence, let for each $k \geq 1$, $\xi^\omega_{k,1}, \xi^\omega_{k,2}, \cdots, \xi^\omega_{k,n_k}$ be independent point processes on S and let*

$$\xi^\omega_k = \sum_{j=1}^{n_k} \xi^\omega_{k,j}.$$

Let $\nu \in \mathcal{M}$ and assume that for every bounded Borel set $A \subset S$, we have

$$(1) \qquad \lim_{k\to\infty} \max_{1\leq j\leq n_k} \mathbb{P}\left\{\xi^\omega_{k,j}(A) \geq 1\right\} = 0,$$

$$(2) \qquad \lim_{k\to\infty} \sum_{j=1}^{n_k} \mathbb{P}\left\{\xi^\omega_{k,j}(A) \geq 1\right\} = \nu(A),$$

and

$$(3) \qquad \lim_{k\to\infty} \sum_{j=1}^{n_k} \mathbb{P}\left\{\xi^\omega_{k,j}(A) \geq 2\right\} = 0.$$

Then ξ^ω_k converges to a Poisson point process on S with intensity ν.

Theorem 3.2.3 is well-known and can be found in the literature e.g. [DV, Ka] as a corollary of more general results on point processes. For completeness, we include a self-contained proof here, following the original arguments of [G].

Proof. We use the standard notation $ab = \sum_{s=1}^m a_s b_s$, for $a, b \in \mathbb{R}^m$ and $|\alpha| = \sum_{s=1}^m \alpha_s$ for $\alpha \in \mathbb{N}^m$. We denote by $\{e_s\}_{s=1}^m$ the standard basis vectors of $\mathbb{R}^m$. Let $A_1, A_2, \cdots, A_m$ be given disjoint bounded Borel sets in S. Let X^ω_k be the random vector

$$X^\omega_k = [\xi^\omega_k(A_1), \xi^\omega_k(A_2), \cdots, \xi^\omega_k(A_m)],$$

and let $\phi_k : \mathbb{R}^m \to \mathbb{C}$ be the corresponding characteristic function

$$\phi_k(t) = \mathbb{E}e^{itX_k^\omega}, \qquad t \in \mathbb{R}^m.$$

According to (3.4), we have to show that for all $t \in \mathbb{R}^m$,

$$\lim_{k\to\infty} \phi_k(t) = \prod_{s=1}^{m} \exp\left(\nu(A_s)(e^{it_s} - 1)\right). \tag{3.5}$$

We set

$$X_{k,j}^\omega = [\xi_{k,j}^\omega(A_1), \xi_{k,j}^\omega(A_2), \cdots, \xi_{k,j}^\omega(A_m)],$$

$$\phi_{k,j}(t) = \mathbb{E}e^{itX_{k,j}^\omega}, \qquad t \in \mathbb{R}^m,$$

and

$$A = \bigcup_{s=1}^{m} A_s.$$

By assumption (1), there is a k_0 such that for $k \geq k_0$,

$$\max_{1\leq j\leq n_k} \mathbb{P}\left\{\xi_{k,j}^\omega(A) \geq 1\right\} < 1/4.$$

Hence for $k \geq k_0$ and $1 \leq j \leq n_k$,

$$\left|\sum_{|\alpha|\geq 1} \mathbb{P}\left\{X_{k,j}^\omega = \alpha\right\}(e^{i\alpha t} - 1)\right| \leq 2\sum_{|\alpha|\geq 1} \mathbb{P}\left\{X_{k,j}^\omega = \alpha\right\} = 2\mathbb{P}\left\{\xi_{k,j}^\omega(A) \geq 1\right\} < 1/2,$$

and we can write

$$
\begin{aligned}
\phi_{k,j}(t) &= 1 + \sum_{|\alpha|\geq 1} \mathbb{P}\left\{X^{\omega}_{k,j} = \alpha\right\}\left(e^{i\alpha t} - 1\right) \\
&= \exp\left(\sum_{|\alpha|\geq 1} \mathbb{P}\left\{X^{\omega}_{k,j} = \alpha\right\}\left(e^{i\alpha t} - 1\right) + E_{k,j}\right),
\end{aligned}
\tag{3.6}
$$

where

$$
E_{k,j} = f\left(\sum_{|\alpha|\geq 1} \mathbb{P}\left\{X^{\omega}_{k,j} = \alpha\right\}\left(e^{i\alpha t} - 1\right)\right),
$$

and $f(z) = \log(1+z) - z$. The function f is analytic in the open disk $\{|z| < 1\}$ and

$$
|f(z)| \leq C\,|z|^2 \qquad \text{for } |z| < 1/2, \tag{3.7}
$$

where $0 < C < \infty$ is a numerical constant. Next, we write

$$
\begin{aligned}
\sum_{|\alpha|\geq 1} \mathbb{P}\left\{X^{\omega}_{k,j} = \alpha\right\}\left(e^{i\alpha t} - 1\right) &= \sum_{|\alpha|=1} \mathbb{P}\left\{X^{\omega}_{k,j} = \alpha\right\}\left(e^{i\alpha t} - 1\right) + F_{k,j} \\
&= \sum_{s=1}^{m} \mathbb{P}\left\{X^{\omega}_{k,j} = e_s\right\}\left(e^{it_s} - 1\right) + F_{k,j} \\
&= \sum_{s=1}^{m} \mathbb{P}\left\{\xi^{\omega}_{k,j}(A_s) = 1\right\}\left(e^{it_s} - 1\right) + G_{k,j} + F_{k,j},
\end{aligned}
\tag{3.8}
$$

where

$$
F_{k,j} = \sum_{|\alpha|\geq 2} \mathbb{P}\left\{X^{\omega}_{k,j} = \alpha\right\}\left(e^{i\alpha t} - 1\right),
$$

and

$$
G_{k,j} = \sum_{s=1}^{m} \left(\mathbb{P}\left\{X^{\omega}_{k,j} = e_s\right\} - \mathbb{P}\left\{\xi^{\omega}_{k,j}(A_s) = 1\right\}\right)\left(e^{it_s} - 1\right).
$$

Hence,

$$
\phi_{k,j}(t) = \exp\left(\sum_{s=1}^{m} \mathbb{P}\left\{\xi^{\omega}_{k,j}(A_s) = 1\right\}\left(e^{it_s} - 1\right) + H_{k,j}\right),
$$

where

$$H_{k,j} = E_{k,j} + F_{k,j} + G_{k,j}.$$

We then have, by independence, that

$$\begin{aligned}\phi_k(t) &= \prod_{j=1}^{n_k} \phi_{k,j}(t) \\ &= \exp\left(\sum_{s=1}^{m}\left(\sum_{j=1}^{n_k} \mathbb{P}\left\{\xi_{k,j}^{\omega}(A_s) = 1\right\}\right)(e^{it_s} - 1) + \sum_{j=1}^{n_k} H_{k,j}\right)\end{aligned} \tag{3.9}$$

The assumptions (2) and (3) imply that

$$\lim_{k\to\infty} \sum_{j=1}^{n_k} \mathbb{P}\left\{\xi_{k,j}^{\omega}(A_s) = 1\right\} = \nu(A_s). \tag{3.10}$$

We claim that

$$\lim_{k\to\infty} \sum_{j=1}^{n_k} H_{k,j} = 0. \tag{3.11}$$

If (3.11) holds, then (3.10), (3.11) and (3.9) together yield the desired conclusion (3.5) and we are done. We now prove (3.11). We have

$$|F_{k,j}| \le 2\mathbb{P}\left\{\xi_{k,j}^{\omega}(A) \ge 2\right\}, \tag{3.12}$$

and the bound (3.7) yields

$$|E_{k,j}| \le C\left(2\sum_{|\alpha|\ge 1} \mathbb{P}\left\{X_{k,j}^{\omega} = \alpha\right\}\right)^2 = 4C\left(\mathbb{P}\left\{\xi_{k,j}^{\omega}(A) \ge 1\right\}\right)^2. \tag{3.13}$$

To estimate $|G_{k,j}|$, note that

$$\left\{X_{k,j}^{\omega} = e_s\right\} \subset \left\{\xi_{k,j}^{\omega}(A_s) = 1\right\},$$

and

$$\left(\left\{\xi^\omega_{k,j}(A_s)=1\right\}\setminus\left\{X^\omega_{k,j}=e_s\right\}\right)\subset\left\{\xi^\omega_{k,j}(A)\ge 2\right\}.$$

Hence

$$|G_{k,j}|\le 2m\mathbb{P}\left\{\xi^\omega_{k,j}(A)\ge 2\right\}. \tag{3.14}$$

We now combine the bounds (3.13), (3.12) and (3.14) to get

$$\begin{aligned}\left|\sum_{j=1}^{n_k}H_{k,j}\right| \le&(2m+2)\sum_{j=1}^{n_k}\mathbb{P}\left\{\xi^\omega_{k,j}(A)\ge 2\right\}\\ &+4C\left(\max_{1\le j\le n_k}\mathbb{P}\left\{\xi^\omega_{k,j}(A)\ge 1\right\}\right)\sum_{j=1}^{n_k}\mathbb{P}\left\{\xi^\omega_{k,j}(A)\ge 1\right\}.\end{aligned}$$

The assumptions (1),(2) and (3) imply that the right hand side of last inequality converges to zero as $k\to\infty$, completing the proof. □

3.2.3 Corollaries of Grigelionis' limit theorem

For the point processes ξ^ω on $S=\mathbb{R}$ arising in the study of eigenvalue statistics, it is sometimes more natural to obtain information about the Poisson integrals $\int_{\mathbb{R}}\operatorname{Im}(t-z)^{-1}d\xi^\omega(t)$, $\operatorname{Im} z>0$, rather than about the events $\{\xi^\omega(A)\ge 1\}$ and $\{\xi^\omega(A)\ge 2\}$. In this subsection, we replace the conditions (2) and (3) of Theorem 3.2.3 by sufficient conditions in terms of the Poisson integrals. We refer the reader to [J] for the general theory of Poisson integrals and their applications to spectral theory.

For a positive Borel measure μ on S and a Borel function $f:S\to[0,\infty)$, we set

$$\mathcal{I}(\mu,f)=\int_{t\ne t'}f(t)f(t')d\mu(t)d\mu(t').$$

If $\mu = \sum_j \delta(t_j)$ is a point measure on S and $f(t) = 1_A(t)$ is the indicator function of a bounded Borel set $A \subset S$, then we have

$$\mathcal{I}(\mu, 1_A) = \sum_{i \neq j} 1_A(t_i) 1_A(t_j) = \mu(A)(\mu(A) - 1),$$

and therefore $\mathcal{I}(\mu, 1_A) \neq 0 \Leftrightarrow \mu(A) \geq 2$. If ξ^ω is a point process on S, then

$$\begin{aligned}\sum_{l \geq 2} \mathbb{P}\{\xi^\omega(A) \geq l\} &= \sum_{l \geq 2} (l-1) \mathbb{P}\{\xi^\omega(A) = l\} \\ &\leq \sum_{l \geq 2} l(l-1) \mathbb{P}\{\xi^\omega(A) = l\} \\ &= \mathbb{E}\mathcal{I}(\xi^\omega, 1_A).\end{aligned}$$

Since

$$\mathbb{P}\{\xi^\omega(A) \geq 1\} = \mathbb{E}\xi^\omega(A) - \sum_{l \geq 2} \mathbb{P}\{\xi^\omega(A) \geq l\},$$

we conclude that the conditions

$$(2') \qquad \lim_{k \to \infty} \sum_{j=1}^{n_k} \mathbb{E}\xi^\omega_{k,j}(A) = \nu(A),$$

and

$$(3') \qquad \lim_{k \to \infty} \sum_{j=1}^{n_k} \mathbb{E}\mathcal{I}(\xi^\omega_{k,j}, 1_A) = 0,$$

together imply conditions (2) and (3) of Theorem 3.2.3. The next step is to replace, in (2') and (3'), the quantity $\mathbb{E}\xi^\omega_{k,j}(A)$ by $\mathbb{E}\int f d\xi^\omega_{k,j}$ for f in a sufficiently rich family F of functions.

Theorem 3.2.4. *For each $k \geq 1$, let $\xi^\omega_{k,1}, \xi^\omega_{k,2}, \cdots, \xi^\omega_{k,n_k}$ be point processes on S and let $\xi^{av}_k = \sum_{j=1}^{n_k} \mathbb{E}\xi^\omega_{k,j}$. Let $\nu \in \mathcal{M}$. Suppose that there is a measure $\mu \in \mathcal{M}$ s.t. that ν and $(\xi^{av}_k)_{k \geq 1}$ are absolutely continuous with respect to μ, with uniformly bounded* *densities, i.e. there is a constant* $0 < C < \infty$ *such that for all bounded Borel sets* $A \subset S$,

$$\nu(A) \leq C\mu(A),$$

and

$$\xi_k^{av}(A) \leq C\mu(A), \qquad k \geq 1.$$

Suppose that $F \subset L_1(S,\mu)$ *is a family of functions such that finite linear combinations of functions in* F *are dense in* $L_1(S,\mu)$ *and such that for every bounded Borel set* $A \subset S$, *there exists* $f \in F$ *with* $f \geq 1_A$. *Suppose that for all* $f \in F$, *we have*

$$(2'') \qquad \lim_{k\to\infty} \int f d\xi_k^{av} = \int f d\nu,$$

and

$$(3'') \qquad \lim_{k\to\infty} \sum_{j=1}^{n_k} \mathbb{E}\mathcal{I}(\xi_{k,j}^{\omega}, f) = 0.$$

Then (2') and (3') hold for all bounded Borel sets $A \subset S$.

Proof. Let A be a bounded Borel set. Let $\varepsilon > 0$. There is a finite linear combination $g = \sum_i c_i f_i$, $f_i \in F$, with $\int |g - 1_A|\, d\mu < \varepsilon$. Then $\left|\int g d\nu - \nu(A)\right| < C\varepsilon$ and $\left|\int g d\xi_k^{av} - \xi_k^{av}(A)\right| < C\varepsilon$. Since $\lim_{k\to\infty} \int g d\xi_k^{av} = \int g d\nu$, we have

$$\nu(A) - 2C\varepsilon \leq \liminf_{k\to\infty} \xi_k^{av}(A) \leq \limsup_{k\to\infty} \xi_k^{av}(A) \leq \nu(A) + 2C\varepsilon,$$

and (2') is obtained after letting $\varepsilon \downarrow 0$. Now let $f \in F$ be such that $f \geq 1_A$. Since, $\mathcal{I}(\xi_{k,j}^{\omega}, 1_A) \leq I(\xi_{k,j}^{\omega}, f)$, (3') follows from (3"). $\square$

The special case when $S = \mathbb{R}$, $\mu = \mathcal{L}$ is the Lebesgue measure on $\mathbb{R}$, $\nu = \lambda\mathcal{L}$ for a $\lambda > 0$ and F is the family of functions $\{\operatorname{Im}(t-z)^{-1}\}_{\operatorname{Im} z > 0}$ yields

Theorem 3.2.5. *Let $(n_k)_{k\geq 1}$ be a natural subsequence, let for each $k \geq 1$, $\xi^\omega_{k,1}, \xi^\omega_{k,2}, \cdots, \xi^\omega_{k,n_k}$ be independent point processes on $\mathbb{R}$ and let*

$$\xi^\omega_k = \sum_{j=1}^{n_k} \xi^\omega_{k,j}.$$

We make the following four hypotheses:

(H0): there is a constant $0 < C < \infty$ such that for all $k \geq 1$ and every bounded Borel set $A \subset \mathbb{R}$,

$$\sum_{j=1}^{n_k} \mathbb{E}\xi^\omega_{k,j}(A) \leq C\mathcal{L}(A).$$

(H1): for every bounded Borel set $A \subset \mathbb{R}$,

$$\lim_{k\to\infty} \max_{1\leq j\leq n_k} \mathbb{P}\left\{\xi^\omega_{k,j}(A) \geq 1\right\} = 0.$$

(H2): there is a constant $0 < \lambda < \infty$ such that for $\operatorname{Im} z > 0$,

$$\lim_{k\to\infty} \sum_{j=1}^{n_k} \mathbb{E} \int_{\mathbb{R}} \operatorname{Im}(t-z)^{-1} d\xi^\omega_{k,j}(t) = \pi\lambda.$$

(H3): for $\operatorname{Im} z > 0$,

$$\lim_{k\to\infty} \sum_{j=1}^{n_k} \mathbb{E} \int_{t\neq t'} \operatorname{Im}(t-z)^{-1} \operatorname{Im}(t'-z)^{-1} d\xi^\omega_{k,j}(t) d\xi^\omega_{k,j}(t') = 0.$$

Then ξ^ω_k converges to a Poisson point process on $\mathbb{R}$ with intensity $\lambda\mathcal{L}$.

Theorem 3.2.5 is implicitly derived in [Mi] and is suitable for applications to eigenvalue statistics of general random discrete Schrödinger operators.

3.3 The density of states in the hierarchical Anderson model

In this subsection, $(\mathbb{X}, \mathtt{d}, \Delta)$ is a homogeneous hierarchical model of degree $\mathtt{n}$. We do not impose any special decay condition on the sequence $(\mathrm{p}_r)_{r\geq 1}$. In particular, Δ can have an arbitrary spectral dimension d_{sp}. The random variables $\{\omega(x)\}_{x\in\mathbb{X}}$ are assumed to be i.i.d.. with a common probability distribution μ^{V}. No further regularity of μ^{V} is assumed. Hence, the probability measure $\mathbb{P}$ on $\Omega = \mathbb{R}^X$ is the product measure

$$\mathbb{P} = \times_{x\in\mathbb{X}}\mu^{\mathrm{V}}.$$

The finite volume approximations to H_ω are defined as follows. We fix a point $x_0 \in \mathbb{X}$ and we consider the increasing sequence of balls

$$B_k = B(x_0, k) \qquad k \geq 0.$$

Each B_k has size $|B_k| = \mathtt{n}^k$. We take the approximating sequence H_k^ω to be the truncated operators

$$H_k^\omega = \sum_{s=1}^{k} \mathrm{p}_s E_s + V_\omega,$$

as we did in (1.5) for the proof of localization. The subspace $l^2(B_k)$ is invariant for H_k^ω. The normalized eigenvalue counting measure μ_k^ω for H_k^ω is given by (3.1). The averaged spectral measure for H_ω is the unique Borel probability measure μ^{av} on $\mathbb{R}$ defined by

$$\int f(t)d\mu^{av}(t) = \mathbb{E}\langle\delta_{x_0}|f(H_\omega)\delta_{x_0}\rangle, \qquad f \in C_0(\mathbb{R}). \tag{3.15}$$

By symmetry, $\int f(t)d\mu^{av}(t) = \mathbb{E}\langle\delta_x|f(H_\omega)\delta_x\rangle$ for all $x \in \mathbb{X}$. The content of the following theorem is that the averaged spectral measure μ^{av} is naturally interpreted as the density of states for H_ω.

Theorem 3.3.1. *For $\mathbb{P}$-a.a. $\omega \in \Omega$, $\mu_k^\omega \to \mu^{av}$ in the weak-* topology as $k \to \infty$. Precisely, there is a set $\widetilde{\Omega} \in \mathcal{F}$ with $\mathbb{P}(\widetilde{\Omega}) = 1$ such that for all $\omega \in \widetilde{\Omega}$ and $f \in C_0(\mathbb{R})$ we have*

$$\lim_{k\to\infty} \int f(t)d\mu_k^\omega(t) = \int f(t)d\mu^{av}(t).$$

We start the proof of Theorem 3.3.1 with resolvent bounds. Since

$$H_r^\omega = H_{r-1}^\omega + \mathrm{p}_r E_r,$$

the resolvent identity yields

$$(H_{r-1}^\omega - z)^{-1} - (H_r^\omega - z)^{-1} = \mathrm{p}_r (H_{r-1}^\omega - z)^{-1} E_r (H_r^\omega - z)^{-1},$$

for $z \in \mathbb{C}\backslash\mathbb{R}$. Therefore:

$$\left\|(H_{r-1}^\omega - z)^{-1} - (H_r^\omega - z)^{-1}\right\| \leq |\operatorname{Im} z|^{-2}\, \mathrm{p}_r, \qquad z \in \mathbb{C}\backslash\mathbb{R}. \tag{3.16}$$

Iterating (3.16) yields for $r < k$,

$$\left\|(H_r^\omega - z)^{-1} - (H_k^\omega - z)^{-1}\right\| \leq |\operatorname{Im} z|^{-2} \sum_{s=r+1}^{k} \mathrm{p}_s, \qquad z \in \mathbb{C}\backslash\mathbb{R}. \tag{3.17}$$

and letting $k \to \infty$,

$$\left\|(H_r^\omega - z)^{-1} - (H_\omega - z)^{-1}\right\| \leq |\operatorname{Im} z|^{-2} \sum_{s=r+1}^{\infty} \mathrm{p}_s, \qquad z \in \mathbb{C}\backslash\mathbb{R}. \tag{3.18}$$

Proposition 3.3.2. *For every $z \in \mathbb{C} \setminus \mathbb{R}$ there is a set $\Omega_z \in \mathcal{F}$, with $\mathbb{P}(\Omega_z) = 1$ and such that for all $\omega \in \Omega_z$, the difference*

$$D_{k,\omega} = \int (t-z)^{-1} d\mu_k^\omega(t) - \int (t-z)^{-1} d\mu^{av}(t),$$

converges to 0 *as* $k \to \infty$.

Proof. Let $\varepsilon > 0$ be given. We take $r = r(\varepsilon, z)$ big enough so that

$$|\mathrm{Im}\, z|^{-2} \sum_{s=r+1}^{\infty} \mathrm{p}_s < \varepsilon/2. \tag{3.19}$$

Then for $r < k$,

$$\begin{aligned} D_{k,\omega} &= |B_k|^{-1} \sum_{x \in B_k} \langle \delta_x | (H_k^\omega - z)^{-1} \delta_x \rangle - \mathbb{E} \langle \delta_{x_0} | (H_\omega - z)^{-1} \delta_{x_0} \rangle \\ &= \left\{ |B_k|^{-1} \sum_{x \in B_k} \langle \delta_x | \left((H_k^\omega - z)^{-1} - (H_r^\omega - z)^{-1} \right) \delta_x \rangle \right\} \\ &\quad + \left\{ |B_k|^{-1} \sum_{x \in B_k} \langle \delta_x | (H_r^\omega - z)^{-1} \delta_x \rangle - \mathbb{E} \langle \delta_{x_0} | (H_\omega - z)^{-1} \delta_{x_0} \rangle \right\} \\ &= I_{k,\omega} + II_{k,\omega} \end{aligned} \tag{3.20}$$

The bounds (3.17) and (3.19) yield $|I_{k,\omega}| < \varepsilon/2$. We proceed with estimating $|II_{k,\omega}|$. Note that B_k is a disjoint union of n^{k-r} balls of radius r,

$$B_k = \bigcup_{j=1}^{\mathrm{n}^{k-r}} B_{k,j},$$

and therefore

$$l^2(B_k) = \bigoplus_{j=1}^{\mathrm{n}^{k-r}} l^2(B_{k,j}).$$

Since each subspace $l^2(B_{k,j})$ is invariant for H_r^ω, we can write

$$|B_k|^{-1}\sum_{x\in B_k}\langle\delta_x|(H_r^\omega-z)^{-1}\delta_x\rangle=\frac{1}{\mathrm{n}^{k-r}}\sum_{j=1}^{n^{k-r}}\mathrm{n}^{-r}\sum_{x\in B_{k,j}}\langle\delta_x|(H_r^\omega-z)^{-1}\delta_x\rangle,$$

and recognize that the right hand side is an average of n^{k-r} i.i.d. bounded random variables. Hence, Kolmogorov's strong law of large numbers yields that there is a set $\Omega_{z,\varepsilon}\in\mathcal{F}$ with $\mathbb{P}(\Omega_{z,\varepsilon})=1$ and such that for all $\omega\in\Omega_{z,\varepsilon}$,

$$\lim_{k\to\infty}|B_k|^{-1}\sum_{x\in B_k}\langle\delta_x|(H_r^\omega-z)^{-1}\delta_x\rangle=\mathrm{n}^{-r}\sum_{x\in\widetilde{B}}\mathbb{E}\langle\delta_x|(H_r^\omega-z)^{-1}\delta_x\rangle,\tag{3.21}$$

where $\widetilde{B}$ is some fixed ball of radius r. The bounds (3.17) and (3.19) yield

$$\left|\langle\delta_x|(H_r^\omega-z)^{-1}\delta_x\rangle-\langle\delta_x|(H_\omega-z)^{-1}\delta_x\rangle\right|<\varepsilon/2,$$

which combined with (3.21) yields

$$\limsup_{k\to\infty}|II_{k,\omega}|<\varepsilon/2.$$

Hence for $\omega\in\Omega_{z,\varepsilon}$, $\limsup_{k\to\infty}|D_{k,\omega}|<\varepsilon$, and the statement follows after taking $\Omega_z=\bigcap_{m=1}^{\infty}\Omega_{z,1/m}$. □

Theorem 3.3.1 is a consequence of Proposition 3.3.2 and a density argument. Let G be a countable dense set in $\mathbb{C}\backslash\mathbb{R}$. Since any function $f\in C_0(\mathbb{R})$ can be uniformly approximated by finite linear combinations of the functions $t\to(t-z)^{-1}$, with z ranging through G, Theorem 3.3.1 follows after taking $\widetilde{\Omega}=\bigcap_{z\in G}\Omega_z$.

3.4 Poisson statistics of eigenvalues in the hierarchical Anderson model

We keep the setup of the previous subsection and we make the following additional assumptions on the hierarchical Anderson model. We assume that there exist constants $C_1 > 0, C_2 > 0$ and $\rho > 1$ such that

$$C_1\rho^{-r} \leq \mathrm{p}_r \leq C_2\rho^{-r},$$

for r big enough. Then, according to Proposition 1.3.2, the spectral dimension Δ is equal to

$$\mathrm{d_{sp}} = 2\frac{\log \mathrm{n}}{\log \rho}. \tag{3.22}$$

We make the assumption that

$$0 < \mathrm{d_{sp}} < 1. \tag{3.23}$$

Concerning the probability distribution μ^{V}, we make the assumption that μ^{V} has a bounded density:

$$d\mu^{\mathrm{V}}(t) = \gamma(t)dt \text{ and } \|\gamma\|_\infty < \infty. \tag{3.24}$$

For our study of fine eigenvalue statistics, we need the following two general estimates for random discrete Schrödinger operators. For both estimates, the density assumption (3.24) plays a crucial role.

Lemma 3.4.1 (Wegner Estimate [W]). *Let M_0 be any self-adjoint operator on $l^2(\mathbb{X})$ and let*

$$M_\omega = M_0 + V_\omega.$$

Then for every bounded Borel measurable function $h : \mathbb{R} \to [0, \infty)$ and $x \in \mathbb{X}$, we have

$$\mathbb{E}\langle \delta_x | h(M_\omega)\delta_x \rangle \leq \|\gamma\|_\infty \int h(t)dt. \tag{3.25}$$

Hence, if ν_ω is the spectral measure for δ_x and M_ω and $\nu^{av} = \mathbb{E}\nu_\omega$ is the corresponding averaged measure, then ν^{av} is absolutely continuous with respect to Lebesgue measure,

$$d\nu^{av}(t) = \upsilon(t)dt,$$

and

$$\|\upsilon\|_\infty \leq \|\gamma\|_\infty .$$

For a bounded operator A on a Hilbert space, we denote by $\operatorname{Im} A$ the self-adjoint operator $(2i)^{-1}(A - A^*)$.

Lemma 3.4.2 (Minami's Estimate [Mi, GV, BHS]). *Let M_0 be any self-adjoint operator on $l^2(\mathbb{X})$ and let*

$$M_\omega = M_0 + V_\omega.$$

Then for every $x, y \in \mathbb{X}$ and $\operatorname{Im} z > 0$, we have

$$\mathbb{E} \det \begin{pmatrix} \langle \delta_x | \operatorname{Im}(M_\omega - z)^{-1}\delta_x \rangle & \langle \delta_x | \operatorname{Im}(M_\omega - z)^{-1}\delta_y \rangle \\ \langle \delta_y | \operatorname{Im}(M_\omega - z)^{-1}\delta_x \rangle & \langle \delta_y | \operatorname{Im}(M_\omega - z)^{-1}\delta_y \rangle \end{pmatrix} \leq \pi^2 \|\gamma\|_\infty^2 \tag{3.26}$$

The proofs of Wegner and Minami estimates are given in the appendix. Wegner estimate yields that μ^{av} is absolutely continuous with respect to Lebesgue measure,

$$d\mu^{av}(t) = \eta(t)dt,$$

and

$$\|\eta\|_\infty \leq \|\gamma\|_\infty .$$

If $e \in \mathbb{R}$ and $\varepsilon > 0$ are given, then in view of Theorem 3.3.1 we expect the number of eigenvalues of $H_k^\omega \restriction l^2(B_k)$ in the interval $(e-\varepsilon, e+\varepsilon)$,

$$\#\left\{i : e_i^{\omega,k} \in (e-\varepsilon, e+\varepsilon)\right\},$$

to have typical size of order $|B_k|\,\mu^{av}(e-\varepsilon, e+\varepsilon)$ for large k. The precise statistical behavior of the eigenvalues $e_j^{\omega,k}$ near e is captured by the rescaled measure $\xi_k^{\omega,e}$ given by (3.2). We make the following regularity assumption on e: for $\operatorname{Im} z > 0$,

$$\lim_{\varepsilon \downarrow 0} \int \operatorname{Im}\,(t - e - \varepsilon z)^{-1} \eta(t)dt = \pi\eta(e). \tag{3.27}$$

For example, if η is continuous at e, then (3.27) holds. However, it is in general a difficult problem to establish the continuity of η for random discrete Schrödinger operators. In the case of the Cauchy random potential (2.4), η is known to be analytic [L]. If the Fourier transform of $\gamma(t)$ decays exponentially, then it is possible [CFS] to prove analyticity of η after increasing the disorder, i.e. replacing V_ω with cV_ω for a sufficiently large $c > 0$. When continuity of η is not available, one appeals to a classical theorem in harmonic analysis (see for example [Ko]), due to Fatou, guaranteeing that (3.27) holds for $\mathcal{L}$-a.a. $e \in \mathbb{R}$.

Finally, we assume that

$$\eta(e) > 0. \tag{3.28}$$

The assumption (3.28) holds for $\mathcal{L}$-a.a. $e \in \operatorname{supp}(\mu^{\mathrm{av}})$.

Our main result on Poisson statistics of eigenvalues is:

Theorem 3.4.3. *Under the present assumptions, $\xi_k^{\omega,e}$ converges to a Poisson point process on $\mathbb{R}$ with intensity $\eta(e)\mathcal{L}$.*

The rest of the subsection is devoted to the proof of Theorem 3.4.3. The main idea is to approximate H_k^ω with H_r^ω for $r < k$, as in the proof of Theorem 3.3.1. This time we choose r to depend on k, $r = r_k$, such that

$$\lim_{k\to\infty} \frac{r_k}{k} = \mathfrak{c}, \tag{3.29}$$

with

$$\mathrm{d_{sp}} < \mathfrak{c} < 1. \tag{3.30}$$

Let

$$\widetilde{e}_1^{\omega,k} \le \widetilde{e}_2^{\omega,k} \le \cdots \le \widetilde{e}_{|B_k|}^{\omega,k},$$

denote the eigenvalues of $H_{r_k}^\omega \restriction l^2(B_k)$ and let

$$\widetilde{\xi}_k^{\omega,e} = \sum_{i=1}^{|B_k|} \delta(|B_k|\,(\widetilde{e}_i^{\omega,k} - e)),$$

be the corresponding rescaled measure near e. Since B_k is a disjoint union of n^{k-r_k} balls of radius r_k,

$$B_k = \bigcup_{j=1}^{\mathrm{n}^{k-r_k}} B_{k,j},$$

we have the corresponding direct sum decomposition

$$H_{r_k}^\omega \restriction l^2(B_k) = \bigoplus_{j=1}^{\mathrm{n}^{k-r_k}} H_{r_k}^\omega \restriction l^2(B_{k,j}).$$

Therefore, the point process $\widetilde{\xi}_k^{\omega,e}$ is the sum of n^{k-r_k} independent point processes,

$$\widetilde{\xi}_k^{\omega,e} = \sum_{j=1}^{\mathrm{n}^{k-r_k}} \widetilde{\xi}_{k,j}^{\omega,e},$$

where

$$\widetilde{\xi}_{k,j}^{\omega,e} = \sum_{l=1}^{\mathrm{n}^{r_k}} \delta(|B_k|\,(\widetilde{e}_l^{\omega,k,j} - e)),$$

and $\widetilde{e}_l^{\omega,k,j}$, $l = 1, \cdots, \mathrm{n}^{r_k}$ are the eigenvalues of $H_{r_k}^{\omega} \restriction l^2(B_{k,j})$.

The proof of Theorem 3.4.3 is organized as follows. We first establish that the point processes $\xi_k^{\omega,e}$ and $\widetilde{\xi}_k^{\omega,e}$ are asymptotically close in the following sense:

Proposition 3.4.4. *For every $f \in L_1(\mathbb{R}, dt)$,*

$$\lim_{k\to\infty} \mathbb{E}\left|\int f d\widetilde{\xi}_k^{\omega,e} - \int f d\xi_k^{\omega,e}\right| = 0. \tag{3.31}$$

Corollary 3.4.5. *Let $A_1, A_2, \cdots, A_m$ be given disjoint bounded Borel sets in $\mathbb{R}$. Let X_k^{ω} and $\widetilde{X}_k^{\omega}$ be the random vectors*

$$X_k^{\omega} = [\xi_k^{\omega,e}(A_1), \xi_k^{\omega,e}(A_2), \cdots, \xi_k^{\omega,e}(A_m)],$$

$$\widetilde{X}_k^{\omega} = [\widetilde{\xi}_k^{\omega,e}(A_1), \widetilde{\xi}_k^{\omega,e}(A_2), \cdots, \widetilde{\xi}_k^{\omega,e}(A_m)].$$

and let $\phi_k, \widetilde{\phi}_k : \mathbb{R}^m \to \mathbb{C}$ be the corresponding characteristic functions

$$\phi_k(t) = \mathbb{E}e^{itX_k^{\omega}}, \widetilde{\phi}_k(t) = \mathbb{E}e^{it\widetilde{X}_k^{\omega}}, \qquad t \in \mathbb{R}^m.$$

Then for all $t \in \mathbb{R}^m$,

$$\lim_{k\to\infty} \left|\phi_k(t) - \widetilde{\phi}_k(t)\right| = 0.$$

Then we establish

Proposition 3.4.6. *The point process $\widetilde{\xi}_k^{\omega,e}$ converges to a Poisson point process on $\mathbb{R}$ with intensity $\eta(e)\mathcal{L}$.*

Proposition 3.4.6 and Corollary 3.4.5 together imply Theorem 3.4.3. The Wegner estimate plays a crucial role in the proof of Propositions 3.4.4 and 3.4.6. For every Borel set $A \subset \mathbb{R}$, we have $\xi_k^{\omega,e}(A) = \sum_{x\in B_k} \langle \delta_x | f(H_k^\omega)\delta_x\rangle$, where

$$f(t) = 1_A(|B_k|(t-e)).$$

Wegner estimate (3.25) yields that for all $x \in B_k$,

$$\mathbb{E}\langle \delta_x | f(H_k^\omega)\delta_x\rangle \leq \|\gamma\|_\infty \int f(t)dt = \|\gamma\|_\infty |B_k|^{-1} \mathcal{L}(A). \tag{3.32}$$

Summing (3.32) over all $x \in B_k$ yields

$$\mathbb{E}\xi_k^{\omega,e}(A) \leq \|\gamma\|_\infty \mathcal{L}(A). \tag{3.33}$$

Similarly

$$\mathbb{E}\widetilde{\xi}_k^{\omega,e}(A) \leq \|\gamma\|_\infty \mathcal{L}(A). \tag{3.34}$$

Proof of Proposition 3.4.4. Step 1: We first prove (3.31) for the family of functions

$$g_z(t) = \operatorname{Im}(t-z)^{-1}, \qquad \operatorname{Im} z > 0.$$

Setting

$$z_k = e + |B_k|^{-1} z, \tag{3.35}$$

we have

$$\int g_z d\widetilde{\xi}_k^{\omega,e} - \int g_z d\xi_k^{\omega,e} = |B_k|^{-1}\,\mathrm{Im}\sum_{x\in B_k}\langle\delta_x|\left((H_{r_k}^{\omega}-z_k)^{-1}-(H_k^{\omega}-z_k)^{-1}\right)\delta_x\rangle.$$

Hence

$$\left|\int g_z d\widetilde{\xi}_k^{\omega,e} - \int g_z d\xi_k^{\omega,e}\right| \le |\mathrm{Im}\, z_k|^{-2}\sum_{s=r_k+1}^{\infty}\mathrm{p}_s = const\,|\mathrm{Im}\, z|^{-2}\left(\frac{\mathrm{n}^{2k}}{\rho^{r_k}}\right).$$

The formulas (3.29) and (3.30) imply that for large enough k, $\rho^{r_k}\ge\rho^{\mathfrak{c}'k}$ where $\mathrm{d_{sp}}<\mathfrak{c}'<\mathfrak{c}<1$. Therefore

$$\frac{\mathrm{n}^{2k}}{\rho^{r_k}}\le\left(\frac{\mathrm{n}^2}{\rho^{\mathfrak{c}'}}\right)^k,$$

and $\frac{\mathrm{n}^2}{\rho^{\mathfrak{c}'}}<1$ because of the formula (3.22). This proves (3.31).

Step 2: To prove (3.31) for general $f\in L_1(\mathbb{R},dt)$, note that finite linear combinations from $\{g_z\}_{\mathrm{Im}\,z>0}$ are dense in $L^1(\mathbb{R},dt)$. Hence given $\varepsilon>0$, there is a finite linear combination

$$g(t)=\sum_{j=1}^{p}a_j\mathrm{Im}\,(t-z^{(j)})^{-1},\qquad \mathrm{Im}\,z^{(j)}>0,$$

with

$$\int_{\mathbb{R}}|f(t)-g(t)|\,dt\le\varepsilon.$$

The triangle inequality

$$\begin{aligned}\mathbb{E}\left|\int f d\xi_k^{\omega,e}-\int d\widetilde{\xi}_k^{\omega,e}(t)\right| &\le \mathbb{E}\int|f-g|\,f d\xi_k^{\omega,e}\\ &+\mathbb{E}\left|\int g d\xi_k-\int g d\widetilde{\xi}_k^{\omega,e}\right|+\mathbb{E}\int|g-f|\,d\widetilde{\xi}_k^{\omega,e},\end{aligned}$$

together with Step 1 and the bounds (3.33) and (3.34) imply

$$\limsup_{k\to\infty} \mathbb{E}\left|\int f d\xi_k^{\omega,e} - \int f d\widetilde{\xi}_k^{\omega,e}(t)\right| \le 2\,\|\gamma\|_\infty\,\varepsilon,$$

and (3.31) follows after letting $\varepsilon \downarrow 0$. □

Proof of Proposition 3.4.6. I suffices to show that $\widetilde{\xi}_k^{\omega,e}$ and the $\widetilde{\xi}_{k,j}^{\omega,e}$ verify the four hypotheses of Theorem 3.2.5.

(H0) holds because of the bound (3.34).

(H1): we need to to establish that for every bounded Borel set $A \subset \mathbb{R}$,

$$\lim_{k\to\infty} \max_{1\le j\le \mathrm{n}^{k-r_k}} \mathbb{P}(\widetilde{\xi}_{k,j}^{\omega,e}(A) \ge 1) = 0. \tag{3.36}$$

Proof. Chebyshev's inequality and the bound (3.32) yield

$$\begin{aligned}
\mathbb{P}(\widetilde{\xi}_{k,j}^{\omega,e}(A) \ge 1) &\le \mathbb{E}\widetilde{\xi}_{k,j}^{\omega,e}(A) \\
&\le \frac{|B_{k,j}|}{|B_k|}\,\|\gamma\|_\infty\,\mathcal{L}(A) \\
&= \mathrm{n}^{r_k-k}\,\|\gamma\|_\infty\,\mathcal{L}(A),
\end{aligned}$$

and (3.36) follows. □

(H2): We need to establish that for all $\operatorname{Im} z > 0$,

$$\lim_{k\to\infty} \mathbb{E}\int \operatorname{Im}(t-z)^{-1} d\widetilde{\xi}_k^{\omega,e}(t) = \pi\eta(e).$$

Proof. We have

$$\mathbb{E}\int \operatorname{Im}(t-z)^{-1}d\tilde{\xi}_k^{\omega,e}(t) = |B_k|^{-1}\,\mathbb{E}\mathrm{Im}\sum_{x\in B_k}\langle\delta_x|(H^\omega_{r_k}-z_k)^{-1}\delta_x\rangle$$

$$= |B_k|^{-1}\,\mathbb{E}\mathrm{Im}\sum_{x\in B_k}\langle\delta_x|\left((H^\omega_{r_k}-z_k)^{-1}-(H^\omega-z_k)^{-1}\right)\delta_x\rangle$$

$$+\,\mathbb{E}\mathrm{Im}\,\langle\delta_{x_0}|(H^\omega-z_k)^{-1}\delta_{x_0}\rangle$$

$$= I_{k,\omega}+II_{k,\omega}.$$

Now $II_{k,\omega}\to\pi\eta(e)$ by 3.27 and $I_{k,\omega}\to 0$, as in the proof of Proposition 3.4.4. □

(H3): We need to establish that for every function $g_z(t)=\operatorname{Im}(t-z)^{-1}$, $\operatorname{Im} z>0$,

$$\lim_{k\to\infty}\sum_{j=1}^{\mathrm{n}^{k-r_k}}\mathbb{E}\mathcal{I}(\tilde{\xi}_{k,j}^{\omega,e},g_z)=0. \tag{3.37}$$

Proof. We have,

$$|B_k|^2\,I(\tilde{\xi}_{k,j}^{\omega,e},g_z)=$$

$$=\left(\sum_{x\in B_{k,j}}\langle\delta_x|\operatorname{Im}(H^\omega_{r_k}-z_k)^{-1}\delta_x\rangle\right)^2-\sum_{x\in B_{k,j}}\langle\delta_x|\left(\operatorname{Im}(H^\omega_{r_k}-z_k)^{-1}\right)^2\delta_x\rangle$$

$$=\sum_{x,y\in B_{k,j}}\det\begin{pmatrix}\langle\delta_x|\operatorname{Im}(H^\omega_{r_k}-z_k)^{-1}\delta_x\rangle & \langle\delta_x|\operatorname{Im}(H^\omega_{r_k}-z_k)^{-1}\delta_y\rangle\\ \langle\delta_y|\operatorname{Im}(H^\omega_{r_k}-z_k)^{-1}\delta_x\rangle & \langle\delta_y|\operatorname{Im}(H^\omega_{r_k}-z_k)^{-1}\delta_y\rangle\end{pmatrix}.$$

Using Minami's estimate (3.26) we get the bounds

$$|B_k|^2\,\mathbb{E}I(\tilde{\xi}_{k,j}^{\omega,e},g_z)\le\pi^2\,\|\gamma\|_\infty^2\,|B_{k,j}|^2,$$

and hence

$$\sum_{j=1}^{n^{k-r_k}} \mathbb{E} I(\widetilde{\xi}_{k,j}^{\omega,e}, g_z) \leq \pi^2 \, \|\gamma\|_\infty^2 \, \mathrm{n}^{-r_k},$$

which yields (3.37). $\square$

Conclusion

In this thesis, we studied in detail the hierarchical Anderson model. We discussed the deterministic spectral properties of the free hierarchical Laplacian and the typical spectral properties of the hierarchical Anderson model. We proved two main results. If the model has a spectral dimension $d_{sp} \leq 4$ then, with probability one, the spectrum of H_ω is dense pure-point. For $d_{sp} < 1$, the energy levels for H_ω are statistically uncorrelated in the thermodynamic limit.

A number of interesting questions remain open. First of all, does spectral localization hold at arbitrary spectral dimension for an arbitrary density $\gamma(t)$ of the random potential? Based on Molchanov's theorem and its generalization, we tend to believe that the answer is yes. Still, it might very well be that $d_{sp} = 4$ is a true critical exponent for the hierarchical Anderson model and that one can observe a qualitative change in the spectral behavior for $d_{sp} > 4$. Concerning Poisson statistics of eigenvalues, it is interesting to compare our result with Minami's theorem. The one-dimensional Anderson model on $\mathbb{Z}$ has Poisson statistics of eigenvalues, at arbitrary disorder. So does the hierarchical Anderson model with spectral dimension $d_{sp} < 1$. A closer look at Minami's paper reveals that the proof of Poisson statistics in dimension one requires, in addition to Aizenman-Molchanov theory, the powerful machinery of Furstenberg theorem and in particular nontrivial regularity results about Lyapunov exponents. On the other hand, the proof of Poisson statistics for the hierarchical Anderson is very simple technically. The resolvent identity alone is a sufficiently powerful tool because of the low spectral dimension assumption and because of the high degree of self-similarity of the model. Does the hierarchical

Anderson model still have Poisson statistics of eigenvalues for spectral dimension $d_{sp} = 1$? What about $d_{sp} > 1$?

Of course, the pessimist may argue as follows. The hierarchical Anderson model remains a toy model missing many important features of the Anderson model on $\mathbb{Z}^d$. Unlike $\Delta_{\mathbb{Z}^d}$, the free hierarchical Laplacian Δ has no a.c. spectrum to start with and it is therefore very unlikely to find a.c. spectrum or eigenvalue repulsion for the randomly perturbed operator. The hierarchical Anderson model is therefore too unrealistic physically. In our opinion, the hierarchical Anderson model is actually a beautiful playground to investigate the mathematical mechanisms responsible for localization and for Poisson statistics. The spectral dimension d_{sp} serves as a *continuous* tuning parameter, whose effect can be immediately observed on the different estimates. We believe that a deeper study of the hierarchical model, as well as other long-range symmetric toy models, has the potential to improve the understanding of the necessary and sufficient conditions for localization and Poisson statistics in general random discrete Schrödinger operators. It may then be possible to understand why these conditions are violated for delocalized models.

Frequently used notations and definitions

Parameters of a hierarchical structure

- $(\mathbb{X}, \mathtt{d})$ is a hierarchical structure
- $B(x,r) = \{y \in \mathbb{X} : \mathtt{d}(y,x) \leq r\}$ is the (closed) ball with radius r and center x
- $\mathtt{N}_r$ is the cardinality of $B(x,r)$
- n_r is the number of balls of radius $r-1$ contained in a ball of radius r
- $\mathcal{P}_r$ is the collection of balls of radius r
- δ_x is the Kronecker delta function at x

Parameters of the hierarchical Laplacian

- E_r is the orthogonal projection on the subspace of $l^2(\mathbb{X})$ consisting on functions that are constant on every closed ball of radius r
- $(\mathrm{p}_r)_{r\geq 0}$ is a sequence with $\mathrm{p}_0 = 0$, $\mathrm{p}_r > 0$ for $r \geq 1$, and $\sum_{r=1}^{\infty} \mathrm{p}_r = 1$
- $\rho > 1$ is the rate of decay of p_r: $\mathrm{p}_r \sim \rho^{-r}$
- $\Delta = \sum_{r=0}^{\infty} \mathrm{p}_r E_r$ is the hierarchical Laplacian
- $\lambda_r = \sum_{s=0}^{r} \mathrm{p}_s$ are the eigenvalues of Δ
- $\Delta_r = \sum_{s=0}^{r} \mathrm{p}_s E_s$ the truncated hierarchical Laplacian
- d_{sp} is the spectral dimension of Δ

Parameters of the hierarchical Anderson Model

- $V_\omega = \sum_{x\in\mathbb{X}} \omega(x) \langle \delta_x | \cdot \rangle \delta_x$ is the random potential
- $H_\omega = \Delta + V_\omega$ is the hierarchical Anderson model
- $H_{\omega,k}$ and also H_k^ω denote the truncated operator $\Delta_k + V_\omega$
- $\mathbb{P}$ is a probability measure on $\Omega = \mathbb{R}^{\mathbb{X}}$
- for the i.i.d. case, μ^{V} is the probability distribution of $\omega(x)$
- $\gamma(t)$ is the density of μ^{V}

- $\eta(t)$ is the density of the averaged spectral measure μ^{av}

Convergence of measures

- $C_0(\mathbb{R})$ is the set of continuous functions $f : \mathbb{R} \to \mathbb{C}$ with $\lim_{|t|\to\infty} |f(t)| = 0$.
- A sequence of Borel measures μ_k on $\mathbb{R}$ is said to converge in the weak-* topology to a measure μ on $\mathbb{R}$ if
$$\lim_{k\to\infty} \int_{\mathbb{R}} f\, d\mu_k = \int_{\mathbb{R}} f\, d\mu,$$
for every $f \in C_0(\mathbb{R})$.

Other symbols and notations

- $\mathbb{N}$ denotes the natural numbers $\{0, 1, 2, \cdots\}$
- $\mathcal{L}$ is the Lebesgue measure on $\mathbb{R}$
- $\delta(e)$ is the Dirac delta mass at e
- 1_A is the indicator function of the set A
- a.a. is an abbreviation for *almost all*
- a.c. is an abbreviation for *absolutely continuous*
- $\Delta_{\mathbb{Z}^{\mathrm{d}}}$ is the discrete Laplacian on $\mathbb{Z}^{\mathrm{d}}$
- $\mathbb{C}_+$ is the upper-half complex plane $\{z : \operatorname{Im} z > 0\}$
- for $S_1, S_2 \subset \mathbb{R}$, $S_1 + S_2$ denotes the set $\{s_1 + s_2 : s_1 \in S_1 \text{ and } s_2 \in S_2\}$

Appendix A: A matrix lemma

Lemma 3.4.7. *Let A be a hermitian $N \times N$ matrix and $v \in \mathbb{C}^N$. Then for all $M > 0$*

$$\mathcal{L}\left(\left\{e : \left\|(A-e)^{-1}v\right\|_2^2 \geq M\right\}\right) \leq 4\sqrt{\frac{N}{M}}\,\|v\|_2$$

Proof. (Following [M3]) Let $\lambda_1, \ldots, \lambda_N$ be the eigenvalues of A and $\psi_1, \ldots, \psi_N$ the corresponding orthonormal basis of eigenvectors. In this basis, we have $v = \sum_{i=1}^N v_i \psi_i$, $\|v\|_2^2 = \sum_{i=1}^N |v_i|^2$ and $(A-e)^{-1}v = \sum_{i=1}^N (\lambda_i - e)^{-1} v_i \psi_i$. For each $i = 1, \ldots, N$, consider the open interval $\Delta_i = (\lambda_i - v_i/\sqrt{M}, \lambda_i + v_i/\sqrt{M})$. The total length of these N intervals is

$$\mathcal{L}\left(\bigcup_{i=1}^N \Delta_i\right) \leq \sum_{i=1}^N m\left(\Delta_i\right) = \frac{2}{\sqrt{M}} \sum_{i=1}^N |v_i| \leq \frac{2}{\sqrt{M}} \sqrt{N}\,\|v\|_2 .$$

Chebyshev inequality yields

$$\begin{aligned}
\mathcal{L}\left(\left\{e \in \mathbb{R}\backslash\bigcup_{i=1}^N \Delta_i : \left\|(A-e)^{-1}v\right\|_2^2 \geq M\right\}\right) &\leq \frac{1}{M} \int_{\mathbb{R}\backslash\bigcup_{i=1}^N \Delta_i} \left\|(A-e)^{-1}v\right\|_2^2 de \\
&= \frac{1}{M} \int_{\mathbb{R}\backslash\bigcup_{i=1}^N \Delta_i} \sum_{i=1}^N \left|\frac{v_i}{\lambda_i - e}\right|^2 de \\
&\leq \frac{1}{M} \sum_{i=1}^N \int_{\mathbb{R}\backslash\Delta_i} \left|\frac{v_i}{\lambda_i - e}\right|^2 de \\
&= \frac{1}{M} \sum_{i=1}^N 2 \int_{|v_i|/\sqrt{M}}^{\infty} \frac{|v_i|^2}{t^2} dt \\
&= \frac{2}{M} \sqrt{M} \sum_{i=1}^N |v_i| \\
&\leq \frac{2}{\sqrt{M}} \sqrt{N}\,\|v\|_2 ,
\end{aligned}$$

and the result follows. □

Appendix B: Wegner and Minami's estimates

Proof of Lemma 3.4.1: (Following [J] section 5.3) We decompose the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ along the x'th coordinate: $\Omega = \mathbb{R} \times \mathbb{R}^{\mathbb{X}\setminus\{x\}}$, $\omega = (t, \omega')$, $\mathbb{P} = \gamma(t)dt \times \mathbb{P}'$. Then

$$M_\omega = M_{(t,\omega')} = M_{(0,\omega')} + t\langle\delta_x|\cdot\rangle\delta_x,$$

and the resolvent identity

$$(M_{(t,\omega')} - z)^{-1} - (M_{(0,\omega')} - z)^{-1} = (M_{(t,\omega')} - z)^{-1}(-t\langle\delta_x|\cdot\rangle\delta_x)(M_{(0,\omega')} - z)^{-1},$$

yields

$$\langle\delta_x|(M_{(t,\omega')} - z)^{-1}\delta_x\rangle = \frac{1}{t - (-\langle\delta_x|(M_{(0,\omega')} - z)^{-1}\delta_x\rangle)^{-1}}.$$

Hence for $\operatorname{Im} z > 0$,

$$\int \operatorname{Im} \langle\delta_x|(M_{(t,\omega')} - z)^{-1}\delta_x\rangle dt = \pi,$$

which implies that

$$\int \langle\delta_x|h(M_{(t,\omega')})\delta_x\rangle dt = \int h(t)dt.$$

Hence, using Fubini's theorem,

$$\begin{aligned}\mathbb{E}\langle\delta_x|h(M_\omega)\delta_x\rangle &= \int_{\mathbb{R}^{\mathbb{X}\setminus\{x\}}} \int \langle\delta_x|h(M_{(t,\omega')})\delta_x\rangle\gamma(t)dtd\mathbb{P}'(\omega') \\ &\leq \|\gamma\|_\infty \int h(t)dt.\end{aligned}$$

□

Proof of Lemma 3.4.2: The main ingredient of the proof is the following calculation: if $A = (a_{i,j})_{i,j=1,2}$ is a 2×2 matrix with $\operatorname{Im} A > 0$, then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \det \operatorname{Im} \left(\begin{pmatrix} t & 0 \\ 0 & u \end{pmatrix} - A \right)^{-1} dtdu \tag{3.38}$$

$$= \pi^2 \frac{\det \operatorname{Im} A}{\sqrt{(\det \operatorname{Im} A)^2 + (\det \operatorname{Im} A)(|a_{1,2}|^2 + |a_{2,1}|^2)/2 + (|a_{1,2}|^2 - |a_{2,1}|^2)^2/16}}.$$

For a detailed derivation of (3.38), we refer the reader to Lemma 2 in [GV] or to [Mi] for the special case $a_{1,2} = a_{2,1}$.

If $x = y$ then (3.26) holds trivially. If $x \neq y$, we decompose the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ along the coordinates x and y:

$$\Omega = \mathbb{R}^2 \times \mathbb{R}^{\mathbb{X}\setminus\{x,y\}}, \omega = (t, u, \omega'), \mathbb{P} = \gamma(u)du \times \gamma(t)dt \times \mathbb{P}'.$$

Consider the 2×2 matrix

$$F_{(t,u,\omega')} = \begin{pmatrix} \langle x|(M_{(t,u,\omega')} - z)^{-1}x\rangle & \langle x|(M_{(t,u,\omega')} - z)^{-1}y\rangle \\ \langle x|(M_{(t,u,\omega')} - z)^{-1}y\rangle & \langle y|M_{(t,u,\omega')} - z)^{-1}y\rangle \end{pmatrix},$$

and note that $\operatorname{Im} F_{(t,u,\omega')} > 0$. We can write

$$M_\omega = M_{(t,u,\omega')} = M_{(0,0,\omega')} + t\langle\delta_x|\cdot\rangle\delta_x + u\langle\delta_y|\cdot\rangle\delta_y,$$

and the resolvent identity implies

$$F_{(t,u,\omega')} = \left(\begin{pmatrix} t & 0 \\ 0 & u \end{pmatrix} + F^{-1}_{(0,0,\omega')} \right)^{-1}. \tag{3.39}$$

Let $A_{\omega'} = -F^{-1}_{(0,0,\omega')}$ and note that $\operatorname{Im} A_{\omega'} > 0$. Then (3.39) and(3.38) yield that for for each fixed ω',

$$\int\int \det \operatorname{Im} F_{(t,u,\omega')} dt du \leq \pi^2.$$

Minami's estimate is now an immediate consequence of Fubini's theorem. □

Appendix C: Minami's proof of Poisson statististics of eigenvalues for the localized Anderson model on $\mathbb{Z}^d$

For a rectangle $B \subset \mathbb{Z}^{\mathrm{d}}$, we denote by H_B^ω the restriction of H_ω to $l^2(B)$ with Dirichlet boundary conditions: i.e. $\langle \delta_x | H_B^\omega \delta_y \rangle = \langle \delta_x | H_\omega \delta_y \rangle$ if both $x, y \in B$, and $\langle \delta_x | H_B^\omega \delta_y \rangle = 0$ otherwise. For $k \geq 1$, let B_k be the rectangle

$$B_k = \left\{ x \in \mathbb{Z}^{\mathrm{d}} : \max_{i=1,\cdots,\mathrm{d}} |x_i| \leq k \right\},$$

and let $H_k^\omega = H_{B_k}^\omega$. As before, $e_1^{\omega,k} \leq e_2^{\omega,k} \leq \cdots \leq e_{|B_k|}^{\omega,k}$, are the eigenvalues of $H_k^\omega \restriction l^2(B_k)$, μ_k^ω is the corresponding normalized counting measure given by (3.1) and $\xi_k^{\omega,e}$ is the rescaled measure near e given by (3.2). We refer the reader to the recent work [KN] for a discussion of the regime where both space and energy are rescaled. The averaged spectral measure for H_ω is given by (3.15) and the Wegner estimate yileds that μ^{av} has a bounded density $\eta(t)$ with respect to $\mathcal{L}$. A basic result for the multi-dimmensional Anderson model is that for $\mathbb{P}$-a.a. $\omega \in \Omega$, the spectrum of H_ω is equal to $[-2\mathrm{d}, 2\mathrm{d}] + \mathrm{supp}(\gamma) = \mathrm{supp}(\mu^{\mathrm{av}})$ and μ_k^ω converges to μ^{av} in the weak-* topology as $k \to \infty$ ([PF, CL, CKFS]).

Theorem 3.4.8. *(Minami, 1996) Assume that there are constants $0 < C < \infty, 0 < D < \infty$ and $0 < s < 1$ such that*

$$\mathbb{E} \left| \langle \delta_x | (H_B^\omega - z)^{-1} \delta_y \rangle \right|^s \leq C e^{-D|x-y|}, \qquad x, y, \in \mathbb{Z}^{\mathrm{d}}, \tag{3.40}$$

for all z with $e_1 < \mathrm{Re}\, z < e_2, \mathrm{Im}\, z \neq 0$ and for all rectangles $B \subset \mathbb{Z}^{\mathrm{d}}$. Assume that $e \in (e_1, e_2)$ verifies the regularity condition (3.27) and that $\eta(e) > 0$. Then $\xi_k^{\omega,e}$ converges to a Poisson point process on $\mathbb{R}$ with intensity $\eta(e)\mathcal{L}$.

We refer the reader to [HM] for a discussion of the set of e for which $\eta(e) > 0$. Condition (3.40) is called fractional-moments localization. It implies that within

(e_1, e_2), for $\mathbb{P}$-a.a. $\omega \in \Omega$ the spectrum of H_ω, if any, is pure-point with exponentially decaying eigenfunctions [AM, ASFH]. For d = 1, condition (3.40) holds for all energy intervals (e_1, e_2) [Mi]. In dimensions d ≥ 2, condition (3.40) is obtained by either moving the energy interval (e_1, e_2) to $\pm\infty$ or by increasing the disorder. The two main techniques for proving that are the multiscale analysis [FS, DK] and the Aizenman-Molchanov theory [AM].

Proof of Theorem 3.4.8. We fix $\alpha \in (0, 1)$ and for each k, we make a partition

$$B_k = \bigcup_{j=1}^{n_k} B_{k,j},$$

where $B_{k,j}$ are disjoint rectangles with side $\sim (2k)^\alpha$. Hence $n_k \sim k^{\mathrm{d}(1-\alpha)}$. Let $\widetilde{e}_l^{\omega,k,j}$, $l = 1, \cdots, |B_{k,j}|$ denote the eigenvalues of $H^\omega_{B_{k,j}} \restriction l^2(B_{k,j})$ and let

$$\widetilde{\xi}_{k,j}^{\omega,e} = \sum_{l=1}^{|B_{k,j}|} \delta(|B_k| \, (\widetilde{e}_l^{\omega,k,j} - e)),$$

$$\widetilde{\xi}_k^{\omega,e} = \sum_{j=1}^{n_k} \widetilde{\xi}_{k,j}^{\omega,e}.$$

Hence the point process $\widetilde{\xi}_k^{\omega,e}$ is the sum of n_k independent point processes $\widetilde{\xi}_{k,j}^{\omega,e}$. As for the hierarchical Anderson model, Theorem 3.4.8 follows from the following two propositions.

Proposition 3.4.9. *For every $f \in L_1(\mathbb{R}, dt)$,*

$$\lim_{k\to\infty} \mathbb{E} \left| \int f d\widetilde{\xi}_k^{\omega,e} - \int f d\xi_k^{\omega,e} \right| = 0. \tag{3.41}$$

Proposition 3.4.10. *The point process $\widetilde{\xi}_k^{\omega,e}$ converges to a Poisson point process on $\mathbb{R}$ with intensity $\eta(e)\mathcal{L}$.*

Proof of Proposition 3.4.9. As in the proof of Proposition 3.4.4, it is enough to prove (3.41) for the family of functions

$$g_z(t) = \operatorname{Im}(t-z)^{-1}, \qquad \operatorname{Im} z > 0.$$

We set

$$z_k = e + |B_k|^{-1} z. \tag{3.42}$$

Then

$$\int g_z d\widetilde{\xi}_k^{\omega,e} - \int g_z d\xi_k^{\omega,e}$$

$$= |B_k|^{-1} \operatorname{Im} \sum_{j=1}^{n_k} \sum_{x \in B_{k,j}} \langle \delta_x | \left((H^{\omega}_{B_{k,j}} - z_k)^{-1} - (H^{\omega}_k - z_k)^{-1} \right) \delta_x \rangle.$$

Let $v_k = \beta \ln k$, where $\beta > 0$ is a fixed big enough constant to be specified later. We set

$$\operatorname{int}(B_{k,j}) = \{ x \in B_{k,j} : dist(x, \partial B_{k,j}) \geq v_k \},$$

and

$$\operatorname{wall}(B_{k,j}) = \{ x \in B_{k,j} : dist(x, \partial B_{k,j}) < v_k \}.$$

Then

$$\mathbb{E} \left| \int g_z d\widetilde{\xi}_k^{\omega,e} - \int g_z d\xi_k^{\omega,e} \right| \leq \mathbb{E} \left| I_{k,\omega} \right| + \mathbb{E} \left| II_{k,\omega} \right|,$$

where

$$I_{k,\omega} = |B_k|^{-1} \operatorname{Im} \sum_{j=1}^{n_k} \sum_{x \in \operatorname{wall}(B_{k,j})} \langle \delta_x | \left((H^{\omega}_{B_{k,j}} - z_k)^{-1} - (H^{\omega}_k - z_k)^{-1} \right) \delta_x \rangle,$$

$$II_{k,\omega} = |B_k|^{-1} \operatorname{Im} \sum_{j=1}^{n_k} \sum_{x \in \operatorname{int}(B_{k,j})} \langle \delta_x | \left((H^{\omega}_{B_{k,j}} - z_k)^{-1} - (H^{\omega}_k - z_k)^{-1} \right) \delta_x \rangle.$$

The Wegner estimate (3.34) yields that

$$\mathbb{E}\left|I_{k,\omega}\right| \leq 2\pi \left\|\gamma\right\|_{\infty} \left|B_k\right|^{-1} \sum_{j=1}^{n_k} \left|\mathrm{wall}(B_{k,j})\right|,$$

and the right hand side converges to zero as $k \to \infty$.

To estimate $\mathbb{E}\left|II_{k,\omega}\right|$, we use the resolvent identity

$$\langle\delta_x| \left((H^{\omega}_{B_{k,j}} - z_k)^{-1} - (H^{\omega}_k - z_k)^{-1}\right) \delta_x\rangle$$

$$= \sum_{(y,y')} \langle\delta_x|(H^{\omega}_{B_{k,j}} - z_k)^{-1}\delta_y\rangle\langle\delta_{y'}|(H^{\omega}_k - z_k)^{-1}\delta_x\rangle,$$

where the sum is over all pairs (y, y'), with $y \in \partial B_{k,j}$, $y' \notin B_{k,j}$ and $|y - y'| = 1$. Hence,

$$\mathbb{E}\left|II_{k,\omega}\right| \leq \left|B_k\right|^{-1} \sum_{j=1}^{n_k} \sum_{x \in \mathrm{int}(B_{k,j})} \sum_{(y,y')} \mathbb{E}\left|\langle\delta_x|(H^{\omega}_{B_{k,j}} - z_k)^{-1}\delta_y\rangle\langle\delta_{y'}|(H^{\omega}_k - z_k)^{-1}\delta_x\rangle\right|. \tag{3.43}$$

For k large enough so that $e_1 < \mathrm{Re}\, z_k < e_2$, we use the main assumption (3.40) together with the bound

$$\left|\langle\delta_x|(H^{\omega}_{B_{k,j}} - z_k)^{-1}\delta_y\rangle\langle\delta_{y'}|(H^{\omega}_k - z_k)^{-1}\delta_x\rangle\right| \leq (\mathrm{Im}\, z_k)^{-2} = (\left|B_k\right| / \mathrm{Im}\, z)^2,$$

to obtain

$$\begin{aligned}
&\mathbb{E}\left|\langle\delta_x|(H^\omega_{B_{k,j}} - z_k)^{-1}\delta_y\rangle\langle\delta_{y'}|(H^\omega_k - z_k)^{-1}\delta_x\rangle\right| \\
&\le (|B_k|/\operatorname{Im} z)^{2(1-s/2)}\mathbb{E}\left|\langle\delta_x|(H^\omega_{B_{k,j}} - z_k)^{-1}\delta_y\rangle\langle\delta_{y'}|(H^\omega_k - z_k)^{-1}\delta_x\rangle\right|^{s/2} \\
&\le (|B_k|/\operatorname{Im} z)^{2(1-s/2)}\left(\mathbb{E}\left|\langle\delta_x|(H^\omega_{B_{k,j}} - z_k)^{-1}\delta_y\rangle\right|^s\right)^{1/2}\left(\mathbb{E}\left|\langle\delta_{y'}|(H^\omega_k - z_k)^{-1}\delta_x\rangle\right|^s\right)^{1/2} \\
&\le (|B_k|/\operatorname{Im} z)^{2(1-s/2)}Ce^{-Dv_k}.
\end{aligned} \tag{3.44}$$

Since, in (3.43), there are $O(k^{\alpha(\mathrm{d}-1)})$ pairs (y,y') for each $B_{k,j}$, the bounds (3.43) and (3.44) yield

$$\begin{aligned}
\mathbb{E}\,|II_{k,\omega}| &\le O(k^{\alpha(\mathrm{d}-1)}\,|B_k|^{2(1-s/2)}\,e^{-Dv_k}) \\
&= O(k^{\alpha(\mathrm{d}-1)+2\mathrm{d}(1-s/2)}e^{-D\beta\ln k})
\end{aligned}$$

Hence, if we choose $\beta > D^{-1}\left(\alpha(\mathrm{d}-1)+2\mathrm{d}(1-s/2)\right)$, then $\mathbb{E}\,|II_{k,\omega}| \to 0$ as $k\to\infty$. $\square$

Proof of Proposition 3.4.10. As in the proof of Propositon 3.4.6, it suffices to show that $\widetilde{\xi}^{\omega,e}_k$ and the $\widetilde{\xi}^{\omega,e}_{k,j}$ verify the four hypotheses of Theorem 3.2.5. The proof of (H0), (H1) and (H3) is the same as in Propositon 3.4.6. It remains to show that (H2) holds, i.e. for $\operatorname{Im} z>0$,

$$\lim_{k\to\infty}\mathbb{E}\int g_z d\widetilde{\xi}^{\omega,e}_k = \pi\eta(e). \tag{3.45}$$

The argument of the proof of Proposition 3.4.9, with H^ω_k replaced by H_ω, yields that

$$\lim_{k\to\infty}\mathbb{E}\left(\int g_z d\widetilde{\xi}^{\omega,e}_k - \int g_z d\mu^{av}\right) = 0, \tag{3.46}$$

and then (3.45) follows from (3.46) and (3.27).

□

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