A FACTORIZATION ALGORITHM WITH APPLICATIONS TO THE LINEAR FILTERING AND CONTROL PROBLEMS

by

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A thesis submitted to the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy

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September, 1981.

FAST ALGORITHMS FOR PROBLEMS IN FILTERING AND CONTROL

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ABSTRACT

In this study, we address the factorization problem in the Hardy H^p spaces, and provide a fast algorithm for its implementation with applications to some important engineering problems. The Thesis is presented in three parts.

In the first part we lay down the technical foundations of the new approach in the scalar case. First, the factorization problem is formulated in the H^p spaces. A formulation with sufficient generality to encompass practically all such engineering problems. Necessary and sufficient conditions for the existence of the spectral factors are derived, and a characterization of the class of functions admitting a canonical factorization is obtained. The reduction method is applied to certain Toeplitz equations in H^2 space to generate a sequence of approximate spectral factors. When the Laguerre basis is used in the reduction method, the Toeplitz equation turns out to a Toeplitz set of linear equations. We also provide an error bound and an estimate for the speed of convergence.

In the second part, the matrix version of all the scalar results is provided and enriched with discussions and extensions. In particular, we have shown that the factorization problem is associated with the solutions of certain Toeplitz equations in H^{2+}_{mxm} spaces. The classical Gohberg-Krein factorization is re-examined within the framework developed here, and the connections between the outer-factorization, the canonical factorization, and inversion of certain Toeplitz operator have also been unveiled.

In Part Three, we generalize the Davis and Barry formula for the feedback gain in the LQR problems. The new setting, equipped with the spectral factorization method, provides fast and efficient algorithms for solving a wide class of LQR problems, rational matrix factorization, and positive polynomials factorization. Our parallel results for the discrete time case are given in brief together with many interesting computational properties.

RESUME

On aborde dans cette étude le problème de factorisation de fonctions dans un espace de Hardy H^p. On met au point un algorithme de solution rapide, avec application à quelques problèmes importants en génie des systèmes. La thèse est présentée sous forme de trois communications autonomes.

Les fondements techniques de la nouvelle approche sont posés dans la première partie, pour le cas monovariable. On formule d'abord le problème de factorisation dans l'espace H^p, avec suffisemment de généralité pour inclure presque toutes les applications de ce problème. On donne des conditions nécessaires et suffisantes pour l'existence de facteurs spectraux, et on obtient un caractérisation de la classe de fonctions admettant une factorisation canonique. Une méthode de réduction est appliquée à certaines équations de type Toeplitz en H^2 afin de générer une séquence convergente de facteurs spectraux approchés. Ces équations s'avèrent linéaires lorsqu'on utilise les fonctions de Laguerre. On donne aussi une bonne d'erreur ainsi qu'un estimé de la rapidité de convergence.

La seconde partie présente la version multivariable de la première, enrichie de quelques extensions. Le problème de factorisation est associé à la solution d'équations de type Toeplitz dans un espace $H^{2+}_{m \times m}$. On revoit l'oeuvre classique de Gohberg et Krein à la lumière de la nouvelle approche, et on fait les liens entre la factorisation externe, la factorisation canonique et l'inversion de certains opérateurs Toeplitz.

La troisième partie est consacrée à la généralisation des résultats de Davis et Barry pour le gain de rétroaction dans le problème linéaire-quadratique. La nouvelle méthode donne lieu à des algorithms de solution efficaces pour une variété de problèmes de factorisation tels le problème LQ , la factorisation de matrices rationelles et de polynômes. On donne en bref des résultats parallèles pour le cas de systèmes discrétisés.

ACKNOWLEDGEMENT

I would like to express my sincere gratitude to Professor Pierre R. Bélanger for his guidance, encouragement, and support during the course of this research. As my Thesis supervisor, his insight, observations, and suggestions helped to establish the overall direction of the research effort and contributed immensely to the success of the work reported here.

It is also a pleasure to acknowledge several discussions and thought-provoking conversations with Professors G. Zames and F. Reza. I owe much as well to Professors P.E. Caines and M. Kaplan, for their valuable comments, discussions and criticisms, which motivated and enriched many points in this work. I benefitted in many ways from hours of active and intellectually stimulating discussions spent with my colleagues, Dr. A.K. El-Sakkary and Y. Ghoneim.

Many thanks are also due to Ms. P. Hyland and Ms. J. Lélievre for their excellent typing of the thesis.

CLAIM OF ORIGINALITY

Two main contributions in control theory and applications are claimed. The first one is a novel formulation of the spectral factorization problem in the Hardy H^p spaces of frequency response functions. The second contribution is the introduction of fast algorithms for solving some of the most important control problems.

The new characterization of the factorization problem leads to the following results:

(i) A new understanding of the relationship between the factorization problem and the inversion of the generalized frequency-domain image of the Wiener-Hopf operator.

(ii) A complete characterization of a class of functions admitting a canonical factorization in H^p_{mxm} .

(iii) The formulation provides, to the first time, a test and a procedure for solving a wide class of Wiener-Hopf equations with unsummable kernels.

(iv) The relation between the outer-factorization of functions, which appears frequently in the modern design of feedback systems, and the spectral factorization is unveiled.

(v) The classical Gohberg-Krein factorization, and its connection with certain Toeplitz operators in H^2 space are re-investigated in the light of the new formulation.

(vi) The new formulation provides as well a rigorous methodology to approach many related open issues in systems theory, e.g., the spectral theory of the linear quadratic regulator problems, some distributed filtering problems, Wiener-Hopf equations with unsummable kernels, and many others.

The second main contribution is the developing of fast and simple algorithms, the first of their kind, for solving a wide class of LQR and filtering problems without solving the Riccati equation, in continuous and discrete time, and for lumped and distributed parameter systems. We have also modified the integral formula of Davis and Barry for the optimal feedback gain. The new formula enables the treatment of unstable systems and provides a prescaling technique for the eigenvalues of the system in such a way to simplify the computation, and to accelerate the convergence of the algorithms. The potentialities of the approach has been demonstrated by providing subalgorithms for the factorization of rational matrices and positive polynomials arising in other contexts than control.

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HISTORICAL BACKGROUND

The relation between the factorization problem and the solution of the algebraic Riccato equation (A.R.E.) has been known for quite a long time. In fact, this relation was utilized to prove the existence and uniqueness of the solution of certain A.R.E. in [6, 9 and 10, p. III] and [3, p. I]. Anderson [12, p. III] reduced the factorization problem to the solution of an A.R.E.. Brockett [8, p. III] was the first one to suggest the solution of the linear Quadratic Regulator problem via the spectral factorization. However, the computation needed to implement this idea in the multivariable case was not simpler than solving the A.R.E.. Inspired by the work of Brockett, Davis and Barry [2, p. I] derived an integral formula which gives directly the optimal feedback gain in terms of the spectral factor of certain positive function, and applied this approach to the solution of a class of distributed parameter systems. Davis et al extended also these results to the solution of the distributed filtering problem [1, p. I], and to a class of open loop unstable distributed parameter systems [5, p. III]. It is expected that this approach may also cover a wide variety of distributed parameter and large scale systems. Unfortunately, only very few numerical methods are available to implement such a factorization. Perhaps the earliest method is the iteration projection method proposed by Masani and Wiener [4 and 5, p. I]. However, the stringent conditions on the class of functions applicable to their method made the method of limited use. F. Stenger [6, Part I] considered the spectral factorization for the class of functions in $L^1(R) \cap L^2(R)$ using a frequency domain approximation scheme in which a function $a(j\omega)$ is expanded in the form

$$a\,(j\omega) \sim \sum_{n=-\infty}^{\infty} a\,(j\,(nh + \frac{1}{2}\,h))\;\; x_n(\omega)$$

where $x_n(\omega)$ is a characteristic function on the frequency interval $(nh,\ (n+1)h)$. The spectral factor of $a(j\omega)$ is then obtained using the classical idea of taking the Log, performing the causal projection, then taking the anti-log. However, the high numerical accuracy needed to handle these characteristic functions and the fact that the structure of the characteristic functions is different for different intervals, and that for a reasonable accuracy the frequency range has to be divided into a large number of intervals, make the method computationally very demanding.

Davis and Dickinson [7, p. I] have recently introduced an iterative method for the spectral factorization of Hermitian positive definite matrices. Although the method has a quadratic convergence rate, the causal projection and the matrix inversion required at each iteration step and at each frequency represent a serious computational drawback. Techniques of spectral factorization of rational matrices are legion [11 - 22, p. III]. A thorough examination of this case has appeared in [13, p. III]. The majority of these techniques, including those of [14 - 18, p. III], rely on frequency domain manipulations in which the problem of factoring a matrix of real rational functions is reduced to factoring an even polynomial or a self-inversive polynomial. Anderson et al suggested to reduce the factorization problem to the solution of a continuous [10, p. III] or a discrete type [11, p. III] matrix Riccati equation.

The theoretical aspects of the factorization problem and its connection with the Wiener-Hopf equation for the class of kernels in $L_1(R)$ was thoroughly investigated in the famous monograph of Krein [10, p. I] . These results were extended afterwards to the matrix case in [3, p. II], and enriched with many interesting results in [4 - 6, p. II] . Finally, these results were formulated in abstract form in [20 and 21, p. II] .

INTRODUCTION

Some of the most important accomplishments in modern filtering and systems theory hinge on the spectral factorization of functions. When the network or system is characterized by a set of state equations, the solution of network or systems synthesis problems is almost invariably formulated in terms of the solution of a matrix Riccati equation, whereas if the system is modeled by its impulse response, one must solve a Wiener-Hopf equation. Similarly in the frequency domain, synthesis reduces to the factorization of a complex matrix function. Indeed, the three problems are equivalent and all reduce to the problem of factoring a matrix valued function.

In this study we present a novel characterization of the factorization problem in the Hardy H^p spaces of the frequency response functions, as well as a fast algorithm for its implementation. A formulation sufficiently general to encompass both the finite and the infinite dimension cases, and sufficiently rich to inspire the interested colleagues of many extensions and ramifications, and moreover with the proper formal language to communicate with the modern trends in the design of feedback systems. It would appear that the fast algorithm developed here is the first of its kind for the linear quadratic regulator LQR and the filtering problems. A unified approach is also provided for the LQR problems, rational matrix functions factorization, and for positive polynomials factorization.

The Thesis is presented in three autonomous papers. The first lays down the technical foundations of the algorithm in the scalar case. First, we investigate the relationship between the generalized frequency domain image of the Wiener-Hopf equation, which is known as the Toeplitz equations, and the spectral factorization formulated directly in the frequency domain-spaces, i.e., the Hardy H^p spaces. A complete characterization of the class of functions admitting a canonical factorization is brought up. The factorization of positive almost every where a.e.w functions and the standard Krein-type factorization are studied as special cases. It is shown that the spectral factors can be obtained by solving certain Toeplitz equations in the Hilbert space H^{2+} using the reduction method. An orthonormal basis in H^2 space is chosen in such a way that the reduced Toeplitz operator turns out to be a Toeplitz matrix, with the advantage of simple structure and the availability of fast algorithms for its inversion and factorization. We introduce as well a novel approach for estimating the speed of conversion of the algorithm in terms of some smoothness conditions on the canonical factors. Finally, some interesting computational aspects are discussed.

The second paper is concerned with the multidimensional case. The argument and the framework are basically the same as in the scalar case. In fact, with the matrix notations brought up at the start of this paper, most of the scalar results are transferred so smoothly and conveniently to the matrix case that no amendments in the proofs are even required, leaving room for comments and discussion. Moreover, for the ultimate convenience most of the theorems in part two are deliberately stated to match corresponding ones in the first paper. However, other new results are reported as well. The relation between the so-called Outer-factorization of a function, which appears frequently in the modern design of feedback systems, and the canonical factorization is unveiled. The standard Gohberg-Krein factorization, and the connection between the canonical factorization and inversion of the related Toeplitz operator are elaborately re-investigated in the realm of the formulation developed here. The results are also enriched by discussions and extensions.

Part Three is dedicated to the illustration of some important control and systems applications in the light of the results brought up in the preceding parts of this study. We generalize the Davis and Barry integral formula for the optimal feedback gain in the LQR problems. The new setting covers a wider class of cost functions and overcomes the difficulty of treating unstable systems. The new formula, equipped with the proposed spectral factorization method, provides fast and efficient algorithms for solving a wide class of LQR problems, rational matrix functions factorization, and positive polynomials factorization. Our parallel results for the discrete time case are given in brief together with many interesting computational properties.

A FACTORIZATION ALGORITHM WITH APPLICATIONS TO THE LINEAR FILTERING AND CONTROL PROBLEMS, THE SCALAR CASE

PART I

I. INTRODUCTION

Recently, there has been an increasing interest in the solution of the filtering [1] and linear quadratic regulator [2, 3]. Using the so-called Canonical Factorization of a function, say $a\ (j\omega)$, in the form

$$[1 + a\ (j\omega)]^{-1} = [1 + G^{+}\ (j\omega)][1 + G^{+}(j\omega)]^{*} \qquad (1.1)$$

where $[I + G^{+}\ (j\omega)]^{\pm 1}$ have analytic continuation in the open right half plane, an approach which avoids completely the need for solving the matrix Riccati equation.

Moreover, this approach may also cover a wide variety of distributed parameter and large scale systems. Unfortunately, only very few numerical methods are available to implement such a factorization. Perhaps the earliest method is the iterative projection scheme proposed by Masani and Wiener [4], [5] who showed that if $a\ (j\omega)$ is the Fourier transform F.T. of some $a^{v}(t) \in L^{1}\ (R) \cap L^{2}\ (R)$ and the $\| a\ (j\omega) \|_{\infty} < 1$, then the function $G^{+}\ (j\omega)$ may be obtained via the formula.

$$G^{+}\ (j\omega) = p\ [a] - p\ [a\ p\ [a]] + p\ [a\ p\ [ap\ [a]]] - \cdots$$

where p is the projection operator from $L^{2}\ (R)$ onto $H^{2+}\ (R)$.

F. Stenger [6] also considered the factorization (1.1) for the Class of Fourier transform of functions in $L^{1}\ (R) \cap L^{2}\ (R)$ using the frequency domain approximation

$$a(j\omega) \sim \sum_{n=-\infty}^{\infty} a\left(j\left(nh + \frac{1}{2}h\right)\right) x_n(\omega) \tag{1.2}$$

where $x_n(\omega)$ is an approximate characteristic function on the frequency interval $(nh, (n+1)h)$. The constructed characteristic functions turn out to take the form

$$x_n(\omega) \sim \sum_{m=0}^{\infty} \frac{r_m}{j\omega + \alpha_{m,n}} + \frac{\overline{r_m}}{j\omega - \overline{\alpha_{m,n}}} \tag{1.3}$$

where r_m and $\alpha_{m,n}$ are certain constants. Then the factor $(1 + G^+)$ is obtained using the classical idea of taking the Log, performing the projection, then taking the antilog. The main advantage of this method is its ability to track rapidly changing frequency responses. However, the high numerical accuracy needed to handle these characteristic functions, and the fact that the structure of the characteristic functions is different for different intervals, and that for a reasonable accuracy the frequency range has to be divided into a large number of intervals, make the method computationally very demanding. Moreover, the method, in its current form, is technically not applicable to the multivariable case.

J. Davis and R.G. Dickinson [7] have recently developed an iterative method, originally due to T. Wilson [8], [9], for the spectral factorization (1) using the formula

$$(1 + G^+_{n+1}) = (1 + G^+_n)(1 + p\,[(1 + G^+_n)^*(1 + a)(1 + G^+_n) - 1])^{-1}$$

The iteration is executed at each frequency and the projection is performed approximately using the Stenger's idea. As these two methods are basically pointwise, they are not suitable for analytic or semi-analytic approximate solutions.

This paper presents the technical foundations, in the scalar case, of a new approach for approximating the canonical factors of a given function. The idea itself can be summarized in few words; traditionally, the canonical factorization is used to solve the Wiener-Hopf equation [10] , here certain generalized Wiener-Hopf equations, called Toeplitz equations, will be utilized to obtain the spectral factors. First, the formulation of the factorization problem is carried out in the $H^p(R)$ spaces. Necessary and sufficient conditions for the existence of the canonical factorization of a given function have been derived. In particular the correspondence between the canonical factors and the solutions of certain equations in $H^{2\pm}(R)$ spaces is established. It is shown that these equations can be solved using the reduction method for operator equations in Hilbert space. An orthonormal basis in $H^2(R)$ space is chosen in such a way that the reduced operator matrix turns out to be a Toeplitz matrix, with the advantages of simple structure and the availability of fast algorithms for its inversion and factorization. We provide also an error estimate and an expression for the speed by which the approximation error decays to zero in terms of some smoothness conditions on the canonical factors. Finally the method is illustrated by a numerical example.

II. THE MAIN RESULT

For illustration, consider the standard finite dimensional infinite time linear regulator problem

$$\dot{x} = A x + B U \qquad (2.1)$$

$$y = C x$$

with the cost function

$$J = \int_0^\infty (U^2 + y^2)\, dt \qquad (2.2)$$

Assume that [A, B, C] is a minimal realization of the transfer function $F(j\omega) = C(SI - A)^{-1} B$, $\mathrm{Re}(\lambda_j(A)) < 0 \quad j = 1, 2, \ldots, \dim X$. Then by standard results [28] the optimal control is given by

$$U(t) = -B^T P x(t) \qquad (2.3)$$

where K is the unique positive definite solution of the algebraic Riccati equation

$$A^T P + PA - PB B^T P + C^T C = 0 \qquad (2.4)$$

Davis [2] has shown that this optimal feedback gain can be found, without solving (4) , using the integral representation

$$PB = \frac{1}{2\pi} \int_{-\infty}^{\infty} (-j\omega I - A)^{-T} C^T C (j\omega I - A)^{-1} B [1 + G(j\omega)]\, d\omega \qquad (2.5)$$

where $(I + G)$ is the minimal phase function satisfying the spectral factorization

$$(1 + a)^{-1} = [I + F^* (j\omega) F (j\omega)]^{-1} = [1 + G (j\omega)][1 + G (j\omega)]^* \tag{2.6}$$

The above integral formula is valid as well for a variety of distributed parameter LQR problems [2] , and their dual distributed filtering problems [1] .

Clearly the main difficulty of the above approach is the factorization (2.6). An efficient factorization method would greatly increase the applicability of the method. It is expected that the computations could be performed for systems of very large dimension, provided A is sparse.

In this paper we address the factorization (2.6), as well as a generalized version of it, in the H^p spaces (theorems 1, 2, 3 and 4), the spaces of the frequency response functions. In particular, we show that the factor G satisfies certain equations in H^2 space (Theorem 2 and 3). The reduction method in Hilbert spaces (Section IV) is applied to generate a sequence $\{\hat{G}_n\}$ of approximating functions (Theorems 6 and 7).

$$\hat{G}_n = \sum_{k=0}^{n} g_{k,n} \phi_k^+ ,$$

where $\{\phi_k^{\pm}\}_{k=0}^{\infty}$ is the Laguerre orthonormal set in $H^{2\pm}$ space.

It is shown that the coefficients $\{g_{k,n}\}_{k=0}^{n}$ can be obtained by solving a Toeplitz set of linear equations (Theorem 7 and its corollary). Levinson's algorithm [24] is applied to generate recursively the sequence of the approximating solutions as shown below. It is proved that, if $G(s)$ is analytic in the closed right half plane, the method converges exponentially (Theorem 8 and its corollaries).

The Algorithm

The factor G of the factorization (2.6), where $(1 + a(j\omega))$ is real and $\text{ess inf } (1 + a) > 0$, may be obtained as follows:

Step 1

Find the coefficients $\{\alpha_k\}_{k=0}^{N}$ of the Laguerre expansion

$$a(j\omega) = \sum_{k=0}^{N} \alpha_k (\phi_k^+ + \phi_k^-)$$

Using th formula

$$\alpha_k = \langle a(j\omega), \phi_k \rangle = \int_{-\infty}^{\infty} \frac{a(j\omega)}{\sqrt{\pi}} \frac{-1}{j\omega-1} \left(\frac{j\omega+1}{j\omega-1}\right)^k d\omega$$

Step 2

Construct the Toeplitz matrix, T_n

$$T_n = \{b_{k-j}\}_{k,j=0}^{n},$$

$$b_0 = 1 + \frac{\alpha_0}{\sqrt{\pi}}$$

$$b_k = \frac{1}{2\sqrt{\pi}} (\alpha_k - \alpha_{k-1})$$

$$b_k = b_{-k}$$

Step 3

Generate the sequence of approximate solutions $\{\hat{G}_n\}$ using Levinson's recursive algorithm for solving a Toeplitz set of linear equations.

Let

$$\alpha_k = - \alpha_k \tag{2.7}$$

$$\lambda_{00} = b_1 / b_0$$

$$g_{00} = \alpha_0 / b_0$$

$$g_{11} \; (b_0 - \lambda_{00} \, b_1) = \alpha_1 - g_{00} \, b_1$$

$$g_{01} = g_{00} - \lambda_{00} \, g_{11}$$

DO 1 $m = 1 , M_{max}$

$$\lambda_{0m} \; (b_0 - \sum_{k=0}^{m-1} \lambda_{k,m-1} \, b_{m-k}) = b_{m+1} - \sum_{k=1}^{m} \lambda_{k-1,m-1} \, b_k$$

$$\lambda_{k,m} = \lambda_{k-1,m-1} - \lambda_{0m} \, \lambda_{m-k,m-1} \; ; \; k=1, 2, \ldots , m$$

$$g_{m+1,m+1} \; (b_0 - \sum_{k=0}^{m} \lambda_{k,m} \, b_{m+1-k}) = \alpha_{m+1} - \sum_{k=0}^{m} g_{k,m} \, b_{m+1-k}$$

$$g_{k,m+1} = g_{k,m} - \lambda_{k,m} \, g_{m+1,m+1} \; ; \; k=0, \ldots , m$$

IF

$$\sum_{k=0}^{m} (g_{k,m+1} - g_{k,m})^2 + g^2_{m+1,m+1} \leq \epsilon , \quad \text{GO TO } 2$$

1 – Continue

2 – STOP

Step 4

Evaluate the approximate optimal gain using the formula

$$PB \cong \frac{1}{2\pi j} \oint_{\Gamma} (SI + A^T)^{-1} C^T C (SI - A)^{-1} B$$

$$[I + \sum_{k=0}^{m+1} \frac{g_{k,m+1}}{\sqrt{\pi}} \frac{1}{s+1} (\frac{s-1}{s+1})^k] \, ds \qquad (2.8)$$

Γ is a rectifiable contour in the R.H.P. enclosing $\sigma(-A_o^T)$. Practically Steps (1) and (2) are inserted in the algorithm (Step 3) so that the coefficients α_k and b_k are computed only whenever needed in the recursion.

III. BACKGROUND [11] , [12]

Let $1 \leq p < \infty$ $(p = \infty)$ and let $L^p(R)$ denote the set of all complex valued Lebesque measurable functions $a(x)$ defined on the real line R such that

$$\int_R |a(x)|^p \, dx < \infty \qquad (\text{ess sup } |a(x)| < \infty)$$

The set $L^p(R)$ are Banach spaces under the norm

$$\|a(x)\|_p = \left[\int_R |a(x)|^p \, dx\right]^{\frac{1}{p}} \qquad (\|a\|_\infty = \text{ess sup } |a(x)|)$$

The Hardy space $H^{p+}(R)$ $(H^{p-}(R))$ is defined to be the class of analytic functions in the open right half plane C^+ (in the open left half plane C^-), such that

$$\sup_{\substack{\sigma > 0 \\ (\sigma < 0)}} \int_R |a(\sigma + j\omega)|^p \, d\omega < \infty$$

The space $H^{\infty+}(R)$ $(H^{\infty-}(R))$ is the class of analytic functions in C^+ (C^-) such that

$$\operatorname*{ess\,sup}_{\substack{\sigma > 0 \\ (\sigma < 0)}} |a(\sigma + j\omega)| < \infty$$

It can be shown that $H^{p\pm}$ spaces $p \geq 1$ are Banach spaces under the norm

$$\|a(x)\|_{H^{p\pm}} = \sup_{\substack{\sigma > 0 \\ (\sigma < 0)}} \left\{ \int_R | a(\sigma + j\omega)|^p \, d\omega \right\}^{\frac{1}{p}}$$

$$\left(\|a(x)\|_{H^{\infty\pm}} = \operatorname{ess} \sup_{\substack{\sigma > 0 \\ (\sigma < 0)}} | a(\sigma + j\omega)| \right)$$

If $f(s) \in H^p$, then it has non-tangential limits at almost every point of the imaginary axis and its boundary value function $\hat{f}(j\omega)$ is an element of $L^p(R)$. Moreover $f(s)$ can always be extracted from its boundary value via Poisson formula,

$$f(x + jy) = \frac{1}{\pi} \int_{-\infty}^{\infty} \hat{f}(j\omega) \frac{x}{x^2 + (y-\omega)^2} d\omega \qquad x > 0$$

As the association $f(s) \leftrightarrow \hat{f}(j\omega)$ is one-to-one and length preserving, from now on we shall not differentiate between a function and its boundary value. We shall admit the use of expressions such as, say "$f(j\omega)$ is an element of $H^p(R)$ space", whenever we really mean $f(j\omega)$ is the boundary value of some $f(s) \in H^p$ or $f(j\omega)$ has an analytic continuation in the space $H^p(R)$.

For $p = 2$, $H^{2\pm}(R)$ are Hilbert spaces with the inner product

$$\langle f_1(s), f_2(s) \rangle_{H^{2\pm}} = \langle f_1(j\omega), f_1(j\omega) \rangle_{L^2(R)}$$

A function $f(s) \in H^{2+}(R)$ $(H^{2-}(R))$ iff its boundary value function is the Fourier transform F.T. of some $L^2(R)$ function vanishing on the negative (positive) axis. Accordingly, functions of class F.T. $\{L^2[0,\infty)\}$ will be called Hardy functions of class $H^{2+}(R)$. In a similar manner $H^{2-}(R)$ is identified with the F.T. $\{L^2(-\infty, 0]\}$. Furthermore, $L^2(R)$ is exactly the direct sum $H^2 \oplus^{\perp} H^{2-}$, i.e., every function $f \in L^2(R)$ is uniquely expressible as

$$f = f^+ + f^- \qquad \text{where } f^{\pm} \in H^{2\pm}(R)$$

$$= p(f) + Q(f)$$

where p and Q are the projection operators taking $L^2(R)$ onto $H^{2\pm}(R)$ respectively.

In particular p may be given by

$$p(f) = f^+(s) = \frac{1}{2\pi j} \int_{-\infty}^{\infty} \frac{f(jx)}{x - s}\, dx$$

$$s = \sigma + j\omega, \quad \sigma > 0$$

- The following is an orthonormal basis in $L^2(R)$

Let

$$\phi_n(j\omega) = \frac{1}{\sqrt{\pi}} \frac{1}{j\omega + 1} \left(\frac{j\omega - 1}{j\omega + 1}\right)^n \qquad n = \ldots 1, 0, 1, 2 \ldots$$

The set $\{\phi_n\}_{n \geq 0}$ spans $H^2(R)$, while the set $\{\phi_n\}_{n < 0}$ spans $H^{2-}(R)$.

- A function $a(j\omega) \in L^{\infty}(R)$ is in $H^{\infty+}(R)$ iff

$$a(s)\, f(s) \in H^{2+}(R) \quad \text{for every } f(s) \in H^{2+}(R)$$

- Let T be the unit circle in the complex plane and let F be any function on T and if f is defined on R by $f(t) = F(e^{it})$ then f is a periodic function of period 2π. The space $L^p(T)$ for $1 \le p < \infty$ $(p = \infty)$ is the class of all complex measurable 2π periodic functions on R equipped with the norm,

$$\|f\|_p = \left\{\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^p \, d\theta\right\}^{\frac{1}{p}}$$

For $p = \infty$

$$\|f\|_\infty = \operatorname*{ess\,sup}_{\theta \in [0,\, 2\pi)} |f(\theta)|$$

The $L^p(T)$ are also Banach spaces.

In particular $L^2(T)$ is a Hilbert space in the inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, \bar{g}(\theta)\, d\theta \qquad f, g \in L^2(T)$$

- The Hardy space $H^{p+}(T)$ $(H^{p-}(T))$ $1 \le p < \infty$ is defined to consist of all analytic functions inside the unit circle (outside the unit circle) for which

$$\|f\|_{H^{p\pm}} = \sup_{\substack{0 \le r < 1 \\ (r > 1)}} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} | f(r e^{j\theta}) |^p \, d\theta \right\}^{\frac{1}{p}} < \infty \qquad 1 \le p < \infty$$

For $p = \infty$

$$\|f\|_{H^{\infty\pm}(T)} = \operatorname*{ess\,sup}_{\substack{\theta \\ 1 > r \ge 0 \\ (r > 1)}} | f(r e^{j\theta}) |$$

- $H^\infty(T) \subset H^p(T) \subset H^s(T) \subset H^1(T) \qquad 1 \le s \le p \le \infty$.
- $H^p(T)$ are also Banach spaces and $H^{2\pm}(T)$ are Hilbert spaces.
- In particular for $p = 1, 2, \infty$ the Hardy spaces $H^p(T)$ can be defined as

$$H^{p\pm}(T) = \left\{ f \in L^p(T) : \int_{-\pi}^{\pi} f(e^{i\theta}) \, e^{in\theta} \, d\theta = 0 \qquad n = 1, 2, \ldots \right\}.$$

- H^p spaces are closed subspaces of the corresponding $L^p(T)$.
- Define the operator E on $L^2(T)$ as follows

$$E(f) = E\{ \sum_{n=-\infty}^{\infty} f_n z^n \} = \sum_{n=0}^{\infty} f_n z^n , \; z = r e^{j\theta}$$

$E(\cdot)$ is a projection operator from $L^p(T)$ onto $H^p(T)$.

IV. FACTORIZATION OF FUNCTIONS IN $H^p(R)$ SPACES

The relation between spectral factorization and the Wiener-Hopf equation is as old as the Wiener-Hopf equation itself. In fact, the solution of W - H equation is indispensably carried in the frequency domain, via the spectral factorization.

It is then imperative to investigate the relation between the generalized frequency domain image of the W - H equation, which is known as the Toeplitz equations, and the spectral factorization formulated directly in the frequency domain spaces, i.e., $H^p(R)$ spaces. This section is dedicated to this purpose. The approach here is inspired by the work of Gohberg and Budjanu [13], [14], on the factorization in abstract Banach algebra. Nevertheless, the results here and the tools are different from [13] and [14]. As a matter of fact the formulation here is basically in the subalgebras $H^{2\pm} \cap H^{\infty\pm}$ which are, in principle, non-Banach. The machinery and some of the results on the Toeplitz operators are due to Davinatz [15], [16] and [17]; and Douglas [11]. The heart of our results in this section is Theorem 2 which relates the canonical factors, defined below, to the solution of certain Toeplitz operators in $H^{2\pm}(R)$. Theorem 3 considers in detail the special case of factorizing functions which are positive almost everywhere, i.e., their essential infimum is greater than zero (a slightly weaker condition than positiveness). The complete characterization of the class of functions which admit a canonical factorization is brought up in Theorem 4. Finally, the standard Krein-type factorization [10] is treated as a special case in Theorem 5.

Definition

An element $(1 + a(j\omega))$, $a(j\omega) \in L^2(R) \cap L^\infty(R)$, is said to have a canonical factorization if it admits representation in the form

$$1 + a(j\omega) = [1 + h^-(j\omega)][1 + h^+(j\omega)] \qquad \text{a.e.w.} \tag{4.1}$$

where $h^\pm(s) \in H^{2\pm}(R) \cap H^{\infty\pm}(R)$, $[1 + h^\pm(s)]$ is invertible for $s \in C^\pm$, and

$$G^\pm(s) \doteq [1 + h^\pm(s)]^{-1} \in H^{2\pm}(R) \cap H^{\infty\pm}(R)$$

Clearly the definition is quite similar to the Standard Krein - factorization. However, here $a(j\omega)$ need not be the Fourier Transform of an $L^1(R)$ function. Moreover, we do not assume any continuity condition on $a(j\omega)$. The uniqueness of the above canonical factorization is established by the following Theorem:

*** Theorem 1

If an element $(1 + a(j\omega))$ admits the canonical factorization (4.1), the factors $G^\pm(j\omega)$ are uniquely defined (a.e.w).

Proof:

Let $G_1^\pm$ be another factor, then it follows from (4.1) and the equality

$$[1 + G^+][1 + G] = [1 + G_1^+][1 + G_1^-]$$

That

$$[1 + G_1^+][1 + h^+] = [1 + G_1^-][1 + h^-]$$

which implies

$$G_1^+ + h^+ + G_1^+ h^+ = G_1^- + h^- + G_1^- h^- \qquad (4.2)$$

Since the subalgebras $H^{2+} \cap H^{\infty+}$ and $H^{2-} \cap H^{\infty-}$ intersect only at the zero element, both sides of (4.2) must be zero.

Thus

$$G_1^+ + h^+ + G_1^+ h^+ = 0$$

$$[1 + G_1^+] = [1 + h^+]^{-1} \doteq [1 + G^+]$$

i.e.,

$$G_1^+ = G^+ \quad .$$

Similarly, it can be shown that $G_1^- = G^-$ ... Q.E.D.

Before proceeding to derive the necessary and sufficient conditions for the existence of the canonical factorization we first state the following elementary lemmas which will be needed in the subsequent proofs.

<u>Lemma 1</u>

If $f(s) \in H^{2+}(R)$ and $f(j\omega) \in L^{\infty}(R)$, then $f(s) \in H^{2+}(R) \cap H^{\infty+}(R)$.

Lemma 2

If $h(s) \in H^{2+}(R) \cap H^{\infty+}(R)$ and $[1 + h]$ has an inverse in $H^{\infty+}(R)$ then there exists a unique function $g(s) \in H^{2+}(R) \cap H^{\infty+}(R)$ such that $[1 + g] = [1 + h]^{-1}$.

Lemma 3

If $x(s) \in H^{2+}(R)$, then $x\left(\frac{1+\xi}{1-\xi}\right) \in H^{2+}(T)$.

The proof of these lemmas follow directly from the properties of H^p spaces; for convenience and completeness they are given in the Appendix A.

We now come to the main Theorem.

*** Theorem 2

For an element $a(j\omega) \in L^2(R) \cap L^\infty(R)$ to admit the canonical factorization (3), it is necessary and sufficient that the two equations

$$G^+ + p\,[a\,G^+] = -p\,[a] \tag{4.3}$$

and

$$G^- + Q\,[a\,G^-] = -Q\,[a] \tag{4.4}$$

have esentially bounded solutions in $H^{2\pm}(R)$ respectively.

<u>Proof:</u>

The sufficiency part;

Suppose that (4.3), (4.4) have solutions G^+ and G^- respectively in $H^{2\pm}(R)$ which are essentially bounded. Then by Lemma 1 $G^{\pm} \in H^{2\pm} \cap H^{\infty\pm}$ equation (4.3) can be written as

$$(1 + G^+) + p\ [a\ (1 + G^+)] = 1$$

which implies that

$$(1 + a)(1 + G^+) = 1 + Y^- \tag{4.5}$$

for some $Y^- \in H^{2-}(R)$, but the left hand side of (4.5) is in $L^\infty(R)$ so Y^- must be $\in H^{2-}(R) \cap H^{\infty-}(R)$ by Lemma 1 .

Similarly

$$(1 + G^-)(1 + a) = 1 + Y^+ \tag{4.6}$$

for some $Y^+ \in H^{2+}(R) \cap H^{\infty+}(R)$.

Multiplying (4.5) by $(1 + G^-)$ and (4.6) by $(1 + G^+)$, we get

$$(1 + G^-)(1 + a)(1 + G^+) = (1 + G^-)(1 + Y^-) = (1 + Y^+)(1 + G^+) \tag{4.7}$$

The second equality implies that

$$G^- + Y^- + G^- Y^- = Y^+ + G^+ + Y^+ G^+ \tag{4.8}$$

but the two algebras $H^{2\pm}(R) \cap H^{\infty\pm}(R)$ intersect only at the zero element. Then it follows from (4.8) that

$$(1 + G^-)(1 + Y^-) = 1 \quad \text{i.e.,} \quad [1 + Y^-] = [1 + G^-]^{-1} \tag{4.10}$$

Similarly

$$[1 + Y^+] = [1 + G^+]^{-1} \tag{4.11}$$

Substituting (4.10) and (4.11) back in (4.8), we get

$$(1 + a) = [1 + Y^-][1 + Y^+] \tag{4.12}$$

together with (4.10) and (4.11) imply that (1 + a) admits canonical factorization.

Necessity part:-

If (1 + a) admits the canonical factorization we have

$$(1 + G^+)(1 + a) = (1 + h^-) \tag{4.13}$$

Taking the projection P of equation (4.13), we come up with

$$G^+ + P\,[a\,G^+] = -\,p\,[a] \tag{4.14}$$

i.e., G^+ satisfies equation (4.3) and is an element of $H^{2+}(R) \cap H^{\infty+}(R)$. Using a similar argument one can show also that G^- satisfies equation (4.4) and the proof is complete. Q.E.D.

** <u>Lemma 4</u>

If an element (1 + a) , where $a \in L^2(R) \cap L^\infty(R)$ admits the canonical factorization (4.1) , then the operators T and Γ defined by

$$T_{(1+a)}(x^+) \triangleq x^+ + P\,[a\,x^+] \qquad x^+ \in H^{2+}(R)$$

and

$$\Gamma_{(1+a)}(x^-) \triangleq x^- + Q\,[a\,x^-] \qquad x^- \in H^{2-}(R)$$

are invertible in $H^{2\pm}(R)$ respectively.

Proof:

Consider the equation

$$y^+ = x + P\,[a\,x] \tag{4.15}$$

we shall prove that (4.15) has exactly one solution $x^+ \in H^{2+}(R)$ for every $y^+ \in H^{2+}(R)$.

Let

$$x_0^+ = (1 + G^+)\,P\,[(1 + G^-)\,y^+] \tag{4.16}$$

Then by direct substitution one can verify easily that x_0^+ is indeed a solution of (4.15). Now suppose that x_1^+ and x_2^+ are two solutions of (4.15), then we must have,

$$(x_1^+ - x_2^+) + P\,[a\,(x_1^+ - x_2^+)] = 0$$

or

$$(1 + a)(x_1^+ - x_2^+) = y^- \qquad \text{for some} \quad y^- \in H^{2-}(R)$$

since $(1 + a)$ has a canonical factorization (4.1) then

$$(1 + h^+)(x_1 - x_2) = (1 + G^-)\ y^- \tag{4.17}$$

but the L.H.S. is in $H^{2+}(R)$ and the R.H.S. is in $H^{2-}(R)$, so we must have both sides equal zero. Thus

$$(1 + h^+)(x_1^+ - x_2^+) = 0$$

but $(1 + h^+)$ is not identically zero which implies $x_1^+ = x_2^+$, i.e., $T_{(1+a)}(\cdot)$ is invertible.

Applying the same argument to the equation

$$y^- = x^- + Q\ [a\ x^-] \tag{4.18}$$

with

$$x_0^- = Q\ [y^-\ (1 + G^+)]\ (1 + G^-)$$

one can verify that x_0^- is indeed a solution of (4.18) and it is unique which implies that $\Gamma_{(1+a)}$ is also invertible. Q.E.D.

Theorem 2 is quite general. In fact it is the corner stone of the subsequent study, from which special cases will be studied and other equivalent necessary and sufficient conditions will be derived. Lemma 4 reveals the relationship between the invertibility of the operators Γ and T and the canonical factorization. Unfortunately, as the invertibility of T and Γ is only a necessary condition for the canonical factorization, further study of the conditions of the invertibility of Γ and T proves to be not very helpful. This difficulty forces us to study separate special cases from which the complete characterization of the class of functions admitting the canonical factorization (4.1) is formulated. It turns out that the following special case is an indispensable factor of any function admitting the canonical factorization (4.1).

*** <u>Theorem 3</u>

For an element $(1 + a)$, $a \in L^2(R) \cap L^\infty(R)$ to admit the canonical factorization

$$1 + a(j\omega) = [1 + h^+(j\omega)]^* [1 + h^+(j\omega)] \quad \text{a.e.w} \tag{4.19}$$

where

$$h^+(j\omega) \in H^{2+}(R) \cap H^{\infty+}(R) ,$$

$$(1 + h^+(s)] \text{ is invertible for } s = \sigma + j\omega \quad \sigma > 0 ,$$

and

$$G^+(s) \doteq [1 + h^+(s)]^{-1} - 1 \in H^{2+}(R) \cap H^{\infty+}(R) .$$

It is Necessary and Sufficient that $(1 + a)$ be real and

$$\text{ess inf } |(1 + a(j\omega))| > 0 .$$

<u>Proof:</u>

The proof of this theorem is inspired by the following result which is due to A. Davinatz [15] and [16] .

Lemma 5 (Davinatz)

Let ψ be a real function $\in L^{\infty}(R)$, and define the operator T_{ψ} on $H^{2+}(R)$ by

$$T_{\psi}(x) = P[\psi x] , \qquad x \in H^{2+}(R)$$

Then the necessary and sufficient condition for T_{ψ} to be invertible operator of $H^{2+}(R)$ onto itself is

$$\text{ess inf} \; | \psi | > 0 .$$

We now come back to the proof of Theorem 3 .

The Necessity Part

From (4.19) it is evident that $(1 + a)$ must be real. As the Facotrization (4.19) is just a special case of the Factorization (4.1), Theorem 2 and Lemma 4 are also applicable to the factorization (4.19). In particular, Lemma 4 is valid, i.e., the operator $T_{(1+a)}(\cdot)$ on $H^{2+}(R)$ is invertible. Then by the Davinatz lemma we conclude that ess inf $| (1 + a) | > 0$.

The Sufficiency Part

Suppose now that $(1 + a)$ is real and ess inf $(1 + a) > 0$. By Theorem 2 , for $(1 + a)$ to admit the canonical factorization (4.19) it is sufficient to show that there exists an element $G^{+} \in H^{2+}(R)$ such that G^{+} and $(G^{+})^{*}$ satisfy respectively the equations

$$x^{+} + P\,[a\,x^{+}] = -P\,[a] \tag{4.20}$$

$$x^{-} + Q\,[a\,x^{-}] = -Q\,[a] \tag{4.21}$$

and G^{+} is essentially bounded.

By the Dawkins Lemma the equation (4.20) has a unique solution in $H^{2+}(R)$, call it G^{+}, i.e.,

$$G^{+} + P\,[a\,G^{+}] = -P\,[a] \tag{4.22}$$

$$(1 + a)(1 + G^{+}) = 1 + Y^{*} \quad \text{for some } y \in H^{2+}(R) \tag{4.23}$$

Taking the complex conjugate of (4.23), we get,

$$(1 + G^{+})^{*}\,(1 + a) = 1 + Y \tag{4.24}$$

Applying now the projector Q to both sides of (4.24)

$$(G^{+})^{*} + Q\,[a\,(G^{+})^{*}] = -Q\,[a] \tag{4.25}$$

i.e., $(G^{+})^{*}$ satisfies equation (4.21). So what is left is to show that G^{+} is $\in L^{\infty}(R)$. To do so multiply equation (4.23) by $(1 + G^{+})^{*}$ and equation (4.2 4) by $(1 + G^{+})$ to obtain

$$\begin{aligned}[1 + G^{+}]^{*}\,[1 + a]\,[1 + G^{+}] &= [1 + G^{+}]^{*}[1 + Y]^{*} \\ &= [1 + Y]\,[1 + G^{+}] \\ &= f\,(j\omega)\end{aligned} \tag{4.26}$$

We shall prove first that $f(j\omega)$ must be a constant a.e.w.

Consider first

$$f(j\omega) = [1 + Y][1 + G^+] \qquad (4.27)$$

substituting

$$j\omega = 1 + e^{j\theta} / 1 - e^{j\theta} , \quad \text{and}$$

invoking Lemma 3, we conclude that $[1 + Y(\frac{1 + e^{j\theta}}{1 - e^{j\theta}})] \in H^{2+}(T)$ and $[1 + G^+(\frac{1 + e^{j\theta}}{1 - e^{j\theta}})] \in H^{2+}(T)$. Thus $f(\frac{1 - e^{j\theta}}{1 - e^{j\theta}}) \in H^{1+}(T)$.

Applying a similar argument to the second equality of (4.26), we see that $f(\frac{1 + e^{j\theta}}{1 - e^{j\theta}}) \in H^{1-}(T)$.

But the two subspaces $H^{1+}(T)$ and $H^{1-}(T)$ intersect on the constant elements – i.e., f must be constant. We now use the L.H.S. of (29)

$$|(1 + G^+)^* [1 + a][1 + G^+]| = |c|$$

$$\rightarrow \quad \text{ess inf } |(1 + a)| \; |(1 + G^+)|^2 < |c| \qquad \text{a.e.w.}$$

i.e., $(1 + G^+)$ is bounded a.e.w. $\rightarrow$ $G^+ \in L^\infty(R)$.

The result follows. Q.E.D.

The next theorem characterizes completely the class of functions admitting the canonical factorization (4.1). To the best of our knowledge we believe that the preceding formulation of the factorization problem and the following characterization are new.

*** **Theorem 4**

For an element $(1 + a)$, $a \in L^2(R) \cap L^\infty(R)$, to admit the canonical factorization (4.1), it is necessary and sufficient that $(1 + a)$ has the representation

$$(1 + a) = (1 + a_1)(1 + a_2) \tag{4.28}$$

where

(i) $(1 + a_1)$ is real, $a_1 \in L^2(R) \cap L^\infty(R)$ and ess inf $|1 + a_1| > 0$.

(ii) $(1 + a_2)^{\pm 1} \in H^{\infty +}(R)$ and $a_2 \in H^{2+}(R) \cap H^{\infty +}(R)$.

Proof

The Necessity Part

Suppose that $(1 + a)$ admits the canonical factorization (4.1), then

$$\begin{aligned} (1 + a) &= (1 + h^-)(1 + h^+) \\ &= \underbrace{(1 + h^-)(1 + h^-)^*}_{A}\underbrace{(1 + G^-)^*(1 + h^+)}_{B} \end{aligned}$$

The second term B can be written as

$$\begin{aligned} B &= 1 + h^+ + (G^-)^* + (G^-)^* h^+ \\ &= 1 + a_2 \end{aligned}$$

Clearly $a_2 \in H^{2+}(R) \cap H^{\infty+}(R)$. Since $(1 + h^+)$ and $(1 + G^-)^*$ both invertible in $H^{\infty+}(R)$ so is $(1 + a_2)$.

Let $(1 + a_1) = (1 + h^-)(1 + h^-)^* = (1 + (h^-)^*)^* (1 + (h^-)^*)$. Thus $(1 + a_1)$ admits the canonical factorization (4.19), then by Theorem 3 , we must have $(1 + a_1)$ real and ess inf $(1 + a_1) > 0$.

The Sufficiency Part

$(1 + a_2)$ admits the canonical factorization (4.1) , so

$$(1 + a) = (1 + h_1^-)(1 + h_1^+)(1 + a_2)$$

let

$$h^+ = h_1^+ + h_1^+ a_2 + a_2$$

$$\rightarrow \quad (1 + a) = (1 + h_1^-)(1 + h^+)$$

Clearly $h^+ \in H^{2+} \cap H^{\infty+}$ i.e., admits canonical factorization.

Q.E.D.

Theorem 4 not only provides a simple test for factorization admissibility but also reduces the factorization problem (4.1) to two easier factorizations, namely (4.28) and (4.19).

Theorem 5 is in fact a well known result (see e.g. [6]). However, it is re-investigated here as a special case of Theorem 2 with independent proof.

*** Theorem 5

Let $a(j\omega)$ be the F.T. of some $\hat{a}(t) \in L^1(R) \cap L^2(R)$. Then, $[1 + a(j\omega)]$ admits the canonical factorization (4.1) iff

(I) $1 + a(j\omega) \neq 0 \qquad \omega \in [-\infty, \infty]$

(II) Index $[1 + a(j\omega)] = -\frac{1}{2\pi} \int d \arg (1 + a) = 0$

Proof

We need the following lemma which is due to M.G. Krein [10].

Lemma 6 (Krein)

Let $\hat{a}(t) \in L^1(R)$, then the equation

$$x(t) + \int_0^\infty \hat{a}(t - \tau)\, x(\tau)\, d\tau = y(\tau) \tag{4.29}$$

has exactly one solution $x(t) \in L^p[0, \infty)$ $(1 \leq p < \infty)$ for every $y(t) \in L^p[0, \infty)$, if and only if the conditions I and II are fulfilled.

The proof of this famous result can be found in [10] and will not be repeated here.

The Necessity Part of Theorem 4

$(1 + a)$ admits the canonical factorization (4.1) then by Lemma 4 the equation

$$x^{+} + p\,[a\,x^{+}] = y^{+} \tag{4.30}$$

has a unique solution $x^{+} \in H^{2+}(R)$ for every $y^{+} \in H^{2}(R)$. Taking inverse Fourier Transform of (4.30) we come up with

$$\hat{x}^{+}(t) + \int_{0}^{\infty} \hat{a}(t-\tau)\, x^{+}(\tau)\, d\tau = \hat{y}^{+}(t) \tag{4.31}$$

The Wiener - Hopf equation (4.29) has a unique solution $\hat{x}(t)$ for every $\hat{y}(t) \in L^{2}_{-}[0,\infty)$. So by the Krein result the two conditions I and II must be fulfilled.

<u>The Sufficiency Part of Theorem 5</u>

Suppose now conditions I and II are fulfilled then by the Krein Theorem the equation

$$\hat{x}(t) + \int_{0}^{\infty} \hat{a}(t-\tau)\, \hat{x}(\tau)\, d\tau = \hat{a}^{+}(t) \tag{4.32}$$

has a unique solution $\hat{x}(t) \in L^{1}[0,\infty) \cap L^{2}[0,\infty)$, where $\hat{a}^{+}(t)$ is the causal part of $\hat{a}(t)$.

Taking the F.T. of (4.32), we see that $x(j\omega)$ satisfies the equation

$$x^{+} + P\,[a\,x^{+}] = P\,[a] \tag{4.33}$$

i.e., equation (4.3) has a solution $x^{+} \in H^{2+}(R)$.

But $x^{+}(j\omega)$ is also the F.T. of an $L^{1}[0,\infty)$ function so

$X^+ (j\omega)$ must be $\in H^{2+} (R) \cap W^+$

where W^+ is the algebra of the F.T. of functions in $L^1 [0, \infty)$ since $W^+ \subset H^{\infty+} (R)$ so finally we conclude that $x^+ \in H^{2+}(R) \cap H^{\infty+} (R)$. By a similar argument we can also construct an element $x^- \in H^{2-} (R) \cap H^{\infty-} (R)$ and satisfying equation (4.4). So by Theorem 2 (1 + a) admits the canonical factorization (4.1). The proof is complete.

V. THE REDUCTION METHOD FOR SOLVING THE EQUATION $y = A x$ IN HILBERT SPACE

As was mentioned earlier, our final objective is to develop an approximation method for generating the spectral factors $G^{\pm}$ of the canonical factorization (4.1). The idea here is to obtain those factors by solving the Toeplitz equations (4.3) and (4.4). One way of doing this is through the so-called Projection Method or the Reduction Method [18], [20].

Let $\mathcal{H}$ be an abstract separable Hilbert space, $U(\mathcal{H})$ be the group of all bounded linear operators on $\mathcal{H}$. Let $\{p_n\}$ be a chain of projections which converges strongly to the identity operator in $U(\mathcal{H})$, i.e.,

$$\lim_{n \to \infty} \| p_n x - x \| \to 0 \qquad \forall \ x \in \mathcal{H} .$$

let say, $p_n [x] = \sum_{k=1}^{n} \phi_k < x , \phi_k >$, for some basis $\{\phi_k\}$ in $\mathcal{H}$,

we call the approximation method of solving the equation

$$A x = y \qquad A \in U(\mathcal{H}) , \qquad y \in \mathcal{H} \tag{5.1}$$

which consists of finding a solution $\hat{x}_n \in p_n \mathcal{H}$ of the equation

$$p_n A p_n x = p_n y \tag{5.2}$$

The Reduction Method

We say that the reduction method relative to $\{p_n\}$ is applicable to the Operator A if beginning with some n_0, the equation (5.2) has a unique solution for any $y \in \mathcal{H}$, and as $n \to \infty$ the solutions $\hat{x}_n$ tend to the solution of equation (5.1). In other words, the reduction method is applicable to A if

(1) A is invertible.

(2) Beginning with sum n_0 the operators $p_n A p_n$ as operators from $p_n \mathcal{H}$ into $p_n \mathcal{H}$ are invertible and the operators $(p_n A p_n)^{-1} p_n$ converge strongly to A^{-1} as $n \to \infty$.

Lemma 7 ([18], Theorem 2.1, pp. 58; see also [20])

For an invertible operator $A \in U(\mathcal{H})$ to admit reduction relative to the chain $\{p_n\}$ it is necessary and sufficient that

$$\|p_n A p_n x\| \geq c \|p_n x\| \qquad (n \geq n_0, \ c > 0) \qquad (5.3)$$

In the next theorem we investigate the possibility of applying this method to a certain class of operators in $H^{2\pm}$ spaces, namely the class of positive definite operators in H^2. The auxiliary lemmas 8 and 9 explain first what is meant by positivity.

*** Theorem 6

Suppose that $\psi \in L^{\infty}(R)$ is a real valued function with ess inf $|\psi| > 0$. Define

$$T_{\psi}(x) = p[\psi x] \quad , \quad x \in H^{2+}(R) \tag{5.4}$$

Then the operator T_{ψ} admits reduction relative to any basis in $H^{2+}(R)$.

Proof

The proof is based on the following two lemmas.

* Lemma 8

Let ψ be a real valued function $\in L^{\infty}(R)$, and define T_{ψ} as above. The spectrum of T_{ψ} is given by

$$\sigma(T_{\psi}) = [\text{ess inf } \psi \, , \ \text{ess sup } \psi] \tag{5.5}$$

The proof of this lemma is quite lengthy, so we preferred to defer it to Appendix B.

* Lemma 9

Let $\psi \in L^{\infty}(R)$ and define T_{ψ} as in (5.4). T_{ψ} is a positive definite operator in $H^{2+}(R)$ iff ψ is real and ess inf $(\psi) > 0$.

Proof of Lemma 9

The Sufficiency Part

ψ is real and ess inf $\psi > 0$

for $x , y \in H^{2+}(R)$ we have

$$< y , T_\psi x > = < y , \psi x > = < \psi y , x >$$
$$= < T_\psi y , x > = < T_\psi^* y , x >$$

i.e., $T_\psi = T_\psi^*$ i.e., T_ψ is self adjoint,

and by Lemma (8)

$$\sigma (T_\psi) = [\text{ess inf } \psi , \text{ ess sup } \psi]$$

but ess inf $\psi > 0 \rightarrow T_\psi$ is positive definite .

The Necessity Part

If T_ψ is positive $\rightarrow T_\psi = T_\psi^*$

Thus

$$< T_\psi x , x > = < x , T_\psi x >$$

$$\rightarrow \int_{-\infty}^{\infty} x \psi x^* d\omega = \int_{-\infty}^{\infty} x \overline{\psi} x^* dx$$

$$\rightarrow \int_{-\infty}^{\infty} x (\psi - \overline{\psi}) x^* dx = 0 \rightarrow \psi = \overline{\psi} \text{ , i.e., } \psi \text{ is real.}$$

Again by Lemma (8)

$$\sigma(T_\psi) = [\text{ess inf } \psi \ , \ \text{ess sup } \psi]$$

Since T_ψ is positive we must have ess inf $\psi > 0$.

Q.E.D.

We now come back to the proof of Theorem 6 .

By Lemma (9) T_ψ is a positive definite operator on $H^{2+}(R)$.

Let $\{p_n\}$ be the chain of projection generated by some basis $\{\phi_k\} \in H^{2+}(R)$. Then

$$< p_n T_\psi p_n x , p_n x > = < T_\psi p_n x , p_n x > \geq \in \| p_n x \|^2 \quad \in > 0$$

i.e, $p_n T_\psi p_n$ is also positive definite, hence satisfies the conditions of Lemma (7) , i.e., admits reduction relative to $\{\phi_k\}$. Since $\{\phi_k\}$ is arbitrary, T_ψ admits reduction relative to any basis in $H^{2+}(R)$, and the proof of Theorem 6 is complete.

VI. AN ERROR BOUND AND AN ESTIMATE FOR THE SPEED OF CONVERGENCE

In the previous sections we have introduced the reduction method and have investigated the possibility of applying this method to solve certain types of Toeplitz equations in H^{2+} (R) space. Now, we are going to apply these results to the equation

$$G^+ + p\,[a\,G^+] = -\,p\,[a] \tag{6.1}$$

to obtain an approximate solution of the spectral factor G^+, we would like also to know: (i) how far is the obtained solution, $\hat{G}_n$, from the actual factor G, and (ii) how fast this $\hat{G}_n$ converges to G^+ with n. Theorem 7 provides an answer to the first question, while Theorem 8 takes care of the second.

*** Theorem 7

Consider the equation

$$G^+ + p\,[a\,G^+] = -\,p\,[a] \tag{6.1}$$

where $a \in L^2(R) \cap L^\infty(R)$, $(1 + a)$ is real and ess inf $(1+a) > 0$. Let $\{\phi_k\}_{k=0}^{\infty}$ be some basis in H^{2+} (R), and define the projectors

$$P_n\,[x] = \sum_{k=0}^{n} \phi_k \,\langle x\,,\ \phi_k \rangle \qquad x \in H^{2+}(R) \tag{6.2}$$

Let $\hat{G}_n$ be the solution of the equation

$$\hat{G}_n + p_n [a \, p_n \hat{G}_n] = - p_n [a] \qquad (6.3)$$

Finally, let W be the isometric mapping which takes $p_n H^{2+}(R)$ into $\mathbb{C}^{n+1}$ in accordance with the formula

$$W \{ \sum_{j=0}^{n} + \alpha_j \phi_j \} = (\alpha_0 , \alpha_1 \ldots \alpha_n)^T \qquad (6.4)$$

Then,

(i) $\hat{G}_n \to G^+$ as $n \to \infty$.

(ii) $\| G - \hat{G}_n \|_2 < c \| (1 - p_n) G^+ \|_2$ for some $c > 0$.

(iii) Under the mapping W the reduced equation (6.3) is representable in $\mathbb{C}^{n+1}$ by the vector equation

$$T_n \underline{g} = \underline{\alpha}$$

where

$$\underline{\alpha} = W [- p_n [a]]$$

$$\underline{g} = W [\hat{G}_n]$$

$$T_n = I_{n+1,n+1} + \{ \langle a \, \phi_k , \phi_j \rangle \}_{k,j=0}^{n}$$ is a positive definite matrix.

Proof

(i) $(1 + a)$ satisfies the conditions of Theorem 6 , so it admits reduction relative to any basis in $H^{2+}(R)$. Then by the reduction admissibility the solutions of the reduced equation (6.3), $\hat{G}_n$, converges to G^+ the solution of (6.1) as $n \to \infty$, i.e., (i) is proved.

(ii) The error

$$E_n \doteq G^+ - \hat{G}_n$$

$$= p_n G^+ - \hat{G}_n + G - p_n G^+$$

$$= \tilde{e}_n + (I - p_n) G^+$$

$$\to \quad \|E_n\|_2 = \|\tilde{e}_n\|_2 + \|(1 - p_n) G^+\|_2 \tag{6.5}$$

Since the two components are orthogonal.

From (6.3)

$$\hat{G}_n + p_n [a p_n \hat{G}_n] = - p_n [a] \pm p_n [(1 + a) G^+]$$

$$\to \quad p_n [(1 + a) p_n [\hat{G}_n - p_n G^+]] = p_n [a (I - p_n)(G^+)]$$

$$\to \quad \|\hat{G}_n - p_n G_n\|_2 \leq \|(p_n T_{(1+a)} p_n)^{-1} p_n\| \|p_n p [a (1 - p_n) G^+\|$$

$$\leq \| (p_n T_{(1+a)} p_n)^{-1} p_n \| \in (n) \|a\|_\infty \|(1 - p_n) G^+\| \tag{6.6}$$

where $0 < \in (n) \leq 1$ and $\in (n) \to 1$ as $n \to \infty$.

Since $p_n T_{(1+a)} p_n$ is invertible operator from $P_n H^{2+} (R)$ into $P_n H^{2+} (R)$ then $(p_n T_{(1+a)} p_n)^{-1} p_n$ are bounded.

Let

$$\sup_{n > 0} \| (p_n T_{(1+a)} p_n)^{-1} p_n \| = \gamma < \infty$$

Then (6.6) can now be written as

$$\| G_n - p_n G_n \|_2 \le \gamma \|a\|_\infty \| (I - p_n) G^+ \| \quad (6.7)$$

Substituting (6.7) into (6.5) we get finally

$$\| E_n \|_2 \le [1 + \gamma \|a\|_\infty] \| (I - p_n) G^+ \| \quad (6.8)$$

which implies (ii) .

(iii) Substituting

$$\hat{G}_n = \sum_{k=0}^{n} \hat{g}_k \phi_k ,$$

$$p_n [a] = \sum_{k=0}^{n} \alpha_k \phi_k ,$$

into the reduced equation (6.3) we get

$$\sum \hat{g}_k \phi_k + \sum_{k=0}^{n} \phi_k \sum_{\ell=0}^{n} \hat{g}_\ell < a \phi_\ell , \phi_k > = - \sum \alpha_k \phi_k$$

which clearly may be split into the following system of linear equations

$$\hat{y}_k + \sum_{\ell=0}^{n} \hat{g}_\ell < a \phi_\ell , \phi_k > = - \alpha_k , \quad k = 0, 1, \ldots n \quad (6.9)$$

or, in a matrix form,

$$T_n \underline{g} = \underline{\alpha}$$

where

$$T_n = I_{n+1,n+1} + \{\langle a\,\phi_\ell\,,\,\phi_k\rangle\}_{\ell,k=0}^{n} \tag{6.10}$$

Finally to show that T_n is positive definite, let $p_n x \neq 0$ and define $\underline{x} = W\{p_n x\}$. Then

$$\underline{x}^T T_n x = \sum_{\ell,k=0}^{n} x_\ell\,\bar{x}_k \langle a\,\phi_\ell\,,\,\phi_k\rangle + \sum_{\ell=0}^{n} x_\ell\,\bar{x}_k$$

$$= \langle p_n x\,,\,p_n T_{(1+a)}\,p_n x\rangle$$

$$= \langle p_n x\,,\,T_{(1+a)}\,p_n x\rangle \quad > 0 \quad ,$$

since $T_{(1+a)}$ is positive definite by Lemma 9 .

Thus T_n is also positive definite, and the proof of Theorem 7 is complete.

The analysis so far is valid for any basis in H^{2+} . A variety of these orthonormal bases were constructed some time ago by Lee and Wiener [19] . From now on we are going to consider only the Laguerre orthonormal basis defined in Section III.

The Laguerre basis is chosen for many reasons, perhaps the most important of which is that the reduced operator matrix (6.10) turns out to be a Toeplitz matrix with the advantage of simple structure and the availability of fast algorithms for its inversion and its U.L. factorization [21], [22] and [23] (of computational order n^2 compared to n^3 for a general matrix). A third advantage is that the entries of this Toeplitz matrix turn out to be the projection coefficients needed in the r.h.s. of equation (6.3), thus constructed at no extra computational cost. These results are established by the following corollary.

Corollary 7.1

Let G^+, $\hat{G}_n$ and T_n be as in Theorem 7. If $\{\phi_k\}$ is the Laguerre orthonormal basis, T_n is a Toeplitz matrix.

Proof

Since $a(j\omega)$ is $\in L^2(R)$, can be expanded as

$$a(j\omega) = \sum_{k=0}^{\infty} \alpha_k \frac{1}{\sqrt{\pi}\,(j\omega+1)} \left(\frac{j\omega-1}{j\omega+1}\right)^k + \alpha_k \left(\frac{-1/\sqrt{\pi}}{j\omega-1}\right) \left(\frac{j\omega+1}{j\omega-1}\right)^k \quad (6.11)$$

but

$$\frac{1}{j\omega+1} \left(\frac{j\omega-1}{j\omega+1}\right)^k = \frac{1}{2} \left[\left(\frac{j\omega-1}{j\omega+1}\right)^k - \left(\frac{j\omega-1}{j\omega+1}\right)^{k+1} \right] \quad (6.12)$$

$$\frac{1}{j\omega-1} \left(\frac{j\omega+1}{j\omega-1}\right)^k = \frac{1}{2} \left[\left(\frac{j\omega+1}{j\omega-1}\right)^{k+1} - \left(\frac{j\omega+1}{j\omega-1}\right)^k \right] \quad (6.13)$$

Upon substituting the relations (6.12) and (6.13) into (6.11), we get

$$a(j\omega) = \frac{\alpha_0}{\sqrt{\pi}} + \sum_{k=1}^{\infty} \frac{1}{2\sqrt{\pi}} (\alpha_k - \alpha_{k-1})(\psi_k^- + \psi_k^+] \quad (6.14)$$

where

where

$$\psi_k^{\pm} = \left(\frac{j\omega-1}{j\omega+1}\right)^{\pm k} .$$

We denote by

$$b_0 = \alpha_0 / \sqrt{\pi}$$

$$b_k = \frac{1}{2\sqrt{\pi}} (\alpha_k - \alpha_{k-1}) \tag{6.15}$$

$$b_k = b_{-k}$$

Consider again the reduced equation (6.3), and substitute a (jω) by the r.h.s. of (6.14), we have

$$\sum_{k=0}^{n} \hat{g}_k \phi_k + P_n \left[\sum_{m=-\infty}^{\infty} b_m \left(\frac{j\omega-1}{j\omega+1}\right)^m \sum_{\ell=0}^{n} \hat{g}_\ell \phi_\ell \right] = \sum_{k=0}^{n} - \alpha_k \phi_k \tag{6.16}$$

splitting (6.16) and using the relation

$$\left(\frac{j\omega-1}{j\omega+1}\right)^m \phi_\ell = \phi_{\ell+m}$$

we come up with the following set of equations

$$\hat{g}_k + \sum_{\ell=0}^{n} b_{k-\ell} \hat{g}_\ell = - \alpha_k \qquad k = 0, 1, 2 \ldots \tag{6.17}$$

Integrating (6.17) again into one vector equation, we get

$$T_n \hat{\underline{g}} = \underline{\alpha}$$

where T_n is given by

$$T_n = I_{n+1,n+1} + \{b_{k-\ell}\}_{k,\ell=0}^{n} \tag{6.18}$$

i.e., T_n is Toeplitz . Q.E.D.

Notice that the matrix T_n is constructed using the relations (6.15) at no extra cost than the computation of the coefficients $\{\alpha_k\}$ which are any way needed to construct the vector $\underline{\alpha}_o$.

We are going now to provide an estimate of the rate of decay of the error as a function of the approximation degree "n" .

Let $C^p(T)$ by the space of p times continuously differentiable functions on the unit circle, and define the mapping V as

$$V\ f(j\omega) = f\left(\frac{1 + e^{j\theta}}{1 - e^{j\theta}}\right) \tag{6.18}$$

Theorem 8

Let G^+ and $\hat{G}_n$ be as in Theorem 7 . However assume further that G^+ satisfies, in addition, the following smoothness property

(a) $F(e^{j\theta}) \doteq V\,[\,(1 + j\omega)\ G^+(j\omega)] \in C^p(T)$

for some $p \geq 1$

(b) $F^{(p)}$ satisfies a Lipschitz condition of order α

$0 < \alpha < 1$

Then

$$\|G^{+} - \hat{G}_n\|_2 < \frac{\text{Constant}}{n^{p+\alpha - \frac{1}{2}}} \tag{6.19}$$

Proof of Theorem 7

Since $G^{+} \in H^{2+}(R)$, its k'th Fourier coefficient w.r.t. the Laguerre orthonormal basis is given by

$$g_k = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} G(j\omega) \frac{-1}{j\omega-1} \left(\frac{j\omega+1}{j\omega-1}\right)^k d\omega$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} [G(j\omega)(1 + j\omega)] \left(\frac{j\omega+1}{j\omega-1}\right)^k \frac{d\omega}{\omega^2+1}$$

Substituting $\frac{j\omega+1}{j\omega-1} = e^{-j\theta}$

$$g_k = \frac{1}{2\sqrt{\pi}} \int_{-\pi}^{\pi} F(e^{j\theta})\, e^{-kj\theta}\, d\theta \tag{6.20}$$

Integrating the r.h.s. of (6.20) by parts

$$g_k = \frac{1}{2\sqrt{\pi}} \left(\frac{j}{k}\right) \{e^{jn\theta} F(e^{j\theta})\Big|_{-\pi}^{\pi} - \int_{-\pi}^{-\pi} e^{-kj\theta} F'(e^{j\theta})\, d\theta \tag{6.21}$$

Resuming integrating by parts up to p times, this gives

$$g_k = \frac{1}{2\sqrt{\pi}} \left(\frac{-j}{k}\right)^p \int_{-\pi}^{\pi} e^{-kj\theta} F^{(p)}(e^{j\theta})\, d\theta \tag{6.22}$$

Now, changing the variables $\theta = \beta + \frac{\pi}{k}$

$$g_k = \frac{1}{2\sqrt{\pi}} \left(\frac{-j}{k}\right)^p \int_{-\pi}^{\pi} e^{-j\pi} e^{-kj\beta} F \left(e^{j\beta} + \frac{\pi}{k} j\right) d\theta \tag{6.23}$$

adding (6.22) and (6.23), and dividing by 2

$$g_k = \frac{1}{2} \cdot \frac{1}{2\sqrt{\pi}} \left(\frac{-j}{k}\right)^p \int_{-\pi}^{\pi} [F^{(p)} (e^{j\theta}) - F^{(p)} (e^{j\theta} + \frac{\pi}{k} j)] e^{-jk\theta} d\theta \tag{6.24}$$

but $F^{(p)}(e^{j\theta})$ satisfies the Lipschitz condition

$$| F^{(p)}(e^{j\theta}) - F^{(p)}(e^{j\theta} + j\frac{\pi}{k}) | < C \left(\frac{\pi}{k}\right)^{\alpha} \qquad 0 < \alpha \leq 1$$

Substituting this condition back into (6.24)

$$|g_k| \leq \frac{1}{2} \frac{1}{2\sqrt{\pi}} \frac{1}{k^p} C \left(\frac{\pi}{k}\right)^{\alpha} 2\pi$$

$$|g_k| \leq \frac{\text{constant}}{k^{p+\alpha}} \tag{6.25}$$

We shall now use (6.25) to get an estimate of the partial sum

$$\Rightarrow \quad \| (1 - P_n) G^+ \|_2^2 = \sum_{k=n+1}^{\infty} |g_k|^2$$

$$\leq \text{constant} \sum_{m=n+1}^{\infty} \frac{1}{m^{2p+2\alpha}} < \int_{n}^{\infty} \frac{\text{constant}}{x^{2p+2\alpha}} dx$$

$$\rightarrow \quad \| (1 - P_n) G^+ \|_2^2 \leq \frac{\text{constant}}{n^{2p+2\alpha-1}} \tag{6.26}$$

Substituting this result into part (ii) of Theorem 7, we finally come up with

$$\|G - \hat{G}_n\| < \frac{\text{constant}}{n^{p + \alpha - \frac{1}{2}}}$$

Q.E.D.

* Corollary 1

If $G^+ (s)$ is analytic in an open right half plane including the $j\omega$ - axis, then there exist $c > 0$ and $0 < \alpha < 1$ such that

$$\|G - \hat{G}_n\|_2 < c\ \alpha^n$$

Proof of Corollary 1

If $G (s)$ is analytic in the closed right half plane, so is $(1 + s) G (s)$ and hence under the conformal mapping $\eta = \frac{s-1}{s+1}$, $F (\zeta)$ $F (\zeta) = V [(s+1) G^+ (s)]$ must be analytic inside the closed unit disk. This implies $F (e^{j\theta})$ is continuous and infinitely differential function or the unit circle. It follows then from (6.25) that

$$k^p |g_k| < C \tag{6.27}$$

where C is constant, for every p and every k , this implies that either all g_k's are identically zeros or g_k be of exponential order, i.e., $|g_k| = \beta \alpha^k$ for some $\beta > 0$ and $0 < \alpha < 1$.
Now the sum of the squares of g_k's turns out to be

$$\|G - \hat{G}_n\|_2^2 < \sum_{n+1} \beta^2 \alpha^{2k} < \beta^2 \int_n^\infty e^{2x \log \alpha} dx$$

$$< \frac{\beta^2}{2 |\log \alpha|} \alpha^{2n} < \text{constant}\ \alpha^{2n}$$

$$\rightarrow \qquad \|G - \hat{G}_n\|_2 \leq c\ \alpha^n$$

Q.E.D.

* Corollary 2

If G^{+} is a rational function with no poles on the imaginary axis. Then

$$\|G - \hat{G}_n\|_2 < c\,\alpha^n$$

Proof

This result is a direct consequence of Corollary 1, since in this case G is obviously an analytic function in the closed R.H.P.

Q.E.D.

VII. THE ALGORITHM AND THE COMPUTATIONAL ASPECTS

In the previous sections we applied the reduction method to obtain a sequence of approximating functions $\hat{G}_n$

$$\hat{G}_n = \sum_{k=0}^{n} g_{k,n} \phi_k \quad , \tag{7.1}$$

and showed that the coefficients $\{g_{k,n}\}$ satisfy a Toeplitz set of linear equations, namely the vector equation

$$T_n \underline{\hat{g}_n} = \underline{\alpha} \tag{7.2}$$

where

$$\underline{\alpha} = [-\alpha_0 , -\alpha_1 \dots ; -\alpha_n]^T ,$$

$$\alpha_k = \langle a(j\omega) , \phi_k \rangle = \int_{-\infty}^{\infty} a(j\omega) \frac{-1}{\sqrt{\pi}} \frac{1}{j\omega-1} \left(\frac{j\omega+1}{j\omega-1}\right)^k d\omega \tag{7.3}$$

T_n turns out to be a positive definite Toeplitz matrix $\{b_{k-j}\}_{k,j=0}^{n}$ where b_k are related to the coefficients α_k's by the simple relation

$$b_0 = 1 + \frac{\alpha_0}{\sqrt{\pi}}$$

$$b_k = \frac{1}{2\sqrt{\pi}} (\alpha_k - \alpha_{k-1}) \tag{7.4}$$

$$b_{-k} = b_k$$

In other words the problem of factorization is reduced to solving the Toeplitz set of linear equation (7.2). Fortunately, there are several fast algorithms for solving the Toeplitz equation (7.2) in general and when T_n is positive definite in particular. For example one may first find the Cholesky decomposition of T_n, using the fast algorithm by Morf [25] and Rissanen [22], then solve (22) backward and forward as usual, or can just invert T_n directly using Justice algorithm [21]. However here we chose to illustrate the method using the recursive formula of Levinson [24] which gives the approximating solution

$$\hat{g}_{n+1} = [g_{0,n+1}, \ldots g_{n+1,n+1}]$$

In terms of the old approximating solution

$$\hat{g}_n = [g_{0,n}, \ldots g_{n,n}]$$

Without resolving the system of equations (7.2) for n+1, and thus enables us to monitor the change of the approximating solutions as n increases or to use a simple error criterion like the one used at the end of Step 3 in the main algorithm given in Section 2.

A final point is that the Laguerre basis can be expressed in general in the form

$$\phi_k(p) = \frac{\sqrt{p}}{\sqrt{\pi}} \frac{1}{j\omega + p} \left(\frac{j\omega - p}{j\omega + p}\right)^k$$

where $p > 0$ is a scale factor.

The parameter p may be chosen to minimize the H^2 error in approximating a given function by a truncated Laguerre expansion, or to minimize the number of terms in the Laguerre expansion for a prescribed error. T.W. Parks [26] introduced a criterion for the choice of "p" which minimizes the maximum truncation error over a certain class of functions. Clowes [27] showed that in the case of rational functions, say $f(s)$, the optimal value of "p" is one of the positive roots of either

$$f_N(p) = 0 \qquad \text{or} \qquad f_{N+1}(p) = 0 \tag{7.5}$$

Clearly, the solution of (7.5) can be quite tedious because the coefficient $f_N(p)$ will be a polynomial of at least $(N+K)$th degree if $f(s)$ has simple K poles. Here we will not attempt a rigorous treatment for the choice of p. Nevertheless some reflections might be useful for the interested readers. Consider the case when $f(s)$ is an analytic function on the $j\omega$ axis. Invoking the argument of the proof of Theorem 8 and its corollaries, one can show that

$$|f_k(p)| < C\,\alpha^k(p) \quad , \quad C > 0 \quad , \quad 0 < \alpha < 1$$

Clearly to minimize the truncation error the rate of decay of $|f_K|$ has to be maintained maximal. In other words α should be minimal. A simple criterion to achieve this could be

$$\min_p \max_j \left| R_j(p) \left(\frac{p_j - p}{p_j + p}\right)^n \right|$$

where $R_j(p)$ is the residue of $f(s)$ at the pole $s = -p_j$, and n is given. The residues $R_j(p)$ represent the relative weightings of the poles $(-p_j)$. A more crude, but simple, estimate of p may be given by $p = \sqrt{|\lambda_{min}|\,|\lambda_{max}|}$, where λ_{min} and λ_{max} are respectively the poles of the minimum and the maximum absolute values in $\mathbb{C}^-$.

Example

Consider the following system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U$$

$$y = [\, 2\sqrt{21} \quad , \quad 2\sqrt{3}\,]$$

It is required to find the optimal feedback law which minimizes

$$J = \int_0^\infty (U^2 + y^2)\, dt \quad .$$

Using the standard methods [28] one can show systematically that the optimal gain is given by

$$U = -[K_1 \; K_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where

$$\underline{K} = [6 \quad 2] \quad .$$

Let us now apply the method proposed in this paper.

Step 1

$$1 + a\,(j\omega) = 1 + F^{*}(j\omega)\,F\,(j\omega)$$

where

$F\,(s)$ is the transfer function of the system

$$F\,(s) = \frac{2\sqrt{3}\;s + \sqrt{7}}{(s+1)(s+4)}$$

Compute the coefficients α_k through the formula

$$\alpha_k = \int_{-\alpha}^{\infty} \frac{(12\;\omega^2 + 7)}{(\omega^2+1)(\omega^2+16)}\;\frac{-1\sqrt{3}}{\sqrt{\pi}}\;\frac{1}{j\omega - 3}\;\left(\frac{j\omega + 3}{j\omega - 3}\right)^k d\omega$$

where the parameter p is taken to be 3 .

$$\alpha_k = 3.6839753\;(-1)^k\left(\frac{1}{2}\right)^k + .7894231\left(\frac{1}{7}\right)^k$$

Step 2

Construct the Toeplitz matrix $\{b_{k-j}\}$

Using

$$b_0 = \frac{\alpha_0}{\sqrt{\pi\,p}} + 1 \qquad b_k = \frac{1}{2\sqrt{\pi\,p}}\,(\alpha_k - \alpha_{k-1})$$

$$b_{-k} = b_k$$

$b_0 = 2.4571424$

$b_1 = -1.0102038$

$b_2 = .434256$

$\vdots$

Step 3

Generate the approximating sequence $\hat{g}_n$ using Levinson's algorithm

$g_{00} = -1.8205597$

$g_{11} = -.0538409$

$g_{01} = -1.8427052$

$g_{22} = -.0936968$

$g_{12} = -.0920051$

$g_{02} = -1.8418362$

Step 4

Substituting in the Davis' formula (2.8) , we come up with the following numerical results

$$\hat{G}_0 = g_{00} \sqrt{\frac{3}{\pi}} \frac{1}{j\omega + 3}$$

$$K = [6.0015226 , 2.0038285]$$

$$\hat{G}_1 = g_{01} \sqrt{\frac{3}{\pi}} \frac{1}{j\omega + 3} + \frac{g_{11} \sqrt{\frac{3}{\pi}}}{j\omega + 3} \left(\frac{j\omega - 3}{j\omega + 3}\right)$$

$$K = [6.0089695 , 2.0028787]$$

$$\hat{G}_2 = \frac{g_{02} \sqrt{\frac{3}{\pi}}}{j\omega + 3} + \frac{g_{12} \sqrt{\frac{3}{\pi}}}{j\omega + 3} \left(\frac{j\omega - 3}{j\omega + 3}\right) + \frac{g_{22} \sqrt{\frac{3}{4}}}{j\omega + 3} \left(\frac{j\omega - 3}{j\omega + 3}\right)^2$$

$$K = [6.0000097 , 2.0000266] .$$

VIII. CONCLUSION

We have presented a simple fast computing - fast converging method for solving the scalar LQR and the filtering problems. The multivariable version will be submitted soon. The method is applicable as well to a wide variety of distributed parameter LQR and their dual filtering problems, and large scale systems. The more general factorization (4.1) may be attacked by reducing it to the two easier factorizations (4.20) and (4.19). Another point to be declared is that Toeplitz equations of the type (4.15) can also be solved by this method. Except for minor amendments all the results are valid. In particular Theorem 6, 7 and 8 are applicable. Considering the algorithm in Section II, only the equality (2.7) should be replaced by

$$\alpha_k = g_k$$

where y_k is the kth Laguerre coefficient of $y^+(j\omega)$.

APPENDIX A

Lemma 1

If $f(s) \in H^{2+}(R)$ and $f(j\omega) \in L^{\infty}(R)$ then $f(s) \in H^{2+}(R) \cap H^{\infty+}(R)$.

Proof

Clearly $f(s) \in H^{\infty+}(R)$ iff $f\left(\frac{1+\zeta}{1-\zeta}\right) \in H^{\infty+}(T)$ so it is sufficient to show that

$$I \doteq \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(\frac{1+e^{j\theta}}{1-e^{j\theta}}\right) e^{jn\theta} \, d\theta = 0 \qquad n = 1, 2, \ldots$$

Consider the integral I, and substitute $\frac{1+e^{j\theta}}{1-e^{j\theta}} = j\gamma$.

$$I = \frac{2}{2\pi} \int_{-\infty}^{\infty} f(j\gamma) \frac{(j\gamma - 1)^{n-1}}{(j\gamma + 1)^{n+1}} \, d\gamma$$

$$= \frac{1}{2} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(j\gamma) \left\{ \frac{-1}{\sqrt{\pi}(j\gamma - 1)} \left(\frac{j\gamma + 1}{j\gamma - 1}\right)^{n} \right\}^{*} d\gamma$$

$$- \frac{1}{2}\sqrt{\pi} \int_{-\infty}^{\infty} f(j\gamma) \left\{ \frac{-1}{\sqrt{\pi}(j\gamma - 1)} \left(\frac{j\gamma + 1}{j\gamma - 1}\right)^{n-1} \right\}^{*} d\gamma$$

$$= \frac{1}{2\sqrt{\pi}} \langle f(j\gamma), \phi_n^- \rangle - \langle f(j\gamma), \phi_{n-1}^- \rangle \qquad \text{(A.1)}$$

where ϕ_n^- is nth Laguerre basis function in $H^{2-}(R)$ but $f(s) \in H^{2+}(R)$, i.e., $f(s) \perp H^{2-}(R)$ which implies that $\langle f(j\gamma), \phi_n^- \rangle = 0$ for $n = 0, 1, \ldots,$ i.e., the r.h.s. of

(A.1) is identically zero for $n = 1, 2, \ldots$ and the result follows.

Q.E.D.

* **Lemma 2**

If $h \in H^{2\pm}(R) \cap H^{\infty\pm}(R)$ and $[1 + h]$ has an inverse in $H^{\infty\pm}(R)$ there exists a unique function $g \in H^{2\pm} \cap H^{\infty\pm}(R)$ such that

$$[1 + g] = [1 + h]^{-1}$$

Proof

Let $g = -[1 + h]^{-1} h$ clearly $g \in H^{2\pm}(R) \cap H^{\infty\pm}(R)$

and

$$[1 + h][1 - [1 + h]^{-1} h] = 1 = [1 - h[1 + h]^{-1}][1 + h]$$

i.e., $1 + g$ is the inverse of $[1 + h]$.

Clearly if $[1 + g_1] = [1 + h]^{-1} = [1 + g] \rightarrow g_1 = g$.

Q.E.D.

* **Lemma 3**

If $x(s) \in H^{2+}(R)$ then $x\left(\frac{1 + \zeta}{1 - \zeta}\right) \in H^{2+}(T)$

Proof

$x(j\omega)$ can be represented as

$$x(j\omega) = \sum_{n=0}^{\infty} x_n \frac{1}{\sqrt{\pi}} \frac{1}{j\omega + 1} \left(\frac{j\omega - 1}{j\omega + 1}\right)^n \qquad \text{(A.2)}$$

where $\sum |x_n|^2 < \infty$

substituting $\left(\frac{j\omega - 1}{j\omega + 1}\right) = e^{j\theta}$. Then (A.2) becomes

$$x\left(\frac{1 + e^{j\theta}}{1 - e^{j\theta}}\right) = 1 + \frac{x_0}{2} + \sum_{n=1}^{\infty} (x_n - x_{n-1})\, e^{jn\theta}$$

but

$$\sum | x_n - x_{n-1}|^2 \leq 2 \sum | x_n|^2 + 2 \sum | x_{n+1}|^2 < \infty$$

so the result follows.

APPENDIX B

The Proof of Lemma 8

The proof of this Lemma is motivated by a similar result for Toeplitz operators on H^{2+} (T) [11] , we first introduce the operators V and U as follows, for any $g(e^{j\theta})$ defined on T , we set

$$V g (e^{j\theta}) = g \left(\frac{j\omega - 1}{j\omega + 1}\right) \tag{B.1}$$

and

$$U g (e^{j\theta}) = \frac{1}{\sqrt{\pi}} \frac{g\left(\frac{j\omega - 1}{j\omega + 1}\right)}{j\omega + 1} \tag{B.2}$$

It is not difficult to conceive that V is an isometric mapping from L^{∞} (T) onto L^{∞} (R) , and U is also an isometric mapping from H^{2+} (T) onto H^{2+} (R) .

Consider now the Toeplitz operator T_{ψ}

$$T_{\psi} (x) = p [\psi x] \qquad x \in H^{2+} (R) \tag{B.3}$$

Let $x = U (g)$, $g \in H^{2+}$ (T)

Then

$$\begin{aligned} U^{-1} T_{\psi} (U g) &= U^{-1} T_{\psi} U (g) = \\ &= U^{-1} p U U^{-1} [\psi U g] \\ &= (U^{-1} p U) [V^{-1} \psi g] \\ &= E [\tilde{\psi} g] = \\ &\doteq T_{\tilde{\psi}} (g) \end{aligned} \tag{B.4}$$

where

- $E = U^{-1} p U$ the projection operator from $L^2(T)$ onto $H^{2+}(T)$,
- $\tilde{\psi}(e^{j\theta}) = V^{-1}\psi = \psi\left(\frac{1 + e^{j\theta}}{1 - e^{j\theta}}\right)$, (B.5)
- $T_{\tilde{\psi}}$ is also a Toeplitz operator defined on $H^{2+}(T)$,
- $T_{\tilde{\psi}} = U^{-1} T_{\psi} U$.

Thus T_{ψ} is clearly invertible iff $T_{\tilde{\psi}}$ is invertible. Consider now T_{ψ} when ψ is real. T_{ψ} is self-adjoint because

$$< T_{\psi} x , x > = < x , T_{\psi}^{*} x > = < \psi x , x > = \langle x , \psi x \rangle = < x, T_{\psi} x >$$

i.e.,

$$T_{\psi}^{*} = T_{\psi^{*}} = T_{\psi} \qquad \text{(B.6)}$$

Hence its spectrum is real. So it is sufficient to show that if $(T_{\psi} - \lambda I)$ is invertible then either $\psi(j\omega) > \lambda \ \forall \omega \in R$ or $\lambda > \psi(j\omega) \ \forall \omega \in R$. Since T_{ψ} is invertible iff $T_{\tilde{\psi}}$ is invertible. So it is sufficient to show that $T_{\tilde{\psi}} - \lambda I$ is invertible implies that $\lambda > \tilde{\psi}(e^{j\theta}) \ \forall \theta \in T$ or $\tilde{\psi}(e^{j\theta}) > \lambda \ \forall \theta \in T$.

If $T_{\tilde{\psi} - \lambda}$ is invertible for λ real then there exists $g \in H^{2+}(T)$ such that

$$(T_{\tilde{\psi}} - \lambda I)\, g = 1 \quad , \quad (1 \in H^{2+}(T))$$

$$\rightarrow \quad (\tilde{\psi} - \lambda I)\, g = 1 + \bar{h} \tag{B.7}$$

for some $\bar{h} \in H^{2-}(T)$ with zero constant term.

Taking the complex conjugate of (B.7), we have

$$(\tilde{\psi} - \lambda I)\, \bar{g} = 1 + h \tag{B.8}$$

multiplying (B.7) by $\bar{g}$ and (B.8) by g we get

$$(\tilde{\psi} - \lambda I)\, |\bar{g}\, g| = (1 + h)\, g = (1 + \bar{h})\, \bar{g} \tag{B.9}$$

but

$$(1 + h)\, g \in H^{1+}(T) \quad , \quad (1 + \bar{h})\, \bar{g} \in H^{1-}(T)$$

Since $H^{1+}(T)$ and $H^{1-}(T)$ intersect only at the constant elements, so we must have

$$(\tilde{\psi} - \lambda)\, |\bar{g}\, g| = \text{constant} = \alpha \; , \quad \alpha \in R$$

Since g is not zero almost a.e.w it follows that $(\tilde{\psi} - \lambda I)$ has the same sign of α, i.e., either $\tilde{\psi} > \lambda \;\; \forall\, \theta \in T$ or $\tilde{\psi} < \lambda \;\; \forall\, \theta \in T$ and the result follows.

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A FACTORIZATION ALGORITHM WITH APPLICATIONS TO THE LINEAR FILTERING AND CONTROL PROBLEMS

THE MULTI-VARIABLE CASE

PART II

I. INTRODUCTION

In the first part of this study [36] we have developed the technical foundations of a fast algorithm for solving the scalar LQR and filtering problems via the so called Canonical Factorization of functions, say $K(j\omega)$, in the form

$$[I + K(j\omega)]^{-1} = [I + G^{+}] [I + G^{+}]^{*} \tag{1.1}$$

where $[I + G^{+}(s)]^{\pm 1}$ is analytic in the open right half plane.

Here the technical foundations of the multivariable version of this algorithm will be considered. As we have alluded to in the first part, very few methods are available to implement the factorization (1.1), and when it comes to the multivariable case, the choice is narrowed down almost to one method, namely the recent one of Davis and Dickinson [1]. In their method, the matrix spectral factor G^{+} is obtained iteratively using the formula

$$(I + G_{n+1}) = (I + G_n) [I + P [(I + G_n)^{*} (I + K) (I + G_n) - I]]^{-1}.$$

The method, as discussed in part I, is point wise, and hence has the ability of tracking rapidly changing frequency responses. However, matrix inversions are required at each iteration step and at each sampled frequency. Nevertheless, the main computational load of this method is the projection p which is performed using Stenger's idea [1,2]. Techniques of spectral factorization of rational matrices are legion [23] - [32]. A thorough examination of this case has appeared in [23]. The majority of these techniques, including those of [23] - [28], rely on frequency domain manipulations in which the problem of factoring a matrix of real rational functions is reduced to factoring an even polynomial or a self-inversive polynomial. If the factorization (1.1) of a real rational function is needed in other contexts than the filtering and control problems, Anderson et al suggested to reduce the factorization problem to the solution of a continuous [29] or discrete type [30] matrix Riccati equation.

Concerning our approach here, the argument and the framework are basically the same as in the scalar case [36]. Moreover, for the ultimate convenience most of the theorems here are deliberately put and stated to match corresponding ones in Part I. In fact, with the matrix notions brought up in section 3, most of the scalar results are transferred so smoothly and conveniently to the matrix case that no proof is even required.

Section 4 is dedicated to the formulation of the factorization problem in H^P spaces of matrix valued functions. Necessary and sufficient conditions for the existence of a canonical factorization of a given matrix function have been derived. In particular the correspondence between the canonical factors and the solutions of certain equations in H^P spaces is established. Some other new results are reported as well in this section. The relation between the so called outer-factorization of a function, which appears frequently in the design of feedback systems [33], [34], and the canonical factorization is derived. The standard Gohberg-Krein factorization [3,4,5&6] is elaborately reinvestigated in the realm of the formulation developed in this section.

The reduction method [6,7] in Hilbert spaces is applied in section 5 to a certain equation in H^{2+} space of matrix valued functions to generate a sequence of approximate spectral factors. An orthonormal basis in H^{2+} space is chosen in such a way that the reduced equation turns out to be a Toeplitz set of linear matrix equation with the advantage of simple structure and the availability of fast algorithms [8,9,10 & 11] for its solution. We provide also an error estimate and an expression for the speed by which the approximation error decays to zero in terms of some smoothness conditions on the canonical factors. Finally, the method is illustrated by a numerical example, and some other extensions and applications are discussed.

II. THE MAIN RESULT

For illustration, consider the standard finite dimensional infinite time linear regulator problem

$$\dot{\underline{x}} = A\,\underline{x} + B\,\underline{U}$$

$$\underline{y} = C\,\underline{x} \qquad (2.1)$$

with the cost function

$$J = \int_{\infty}^{\infty} \underline{U}^T\,\underline{U} + \underline{y}^T\,Q\,\underline{y}\;d\,t \qquad (2.2)$$

Assume that [A,B,C] is a minimal realization of the transfer function

$F\,(S) = C\,(SI - A)^{-1}\,B$, $Re\,(\lambda_j(A)) < 0$, $j = 1,2,\ldots \dim x$.

Then by standard results [13], the optimal control is given by

$$\underline{U}\,(t) = -B^T\,\underline{P}\;\underline{x}\,(t) \qquad (2.3)$$

where $\underline{P}$ is the unique positive definite solution of the algebraic Riccati equation

$$A^T\,\underline{P} + \underline{P}\,A - \underline{P}\,B\,B^T\,\underline{P} + C^T\,Q\,C = 0 \qquad (2.4)$$

Davis and Barry [12] have shown that this optimal feedback gain may be found, without solving (2.4), using the integral formula

$$P\,B = \frac{1}{2\pi}\int_{-\infty}^{\infty} (-j\omega I - A^T)^{-1}\,C^T\,Q\,C\,(j\omega I - A)^{-1}\,B\,[I + G(j\omega)]\;d\,\omega \qquad (2.5)$$

where $[I + G]$ is the minimal phase function (i.e. $[I + G(s)]^{\pm 1}$ are analytic in the open right half plane) satisfying the factorization

$$[I + K(j\omega)]^{-1} = [I + F^{*}(j\omega)\ Q\ F(j\omega)]^{-1} = [I + G(j\omega)] \cdot [I + G(j\omega)]^{*} \tag{2.6}$$

where

$$K(j\omega) = F^{*}(j\omega)\ Q\ F(j\omega)$$

The above integral formula is valid as well for a variety of distributed parameter LQR problems [12], and their dual distributed filtering problems [14]. Clearly the main difficulty of the above approach is the factorization (2.6). In this paper we address the factorization (2,6), as well as a generalized version of it, in the Hardy H^{P} spaces of matrix valued functions (Theorems 1,2,3, and 4). We show that the factor G satisfies certain Toeplitz equations in the space $H^{2+}_{mxm}(R)$ (Theorems 2,3). The reduction method in Hilbert spaces (see section II of part I) is applied to generate a sequence $\hat{G}_n$ of approximating functions (Theorems 8 & 9)

$$\hat{G}_n = \sum_{k=0}^{n} g_{K,n}\ \phi_k^{+}$$

where $\{\phi_k^{\pm}\}_0^{\infty}$ is the Laguerre orthonormal basis in $H^{2\pm}(R)$. It is shown that the matrix coefficients $\{g_{K,n}\}_{K=0}^{n}$ can be obtained by solving a Toeplitz set of linear matrix equations (Theorem 9 & Lemma 9). The The Akaike-Levinson algorithm [8] is applied to generate recursively the sequence of the approximating solutions as shown below. It is proved that, if G(s) is analytic on the closed right half plane, the method converges exponentially with n, the degree of approximation (Theorem 10 and its corollary).

The Algorithm

The factor G of the factorization (2.6), where $[I + K(j\omega)]$ is Hermitian and positive definite (a.e.w.), may be obtained as follows:

Step 1

Find the matrix coefficients A_k's of the Laguerre expansion of $K(j\omega)$ using the formula

$$A_k = \langle K(j\omega),\ \phi_K I \rangle = \int_{-\infty}^{\infty} K(j\omega) \left[\frac{1}{\sqrt{\pi}\,(j\omega+1)} \left(\frac{j\omega-1}{j\omega+1}\right)^k\right]^* d\,\omega$$

$k = 0,1,2,\ldots$

Step 2

Construct the block Toeplitz matrix

$$T_n = \{ B_{k-j} \}_{k,j=0}^{n} ,$$

$$B_0 = I + \frac{1}{2\sqrt{\pi}} (A_0 + A_0^T)$$

$$B_k = \frac{1}{2\sqrt{\pi}} (A_k - A_{k-1})$$

$$B_{-k} = B_k^T$$

Step 3

Generate a sequence of approximate solutions $\{G_n\}$ using the Akaike-Levinson recursive algorithm for solving a Toeplitz set of linear matrix equations as follows:

Let

$$g_{0,0} = -B_0^{-1} A_0$$

$$E_{1,1} = -B_1 B_0^{-1}$$

$$D_{1,1} = -B_1^T B_0^{-1}$$

$$S_1 = [B_0 - B_1 B_0^{-1} B_1^T]^{-1}$$

$$q_1 = [B_0 - B_1^T B_0^{-1} B_1]^{-1}$$

DO 1 n=1 , NMAX

$$g_{n,n} = q_n^T \left(-A_n - \sum_{m=1}^{n} B_{n+1-m} \; g_{m-1,n}\right)$$

$$g_{m-1,n} = g_{m-1,n-1} + D^T_{n-m+1,n} \; g_{n,n}$$

$$m = 1,2,\ldots,n$$

$$D_{n+1,n+1} = -\left(B_{n+1} + \sum_{j=1}^{n} D_{n-j+1,n} \; B_j\right) S_n$$

$$D_{m,n+1} = D_{m,n} + D_{n+1,n+1} \; E_{n-m+1,n}$$

$$m = 1,2,\ldots,n$$

$$E_{n+1,n+1} = -\left[B^T_{n+1} + \sum_{j=1}^{n} E_{j,n} \; B^T_{n-j+1}\right] q_n$$

$$E_{m,n+1} = E_{m,n} + E_{n+1,n+1} \; D_{n-m+1,n}$$

$$q^{-1}_{n+1} = q_n^{-1} - D_{n+1,n+1} \; E_{n+1,n+1} \; q_n^{-1}$$

$$S^{-1}_{n+1} = S_n^{-1} - E_{n+1,n+1} \; D_{n+1,n+1} \; S_n^{-1}$$

$$\text{IF} \sum_{\ell=0}^{n} \| g_{\ell,n} - g_{\ell,n+1} \|_E + \| g_{n+1,n+1} \|_E < \varepsilon$$

GO TO 2

1 - CONTINUE

2 - STOP

Step 4

Evaluate the approximate optimal feedback gain using the formula

$$K_n \cong \frac{1}{2\pi j} \oint_{\Gamma} (SI + A^T)^{-1} C^T Q C (SI - A)^{-1} B \cdot$$

$$[I + \sum_{\ell=0}^{n} g_{\ell,n} \frac{1}{\sqrt{\pi}\,(S+1)} (\frac{S-1}{S+1})^{\ell}] \, d S$$

Γ is a rectifiable contour in the r.h.p. enclosing $\sigma(-A^T)$. Practically steps (1) and (2) are inserted in step 3 of the algorithm so that the matrix coefficients A_k and $\bar{B}_k$ are computed whenever needed in the recursion.

III. BACKGROUND [15-19]

- The space $\mathbb{C}_{mxm}$ consists of all mxm matrices with complex entries with either norm

$$|A|_I = \underset{x \neq 0}{\text{l.u.b}} \frac{|Ax|}{|x|}$$,if A is considered as an operator on some Banach space of vectors in $\mathbb{C}_{mx1}$

or $$|A|_E^2 = \text{Tra}\,(AA^*) = \sum_{p,q}^{m} |a_{p,q}|^2$$

In this case $\mathbb{C}_{mxn}$ is a Hilbert space under the innerproduct

$$\langle A,B \rangle = \text{Tra}\,(A\,B^*) = \sum_{p,q} a_{pq}\,\bar{b}_{pq}$$

- The sets $L^p_{mxm}(T)$ $(L^p_{mxm}(R))$ $p \geq 1$ consists of all mxm matrix-valued functions $F = \{f_{ij}\}$, $F:T \to \mathbb{C}_{mxm}$ $(F:R \to \mathbb{C}_{mxm})$, with complex valued entries $f_{ij}(x) \in L^p(T)$ $(L^p(R))$.

- $F(x) \in L^p_{mxm}(T)$ $(L^p_{mxm}(R))$, $p \geq 1$ iff F has measurable entries and $|F(x)|_E \in L^p(T)$ $(L^p(R))$. For $p \geq 1$, $L^p_{mxm}(T)$ $(L^p_{mxm}(R))$ is a Banach space under the norm $\|F\|_p$ = scalar L^p norm of $|F(x)|_E$.

- $L^2_{mxm}(T)$ $(L^2_{mxm}(R))$ is a Hilbert space under the inner product

$$\langle F,G \rangle_{L^2} = \int_{T\,(R)} \text{Tra}\,[F(x)\,G^*(x)]\,dx$$

. The space $H^{p\pm}_{m\times\ell}(T)$ $(H^{p\pm}_{m\times\ell}(R))$ is the subset of $L^{p}_{m\times\ell}(T)$ $(L^{p}_{m\times\ell}(R))$ consisting of all matrix valued functions with entries in $H^{p\pm}(T)$, $(H^{p\pm}(R))$

. In particular $H^{2\pm}_{m\times\ell}(T)$ $(H^{2\pm}_{m\times\ell}(R))$ is a Hilbert space with the inner product of any two elements F and G in the space defined by the $L^{2}_{m\times\ell}$ inner product of the boundary value functions of F and G, and norm

$$\| F \|^2_{H^{2\pm}(T)} = \sup_{\substack{0 < r < 1 \\ (r > 1)}} \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{Tra}\, [F(re^{i\theta})\, F^*(re^{i\theta})]\, d\theta$$

$$\| F \|^2_{H^{\pm 2}(R)} = \sup_{\substack{\sigma > 0 \\ (\sigma < 0)}} \int_{-\infty}^{\infty} \mathrm{Tra}\, [F(S)\, F^*(S)]\, d\hat{\omega}$$

$$S = \sigma + j\omega$$

. $H^{\infty\pm}_{m\times m}(T)$ $(H^{\infty\pm}_{m\times m}(R))$ is a Banach algebra with the norm

$$\| F \|_{H^{\infty\pm}(T)} = \operatorname*{ess\,sup}_{\substack{0 < r < 1 \\ (r > 1)}} |F(re^{i\theta})|_E$$

$$\| F \|_{H^{\infty\pm}(R)} = \operatorname*{ess\,sup}_{\substack{\sigma > 0 \\ (\sigma < 0)}} |F(S)|_E, \quad S = \sigma + j\omega$$

. As in the scalar case, every function $F \in L^{2}_{m\times\ell}(R)$ can be expressed as

$$F = F_+ + F_- = P[F] + Q[F] \quad \text{where } F_\pm \in H^{2\pm}_{m\times\ell}(R),$$

The projectors $P\ (Q) : L^{2}_{m\times\ell}(R) \to H^{2+}_{m\times\ell}(R)$ $(H^{2-}_{m\times\ell}(R))$, $P[X] \triangleq \{P(x_{ij})\}_{\substack{i=1,m \\ j=1,\ell}}$

The projector Q is defined analogously.

The following lemmas are stated without proof. They are simple extensions of their scalar mates given in Part I.

Lemma 1

If $F(S) \in H^{2+}_{mxm}(R)$ and $F(j\omega) \in L^{\infty}_{mxm}(R)$, then $F(S) \in H^{2+}_{mxm}(R) \cap H^{\infty+}_{mxm}(R)$

Lemma 2

If $H(S) \in H^{2+}_{mxm}(R) \cap H^{\infty+}_{mxm}(R)$ and $[I + H(S)]$ has an inverse in $H^{\infty+}_{mxm}(R)$, then there exists a unique (a.e.w) matrix function $G(S) \in H^{2+}_{mxm}(R) \cap H^{\infty+}_{mxm}(R)$ such that $[I + G] = [I + H]^{-1}$

Lemma 3

If $X(S) \in H^{p+}_{mx\ell}(R)$ then $X\left(\frac{1+\xi}{1-\xi}\right) \in H^{p+}_{mx\ell}(T)$

$p \geq 1$.

IV. FACTORIZATION OF MATRIX-VALUED FUNCTIONS IN H^p_{mxm} (R) SPACES

In this section we investigate the relation between the generalized frequency domain image of matrix Wiener-Hopf equation, which is known as Toeplitz equations, and the spectral factorization formulated directly in the frequency domain spaces; i.e. the Hardy H^p spaces. The approach here is inspired by the work of Gohberg and Budjanu [20,21]. While the machinery and some of the results are motivated by the results of Pousson on Toeplitz operates [17,18] on the circle group. The heart of our results in this section is Theorem 2 which relates the canonical factors, defined soon, to the solutions of certain Toeplitz equations in $H^{2\pm}_{mxm}$ (R). Theorem 3 considers in detail the special case of factorizing a matrix valued function which is positive definite almost everywhere, i.e. Hermitian and its ess inf det (.) > 0. The complete characterization is brought up by Theorem 4. We have also studied the relation between the canonical factorization of a function and its outer-factorization, from which new necessary and sufficient tests for the existence of the canonical factorization are brought up.

Definition:

An element $I + K(j\omega)$, $K(j\omega) \in L^2_{mxm}(R) \cap L^\infty_{mxm}(R)$, is said to have a right canonical factorization if it admits representation in the form

$$[I + K(j\omega)] = [I + H^-(j\omega)]\,[I + H^+(j\omega)] \text{ a.e.w.} \tag{4.1}$$

where

$$H^{2\pm}(S) \in H^{2\pm}_{mxm}(R) \cap H^{\infty\pm}_{mxm}(R),$$

$$[I + H^{\pm}(S)]^{-1} \in H^{\infty\pm}_{mxm}(R), \text{ and}$$

$$G^{\pm}(S) \doteq [I + H^{\pm}(S)]^{-1} - I \in H^{2\pm}_{mxm}(R) \cap H^{\infty\pm}_{mxm}(R)$$

The uniqueness of the above canonical factorization is established by the following theorem,

<u>Theorem 1</u>

If an element $I + K(j\omega)$, $K(j\omega) \in L^2_{mxm}(R) \cap L^{\infty}_{mxm}(R)$ admits the canonical factorization (4.1) the factors $G^{\pm}$ are uniquely defined (a.e.w.). The proof of this Theorem is basically the same as its scalar version (I, theorem 1).

The following theorem is the corner stone of the subsequent study, from which special cases will be studied and other equivalent necessary and sufficient conditions will be derived.

<u>Theorem 2</u>

For an element $I + K(j\omega)$, $K(j\omega) \in L^2_{mxm}(R) \cap L^{\infty}_{mxm}(R)$ to admit the canonical factorization (4.1), it is necessary and sufficient that the two equations

$$G^+ + P\,[K\,G^+] = -P[K] \tag{4.2}$$

and

$$G^- + Q\,[G^-\,K] = -Q[K] \tag{4.3}$$

in the Hilbert spaces $H^{2\pm}_{mxm}(R)$, have solutions in $H^{2\pm}_{mxm}(R) \cap H^{\infty\pm}_{mxm}(R)$

respectively.

Except for a few obvious minor changes, the proof of this theorem is identical to that of the scalar case (I, theorem 2).

Studying the solvability of such equations in $H^{2+}_{mxm}(R)$ is not an easy task. However, with only a moment of contemplation we can realize that these two equations are no more than a set of parallel m-vector equations in $H^{2\pm}_{m}(R)$. In other words, it is sufficient to study the two equations (4.2) and (4.3) in the vector spaces $H^{2\pm}_{m}(R)$ rather than in the matrix spaces $H^{2\pm}_{mxm}(R)$. This fact is put forward in formal terms in the following two lemmas.

Lemma 4

Define two operators T_Ψ and $T^{\cup}_{\Psi}$ in the following way

$$T_\Psi [X] = P[\Psi X] \qquad X \in H^{2+}_{mxm}(R) \tag{4.4}$$

$$T^{\cup}_{\Psi} [\underline{x}] = P[\Psi \underline{x}] \qquad \underline{x} \in H^{2+}_{mx1}(R) \tag{4.5}$$

where

$$\Psi \in L^{\infty}_{mxm}(R),$$

T_Ψ is invertible iff $T^{\cup}_{\Psi}$ is invertible

Proof:

From the definition, it is clear that T_Ψ can be expressed as

$$T_\Psi [X] = [T^{\cup}_{\Psi}\underline{x}_1, T^{\cup}_{\Psi}\underline{x}_2, \ldots, T^{\cup}_{\Psi}\underline{x}_m]$$

where $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m$ are the columns of X let T_Ψ be invertible, accordingly

$$T_\Psi^{-1}\,[\overset{\cup}{T}_\Psi\,\underline{x}_1,\ \overset{\cup}{T}_\Psi\,\underline{x}_2,\dots,\ \overset{\cup}{T}_\Psi\,\underline{x}_m] = [\underline{x}_1,\ \underline{x}_2,\dots,\underline{x}_m]$$

$$= [A\,\overset{\cup}{T}_\Psi\,\underline{x}_1,\ A\,\overset{\cup}{T}_\Psi\,\underline{x}_2,\dots,\ A\,\overset{\cup}{T}_\Psi\,\underline{x}_m]$$

for some operator A, and for every $X \in H^{2+}_{mxm}$. It follows that $A\,\overset{\cup}{T}_\Psi = I$, i.e., $A = (\overset{\cup}{T}_\Psi)^{-1}$. Thus $\overset{\cup}{T}_\Psi$ is invertible conversely, let $\overset{\cup}{T}_\Psi$ be invertible and define an operator B as

$$B\,[X] = [(\overset{\cup}{T}_\Psi)^{-1}\,\underline{x}_1,\ (\overset{\cup}{T}_\Psi)^{-1}\,\underline{x}_2,\dots,\ (\overset{\cup}{T}_\Psi)^{-1}\,\underline{x}_m]$$

Thus $B\,T_\Psi\,[X] = [\underline{x}_1,\ \underline{x}_2,\dots,\ \underline{x}_m] = X$, for every $X \in H^{2+}_{mxm}$ i.e T_Ψ is invertible and $B = T_\Psi^{-1}$. Q.E.D.

<u>Lemma 5</u>

Define two operators Γ_Ψ and $\overset{\cup}{\Gamma}_\Psi$ in the following way

$$\Gamma_\Psi\,[X] = Q\,[X\Psi] \qquad X \in H^{2-}_{mxm}\,(R) \tag{4.6}$$

$$\overset{\cup}{\Gamma}_\Psi\,[\underline{x}] = Q\,[\underline{x}^T\,\Psi] \qquad \underline{x} \in H^{2-}_{m}\,(R) \tag{4.7}$$

where $\Psi \in L^{\infty}_{mxm}\,(R)$

Then, Γ_Ψ is invertible iff $\overset{\cup}{\Gamma}_\Psi$ is invertible.

Except for obvious changes, the proof is the same as lemma 4.

Using the above facts, we are going now to show in lemma 6, that (I + K) admits the canonical factorization (1.1), the two equations must have unique solutions. The main conclusion at this point is that the canonical factors can be obtained by solving (4.2) and (4.3) directly in $H^{2\pm}_{mxm}$ (R), and as these two solutions are the only solutions in

$H^{2\pm}_{mxm}$ (R), and as these two solutions are the only solutions in $H^{2\pm}_{mxm}$ (R) they also turn out to be in $H^{2\pm}_{mxm}$ (R) $\cap$ $H^{\infty\pm}_{mxm}$ (R) as well.

Lemma 6

If an element I + K, K ε L^2_{mxm} (R) $\cap$ L^{∞}_{mxm} (R), admits the canonical factorization (4.1), then the operators T and Γ defined by

$$T_{(1+K)}\ (X^+) = X^+ + P\ [K\ x^+], \quad X^+ \varepsilon\ H^{2+}_{mxm}\ (R) \tag{4.8}$$

$$\Gamma_{(1+K)}\ (X^-) = X^- + Q\ [X^- K] \quad X^- \varepsilon\ H^{2-}_{mxm}\ (R) \tag{4.9}$$

are invertible in $H^{2\pm}_{mxm}$ (R) respectively

Proof:

To prove that T is invertible, it is sufficient, by lemma 4, to show that the equation

$$\underline{y}^+ = \underline{x}^+ + P\ [K\ \underline{x}^+] \tag{4.10}$$

has exactly one solution $\underline{x}^+$ ε H^{2+}_{mx1} (R) for every $\underline{y}^+$ ε H^{2+}_{mx1} (R).

Let $$\underline{x}^+_0 = [I + G^+]\ P\ [(I + G^-)\ \underline{y}^+] \tag{4.11}$$

Then by direct substitution one can verify easily that $\underline{x}^+_0$ is indeed a solution of (4.6). Now suppose that $\underline{x}^+_1$ and $\underline{x}^+_2$ are two solutions of (4.6), then we must have

$$(\underline{x}^+_1 - \underline{x}^+_2) + P\ [K\ (\underline{x}_1 - \underline{x}_2)] = 0$$

or $$[I + K]\ (\underline{x}^+_1 - \underline{x}^+_2) = \underline{y}^- \text{ for some } y^-\ \varepsilon\ H^{2-}_{mx1}(R) \tag{4.12}$$

since $(I + K)$ has a canonical factorization (4.1), then

$$(I + H^{+})\,(\underline{x}_1 - \underline{x}_2) = [I + G^{-}]\,\underline{y}^{-}$$

but the L.H.S. is in $H^{2+}_{m\times 1}$ (R) and the R.H.S. is in $H^{2-}_{m\times 1}$ (R), so we must have both sides equal zero, i.e.

$$[I + H^{+}(S)]\,(\underline{x}_1(S) - \underline{x}_2(S)) = 0$$

but $[I + H^{+}(S)]$ is nonsingular for $S \in \mathbb{C}^{+}$, hence $\underline{x}_1^{+} = \underline{x}_2^{+}$ and we conclude that T is invertible.

Similarly, by the aid of lemma 5, and an argument as above we can prove that Γ is invertible as well. Q.E.D.///

Lemmas 4,5, and 6 unveil the relationship between the invertibility of the operators Γ and T and the canonical factorization. Unfortunately, as we have seen, the invertibility of T and Γ is only a necessary condition for the existence of the canonical factorization (4.1). Accordingly a further study of the conditions of the invertibility of T and Γ will not be very helpful. This difficulty forces us to study separate special cases from which the complete characterization of the class of functions admitting the canonical factorization (4.1) is formulated. It turns out that the following special case be an indispensable factor of any function admitting the canonical factorization (4.1).

Theorem 3

For an element $I + K$, $K \in L^{2}_{mxm}$ (R) $\cap$ L^{∞}_{mxm} (R) to admit the canonical factorization

$$[I + K(j\omega)] = [I + H^+(j\omega)]^* [I + H^+(j\omega)] \quad \text{a.e.w.} \qquad (4.13)$$

where $H^+(S) \in H^{2+}(R) \cap H^{\infty+}(R)$, $[I + H^+(S)]^{-1} \in H^{\infty+}_{mxm}(R)$, and

$$G^+(S) \doteq [I + H^+(S)]^{-1} - I \in H^{2+}_{mxm}(R) \cap H^{\infty+}_{mxm}(R)$$

It is necessary and sufficient that $[I + K(j\omega)]$ be Hermitian and positive definite (a.e.w.).

Like the scalar case, we need another lemma on Toeplitz operators to complete the proof of theorem 3. Unfortunately, we don't have ready result on the invertibility of Toeplitz operators on the line, as we had in the scalar case. However, we maintain that the following results, originally established on the circle group, are applicable to the Toeplitz operators on $H^{2+}_{mx1}(R)$ (hence on $H^2_{mxm}(R)$ as well).

Lemma 7

Let Ψ be a matrix valued function $\in L^\infty_{mxm}(R)$, define the operators T_Ψ and $T^\cup_\Psi$ as in (4.4) and (4.5) respectively then

i) If T_Ψ $(T^\cup_\Psi)$ is invertible, then $\Psi^{-1} \in L^\infty_{mxm}(R)$

ii) Suppose Ψ is positive definite (a.e.w.) then T_Ψ $(T^\cup_\Psi)$ is invertible.

The proof of this lemma is postponed to Appendix B.

Proof of theorem 3

The necessity part

From (4.13) it is clear that I + K must be Hermitian and at least positive semi-definite a.e.w.

As the factorization (4.13) is just a special case of the factorization (4.1), theorem 2 and lemma 6 are also applicable, i.e.

The operator $T_{(I+K)}$ on $H^2_{mxm}(R)$ is invertible. Then by part (i) of the last lemma we conclude that det $(I + K)$ does not vanish almost everywhere, i.e. $(I + K)$ is positive definite a.e.w.

The sufficiency part

Suppose now that $I + K(i\omega)$ is Hermitian and positive definite almost everywhere. According to theorem 2, for $1 + K$ to admit the canonical factorization (4.13) it is sufficient to show that there exists an element $G^+ \in H^2_{mxm}(R)$ such that G^+ and $(G^+)^*$ satisfy respectively the equations

$$X^+ + P[K X^+] = -P[K] \qquad (4.14)$$

$$X^- + Q[X^- K] = -Q[K] \qquad (4.15)$$

and $G^+ \in H^{\infty+}_{mxm}(R)$. By lemma 7 (ii) the equation (4.14) has a unique solution in $H^{2+}(R)$, call it G^+, i.e.

$$G^+ + P[K G^+] = -P[K]$$

$$\Rightarrow \quad (I + K)(I + G^+) = I + Y^* \text{ for some } Y \in H^{2+}_{mxm}(R) \qquad (4.16)$$

Taking the complex conjugate of (4.16), we get

$$(I + G^+)^* (I + K) = I + Y \qquad (4.17)$$

we apply now the projector Q to both sides of (4.17), we come up with

$$(G^+)^* + Q[(G^+)^* K] = -Q[K] \qquad (4.18)$$

i.e. $(G^+)^*$ satisfies equation (4.15). So what is left is to show that G^+ is $\in H^{\infty+}_{mxm}(R)$ or equivalently $\in L^{\infty}_{mxm}(R)$.

To do so multiply equation (4.16) by $(I + G^+)^*$ and equation (4.17) by

$(I + G^+)$ to obtain

$$[I + G^+]^* [I + K] [I + G^+] = [I + G^+]^* [1 + Y]^*$$

$$= [I + Y] [I + G^+]$$

$$= f(j\omega) \qquad (4.19)$$

we shall prove first that $f(j\omega)$ must be a constant a.e.w. Consider first the equality

$$f(j\omega) = [I + Y][1 + G^+]$$

substituting $j\omega = \frac{1 + e^{i\theta}}{1 - e^{i\theta}}$, and recalling lemma 3, we realize that

$[I + Y(\frac{1 + e^{i\theta}}{1 - e^{i\theta}})] \in H^{2+}_{mxm}(T)$, and $[I + G^+(\frac{1 + e^{i\theta}}{1 - e^{i\theta}})] \in H^{2+}_{mxm}(T)$.

Thus $f(\frac{1 + e^{i\theta}}{1 - e^{i\theta}}) \in H^{1+}_{mxm}(T)$.

Applying a similar argument to the second equality of 4.19 we see that

$$f(\frac{1 + e^{i\theta}}{1 - e^{i\theta}}) \in H^{1-}_{mxm}(T).$$

But the two subspaces $H^{1+}_{mxm}(T)$ and $H^{1-}_{mxm}(T)$ intersect only on the constant elements, i.e., $f(j\omega) = C$ a constant matrix. We now use the L.H.S. of (4.19)

$$[I + G^+]^* [I + K] [I + G^+] = C \quad \text{a.e.w} \qquad (4.20)$$

Applying the trace operator to both sides of 4.20, and using the matrix identity,

$$\text{Tra } [B^* A B] = \sum_{j=1}^{m} \underline{b}j^* A \underline{b}j$$

where $\underline{b}j$ is the jth column of the matrix B, we see that

$$\sum_{\ell} \underline{J}^*\ell \, [I + K(j\omega)] \, \underline{J}\ell = \text{Tra } [C] \qquad \text{a.e.w.}$$

where $\underline{J}\ell$ is the ℓth column of $[I + G^+]$

$$\rightarrow \qquad \lambda_{min} \, [I + K(j\omega)] \, \| (I + G^+) \|_E \leq \text{Trace}[C]$$

Since $I + K(j\omega)$ is positive definite a.e.w., we have

$$0 < \text{ess inf } \lambda\text{min } [I + K(j\omega)] \, \| (I + G^*) \|_E \leq \text{Trace } [C] \qquad \text{a.e.w.}$$

$\rightarrow$ $\| (I + G^+) \|_E$ is bounded almost everywhere i.e. $G^+ \varepsilon L^{\infty}_{mxm}$ (R)

and the result follows. Q.E.D.

The following theorem completely characterizes the class of functions admitting the canonical factorization (4.1).

Theorem 4

For an element $I + K(j\omega)$, $K(j\omega) \, \varepsilon \, L^2_{mxm} (R) \cap L^{\infty}_{mxm} (R)$, to admit the canonical factorization (4.1), it is necessary and sufficient that $(I + K)$ has the representation

$$I + K = [I + K_1] \, [I + K_2] \qquad (4.22)$$

where

i) $I + K_1$ is Hermitian positive definite (a.e.w.), and

$K_1 \, \varepsilon \, L^{\infty}_{mxm} (R) \cap L^2_{m \times m}(R)$

ii) $K_2 \, \varepsilon \, H^{2+}_{mxm} (R) \cap H^{\infty+}_{mxm}(R)$ with $[I + K_2]^{-1} \, \varepsilon \, H^{\infty+}_{mxm} (R)$

Proof:

Apart from the routine amendments of notations, the proof is essentially the same as in the scalar case (I, theorem 4).
Theorem 4 not only provides a simple test for factorization admissibility but also reduces the factorization problem (4.1) to two possibly easier ones, namely (4.22) and (4.13).

We are going now to two new important theorems which relate the canonical factorization (4.1) to the so-called outer-factorization of functions. In theorem (5) we prove the existence of the outer-factorization by construction, using the properties of the canonical factorization (4.13), while theorem 6 imposes certain necessary and sufficient conditions on the outer-factorization for the function to admit the canonical factorization (4.1).

The Outer-Factorization of Nonsingular Functions

Theorem 5

Suppose that $I + K(j\omega)$, $K \in L^2_{mxm}(R) \cap L^\infty_{mxm}(R)$, is invertible in $L^\infty_{mxm}(R)$, then there exists a unique factorization such that

$$I + K = [I + U]\ [I + J] \qquad (4.23)$$

where

- $I + U$ is unitary
- $[I + J]^{\pm 1} \in H^{\infty +}_{mxm}(R)$ and $J \in H^2_{mxm}(R) \cap H^{\infty +}_{mxm}(R)$

Proof

$(I + K)^* (I + K)$ is positive definite a.e.w., then by theorem 3 it admits the canonical factorization (4.13)

$$(I + K)^* (I + K) = (I + \gamma^*) (I + \gamma) \tag{4.24}$$

$$\rightarrow \quad (I + K) = [(I + K)^{*-1} (I + \gamma^*)] [I + \gamma]$$

$$= [I + U] [I + J] \tag{4.25}$$

i.e. $(I + K)$ admits the factorization (4.23)

To prove the uniqueness of the factorization (4.23), suppose there exist another factorization

$$I + K = [I + U_1] [I + J_1] \tag{4.26}$$

$$\text{and } U_1 \neq U, \; J_1 \neq J$$

It follows that

$$(I + K)^* (I + K) = (I + J_1)^* (I + J_1) = (I + J)^* (I + J)$$

but the canonical factorization of $(I + K)^* (I + K)$ is unique by theorem 1, so we must have $J_1 = J$ (a.e.w.)

Also from (4.25) and (4.26) we get

$$0 = (U_1 - U) (I + J), \tag{4.27}$$

but $(I+J)$ is nonsingular a.e.w., so we conclude that $U_1 = U$ a.e.w.

Q.E.D.

Theorem 6

Suppose that $I + K$, $K \in L^2_{mxm}(R) \cap L^\infty_{mxm}(R)$, admits the factorization (4.23) then the necessary and sufficient condition for $I + K$ to admit the canonical factorization (4.1) is that there exists a function $Z \in L^2_{mxm}(R) \cap L^\infty_{mxm}(R)$, admitting the factorization (4.1) and such that

$$I + U = (I + Z)\ (I + Y)^{-1} \tag{4.28}$$

with

$$(I + Z)^{*}\ (I + Z) \ = \ (I + Y)^{*}\ (I + Y) \tag{4.29}$$

Proof of theorem 6

The necessity part

Suppose that (I + K) admits the canonical factorization (4.1) Since $(I + K)^{*}\ (I + K)$ is positive definite a.e.w., it admits the canonical factorization (4.13),

$$(I + K)^{*}\ (I + K) \ = \ [I + Y]^{*}\ [I + Y] \tag{4.30}$$

which implies

$$\begin{aligned} I + K \ &= \ [(I + K)^{*^{-1}}\ (I + Y)^{*}]\ [I + Y] \\ &= \ [(I + K)\ (I + Y)^{-1}]\ [I + Y] \\ &= \ [I + U]\ [I + Y] \end{aligned} \tag{4.31}$$

Clearly the outer-factorization (4.31) of (I + K) satisfies the necessity condition

I + K has the representation

$$[I + K] \ = \ [(I + Z)\ (I + Y)^{-1}]\ [I + J] \tag{4.32}$$

but (I + Z) admits the canonical factorization (4.1), so (4.32) becomes

$$\begin{aligned} I + K \ &= \ (I + H_z^{-})\ (I + H_z^{+})\ (I + Y)^{-1}\ (I + J) \\ &= \ [I + H_z^{-}]\ [(I + H_z^{+})\ (I + Y)^{-1}\ (I + J)] \\ &= \ [I + H_k^{-}]\ [I + H_k^{+}] \end{aligned} \tag{4.33}$$

i.e. I + K admits the canonical factorization (4.1).

The proof of theorem 6 is complete.

In the next theorems we shall reinvestigate the traditional Gohberg and Krein factorization [3 - 6] of matrix valued functions in the realm of the formulation developed in this section.

Theorem 7

Suppose that $K(j\omega)$ is the Fourier transform of some $K^{v}(t) \in L^{1}_{mxm}(R) \cap L^{2}_{mxm}(R)$, then for $I + K$ to admit the canonical factorization (4.1), it is necessary and sufficient that the operator T_{I+K} defined in (4.4) be invertible on $H^{2+}_{mxm}(R)$

Proof of theorem 7

The necessity follows directly from lemma 6.

The sufficiency

Suppose now T_{I+K} is invertible. Since the Fourier transform 'F' is an isometric mapping from $L^{2}_{mx1}(R)$ onto $L^{2}_{mx1}(R)$, so the operator $F^{-1}\, T^{\upsilon}_{I+K}\, F$ $(F^{-1}\, T_{I+K}\, F)$ is invertible iff T^{υ}_{I+K} (T_{I+K}) is invertible in $H^{2+}_{mx1}(R)$ $(H^{2+}_{mxm}(R))$. It follows then the Wiener-Hopf equation

$$\underline{x}^{v}(t) + \int_{0}^{\infty} k^{v}(t-z)\, \underline{x}^{v}(t)\, d\, t = \underline{y}^{+}(t) \qquad **$$

is uniquely solvable in $L^{2}_{mx1}[0,\infty)$. Now by well known results of Gohberg and Krein [3 , theorem 2.1] the equation (**) must also be solvable in every $L^{p}_{mx1}[0,\infty)$ for every $\underline{y}^{+}(t) \in L^{p}_{mx1}(R)$, $1 \leqslant p < \infty$. In particular, $\underline{x}^{v}(t) \in L^{1}_{mx1}[0,\infty) \cap L^{2}_{mx1}[0,\infty)$ whenever $\underline{y}^{+}(t) \in L^{1}_{mx1}[0,\infty) \cap L^{2}_{mx1}[0,\infty)$. We now observe that the Fourier transform of the equation

$$X + P\,[K\,X] \;=\; -\,P\,[K] \qquad (4.34)$$

is in fact a system of m-vector equations in $L^1_{mx1}\,[°,\infty) \cap L^2_{mx1}\,[°,\infty)$

which implies that $X^v(t)\ \varepsilon\ L^1_{mxm}\,[°,\infty) \cap L^2_{mxm}\,[°,\infty)$, i.e.,

$$X\ \varepsilon\ H^{2+}_{mxm}(R) \cap W^+_{mxm} \subset H^{2+}_{mxm}\,(R) \cap H^{\infty+}_{mxm}\,(R),$$

where W^+_{mxm} is the algebra of the Fourier transform of matrix valued functions in $L^1_{mxm}\,[°,\infty)$.

We have now concluded that the equation (4.2) has a solution $X\ \varepsilon\ H^2_{mxm}\,(R) \cap H^{\infty+}_{mxm}\,(R)$. We are going now to show that equation (4.3) has also a solution in $H^{2-}_{mxm}\,(R) \cap H^{\infty-}_{mxm}\,(R)$. Now since T_{I+K} is invertible, so is $T_{I+K}{}^*$.

Using a similar argument as above we conclude that the equation

$$Y + P\,[K^*\,Y] \;=\; -P\,[K^*] \qquad (4.35)$$

must have a solution $Y\ \varepsilon\ H^{2+}_{mxm}\,(R) \cap H^{\infty+}_{mxm}\,(R)$. Taking the complex conjugator transpose of both sides of (4.35) we get

$$(Y)^* + Q\,[(Y^*)\,K] \;=\; -Q[K] \qquad (4.36)$$

i.e. (4.3) has a solution in $H^{2-}_{mxm}\,(R) \cap H^{\infty-}_{mxm}\,(R)$. Then by theorem 2, I + K admits the canonical factorization (4.1).

... Q.E.D.

<u>Corollary 7.1</u>

Suppose that I + K, $K(j\omega)\ \varepsilon\ L^2_{mxm}\,(R) \cap L^\infty_{mxm}\,(R)$ has a representation in the form

$$I + K = [I + K_1]\ [I + K_2] \tag{4.37}$$

- $K_2 \in H^{2+}_{mxm}(R) \cap H^{\infty+}_{mxm}$, $[I + K_2]^{\pm 1} \in H^{\infty+}_{mxm}(R)$
- $K_1(j\omega)$ is the Fourier transform of some $\overset{v}{k}_1(t) \in L^1_{mxm}(R) \cap L^2_{mxm}(R)$.

Then the necessary and sufficient condition for I + K to admit the canonical factorization (4.1) is that T_{I+K} be invertible.

Proof:

The necessity part follows from lemma 6. Suppose now T_{I+K} is invertible, since $T_{I+K} = T_{I+K_1} \cdot T_{I+K_2}$: T_{I+K_1} must be invertible.

Then by theorem 7 it admits the canonical factorization 4.1, thus

$$I + K = [I + H_1^-]\ [I + H_1^+]\ [I + K_2]$$

$= [I + H^-]\ [I + H^+]$ i.e. I + K admits the factorization (4.1). Q.E.D.

<u>Lemma 8</u>

Suppose that I + K, $K(j\omega) \in L^2_{mxm}(R) \cap L^\infty_{mxm}(R)$, admits the outer factorization (4.23). Then the necessary and sufficient conditions for T_{I+K} to be invertible is that there exists a matrix valued function $B(S) \in H^{\infty+}_{mxm}(R)$, and $B^{-1}(S) \in H^{\infty+}_{mxm}$, such that

$$\| B - (I + U) \|_\infty < 1 \tag{4.37}$$

where $\| A \|_\infty = \text{ess} \sup_w \max_j (\lambda_j (A A^*))^{1/2}$

The proof is given in the Appendix B

V. APPROXIMATING THE CANONICAL FACTORS

In part I of this study we have introduced the reduction method for solving linear equations in Hilbert spaces, and have applied the method to solve the equation

$$G^{+} + P\,[K\,G^{+}] = -P\,[K] \tag{5.1}$$

in the Hilbert space $H^{2+}(R)$ in the scalar case.

We are going now to extend this result to solve (5.1) in the Hilbert space $H^{2+}_{mxm}(R)$ of matrix valued functions.

We first show that if $(I + K)$ is Hermitian positive definite (a.e.w.), the operator T_{I+K} admits reduction relative to any basis in $H^{2+}_{mxm}(R)$.

Next, in theorem 9, we apply the reduction method to generate a sequence of approximating solutions

$$G_n^{+} = \sum_{k=0}^{n} g_{k,n}\,\phi_k, \qquad g_{k,n} \in ¢_{mxm}$$

and $\{\phi_k I\}$ is an appropriate orthonormal basis in $H^{2+}_{mxm}(R)$. The matrix coefficients $g_{k,n}$ turn out to satisfy a set of linear matrix equations. The situation is simplified further by choosing $\{\phi_k\}^{\infty}$ to be the Laguerre orthonormal basis. In this case the block matrix equations come down to a Toeplitz set of linear matrix equations with the advantage of the availability of fast algorithms for its solution [8-11]. We also derive an error bound and an estimate of the speed of convergence. Theorem 8 is the counter part of theorem 6 part I. However, we provide here a direct and simpler proof. Theorems 8 and 9 are stated here without proof. Their proof follows simply from their scalar mates once the notions developed in section 3 is mechanized.

Theorem 8

Suppose that $\psi \in L^{\infty}_{mxm}$ (R) is Hermitian positive definite (a.e.w), then the Toeplitz operator T_{ψ} admits reduction relative to any basis in $H^{2+}_{mxm}(R)$.

Proof

The proof consists of two steps; in the first we show that T_{ψ} must be a positive definite operator, second, by lemma 7, part I the result follows. Let us now prove that T_{ψ} is indeed a positive definite operator on H^{2+}_{mxm} (R).

Definition: A Toeplitz operator T_{ψ}, $\psi \in L^{\infty}_{mxm}$ (R), is said to be positive definite if

$$\text{Tra} < x, T_{\psi} [x] > = \text{Tra} < T_{\psi} [x], x > \geq \varepsilon > 0$$

for every $x \in H^{2+}_{mxm}$ (R). (5.2)

Now if ψ is Hermitian positive definite then

$$\text{Tra} < x, T_{\psi} [x] > = \text{Tra} \int_{-\infty}^{\infty} x^{*} \psi\, x\, d\omega = \text{Tra} \int_{-\infty}^{\infty} x^{*} \psi^{*} x\, d\omega$$

$$= \int_{-\infty}^{\infty} \sum_{k=1}^{m} \underline{x}_k^{*} \psi\, \underline{x}_k\, d\omega$$

where $\underline{x}_k$ is the kth column of the matrix x

$$\text{Tra} \langle x, T_\psi [x] \rangle \geq \int_{-\infty}^{\infty} \sum_{k=1}^{m} x_k^* (j\omega) \; \lambda_{min} (\psi(j\omega)) \, x_k(j\omega) \; d\omega$$

$$\geq \operatorname*{ess\,inf}_{\omega} \lambda_{min} (\psi(j\omega)) \; \| x \|^2_{H^{2+}_{mxm}}$$

$$> 0$$

Where the last step stems directly from the hypothesis, we conclude that Toeplitz operators constructed in the above way must be positive definite.

We can now see that, if $\{P_n\}$ is the chain of projections generated by some orthonormal basis $\{\phi_k I\} \subset H^{2+}_{mxm}$ (R),

$$\text{Tra} \langle P_n T_\psi P_n [x], P_n [x] \rangle = \text{Tra} \langle T_\psi P_n [x], P_n [x] \rangle$$

$$> \operatorname*{ess\,inf}_{\omega} \lambda_{min} (\psi(j\omega)) \; \| P_n x \|^2$$

$$> 0$$

Then by lemma 7 of part I, T_ψ admits reduction to the basis $\{\phi_k I\}$. Since $\{\phi_k I\}$ is arbitrary, T_ψ admits reduction relative to any basis $\{\phi_k I\}$ in H^{2+}_{mxm} (R).

We now come to the main result of this section.

<u>Theorem 9</u>

Consider the equation

$$G^+ + P [K G^+] = -P [K] \tag{5.1}$$

where $K \in L^2_{mxm}(R) \cap L^\infty_{mxm}(R)$, $[I + K]$ is Hermitian positive definite a.e.w.

Let $\{\phi_k I\}_{k=0}^{\infty}$ be some basis in H^{2+}_{mxm} (R), and define the projectors

$$P_n [x] = \sum_{k=0}^{n} \phi_k < x, \phi_k I > = \sum_{k=0}^{n} \phi_k \{(x_{\ell,r}, \phi_k)\}_{\ell,r=1}^{m}$$

$$x \in H_{mxm}^{2+} (R)$$

Let G_n be the solution of the equation

$$G_n + P_n [K P_n [\hat{G}_n]] = -P_n [K] \tag{5.3}$$

finally, let W be the isometric mapping which takes $P_n H_{mxm}^{2+}$ into Φ_{mxm}^{n+1}, the space of all (1+n) - Tuples constant matrices with complex entries equipped with the norm

$$< \{A_i\}_{i=0}^{n} , \{B_i\}_{i=0}^{n} > = \sum_{i=0}^{n} \text{Tra } [A_i B_i^*]$$

in accordance with the formula

$$W \{ \sum_{i=0}^{n} A_i \phi_i\} = \text{Column } (A_0, A_1, \ldots, A_n)$$

Then,

i) $G_n \rightarrow G$ as $n \rightarrow \infty$

ii) $\| G - G_n \|_{H_{mxm}^2} \leq C \| (I - P_n) [G] \|_{H_{mxm}^2} , C > 0$

iii) Under the mapping W the reduced operator equation is representable in Φ_{mxm}^{n+1} by the block matrix equation

$$T_n \underline{g} = \underline{A} \tag{5.3b}$$

where $\underline{A} = -W [P_n [K]]$

$\underline{g} = W [G_n]$

$$T_n = I_{m(n+1) \times m(n+1)} + \{ < K \phi_k , \phi_j I > \}_{k,j=0}^{n}$$

and T_n is a positive definite Block matrix.

The outlines of the proof of this theorem is identical to its scalar counterpart, and will not be repeated. Theorem 9 is the main result in this section. It maintains that the solutions of the reduced equation (5.3) converge in the mean square sense to the spectral factor G. Moreover, the approximating solutions G_n may be obtained directly by solving a positive definite set of linear matrix equations (5.3b). The next lemma, 9, simplifies the problem furthermore. It shows that if we take the basis to be the Laguerre orthonormal functions in $H^2_{mxm}(R)$, we come up with a Toeplitz set of linear matrix equations as well as a surprisingly simple explicit construction of the block matrix T_n in terms of the Laguerre coefficients $\{A_n\}$.

Lemma 9

Let G, G_n, and T_n be as in theorem 9 . If $\{\phi_k I\}$ is the Laguerre orthonormal basis, then T_n is a block Toeplitz matrix.

Proof of lemma

Since $k(j\omega) \in L^2_{mxm}(R)$, it can be expanded in the form

$$K(j\omega) = \sum_{\ell=0}^{\infty} A_\ell \frac{1}{\sqrt{\pi}\,(j\omega+1)} \left(\frac{j\omega-1}{j\omega+1}\right)^{\ell} + A_\ell^* \frac{1}{\sqrt{\pi}} \frac{-1}{(j\omega-1)} \left(\frac{j\omega+1}{j\omega-1}\right)^{\ell} \tag{5.4}$$

but $$\frac{1}{j\omega+1}\left(\frac{j\omega-1}{j\omega+1}\right)^{\ell} = \frac{1}{2}\left[\left(\frac{j\omega-1}{j\omega+1}\right)^{\ell} - \left(\frac{j\omega-1}{j\omega+1}\right)^{\ell+1}\right] \doteq \frac{1}{2}\left[\Psi_\ell - \Psi_{\ell+1}\right] \tag{5.5}$$

$$\frac{1}{j\omega-1}\left(\frac{j\omega+1}{j\omega-1}\right)^{\ell} = \frac{1}{2}\left[\left(\frac{j\omega+1}{j\omega-1}\right)^{\ell+1} - \left(\frac{j\omega+1}{j\omega-1}\right)^{\ell}\right] = \frac{1}{2}\left[\Psi_{-\ell-1} - \Psi_{-\ell}\right] \tag{5.6}$$

Upon substituting the relations (5.5) and (5.6) into (5.4), we get

$$K(j\omega) = \frac{1}{2}(A_0 + A_0^*) + \sum_{\ell=1}^{\infty} \frac{1}{2}(A_\ell - A_{\ell-1})\,\Psi_\ell$$

$$+ \sum_{\ell=1}^{\infty} \frac{1}{2}(A_\ell^* - A_{\ell-1}^*)\,\Psi_{-\ell}$$

$$= \sum_{\ell-\infty}^{\infty} B_\ell\,\Psi_\ell \tag{5.7}$$

with

$$B_0 = \frac{1}{2}(A_0 + A_0^*)\frac{1}{\sqrt{\pi}}$$

$$B_\ell = \frac{1}{2}(A_\ell - A_{\ell-1})\frac{1}{\sqrt{\pi}} \tag{5.8}$$

$$B_{-\ell} = B_\ell^*$$

Consider again equation (5.4) and substitute $K(j\omega)$ by the r.h.s. of (5.7)

Then

$$\sum_{\ell=0}^{n} g_{\ell,n} + P_n\left[\left(\sum_{s=\infty}^{\infty} \Psi_s\,B_s\right) \sum_{r=0}^{n} g_{r,n}\,\phi_r\right] = -\sum_{\ell=0}^{n} A_\ell\,\phi_\ell \tag{5.9}$$

splitting (5.9) and using the relation

$$\Psi_\ell\,\phi_r = \phi_{r+\ell} \qquad (\phi_{-k} = \bar{\phi}_{-k+1})$$

we come up with

$$g_{\ell,n} + \sum_{r=0}^{n} B_{\ell-r}\,g_{r,n} = -A_\ell \qquad \ell = 0, 1, \ldots n \tag{5.10}$$

rewriting (5.10) in a compact Block matrix equation, we have

$$T_n \underline{g} = -\underline{A}$$

where

$$T_n = \begin{bmatrix} I + B_0 & B_1^* & B_2^* & \cdots & B_n^* \\ B_1 & I + B_0 & B_1^* & \cdots & \\ B_2 & B_1 & I + B_0 & \cdots & \\ \vdots & & & & \\ B_n & B_{n-1} & & \cdots & I + B_0 \end{bmatrix} \qquad (5.11)$$

$\underline{A}$ = column $(A_0, A_1, \ldots, A_n)$

$\underline{g}$ = column $(g_{0,n}, g_{1,n}, \ldots, g_{n,n})$

i.e. The reduced equation is melted down to a Toeplitz set of linear matrix equations, and the matrix T_n is given explicitly by (5.8) and (5.11).

Next, we provide an estimate for the speed of decay of the error by theorem 10 and its corollary. Here again they are stated without proof as they follow directly from their scalar mates. See part I.

Define the space $C^p_{m\times m}$ (T) to be the space of p times continuously differentiable matrix valued functions on the unit circle, and the mapping V by

$$V\, f(x) = f_C\,(e^{i\theta}) = f\left(\frac{1+e^{i\theta}}{1-e^{i\theta}}\right)$$

We can now derive an estimate of the speed of conergence of G_n to G when the reduction method is applied with respect to the Laguerre basis in $H^{2+}_{mxm}(R)$.

Theorem 10

Let G and G_n be as in theorem 9. However, assume, in addition, G satisfies the following smoothness conditions,

1) $F_C(e^{i\theta}) \doteq V\,[(1+j\omega)\,G(j\omega)] \in C^p_{mxm}(T)$

2) $F_C^{(p)}(e^{i\theta})$ satisfies the Lipschitz condition

$$| F^{(p)}(e^{i\theta+ih}) - F^{(p)}(e^{i\theta}) |_E \leqslant c\,| h |^{\alpha}$$

for some $0 < \alpha \leqslant 1$ $\qquad$ $c > 0$

and $| A |_E^2 = \mathrm{Tra}\,(A A^*)$

Then

$$\| G - G_n \|_{H^{2+}_{mxm}(R)} \leqslant \frac{\text{constant}}{N^{p+\alpha-\frac{1}{2}}}$$

Corollary 10.1

If G(s) is analytic in an open right half plane including the $j\omega$-axis, then there exist constants $c < 0$ and $0 < \alpha < 1$ such that

$$\| G - G_n \| \leqslant c\,\alpha^n$$

The above corollary indicates that the approximation error decays, in the average, exponentially with the degree of approximation, which is a fairly fast convergence rate.

VI. THE ALGORITHM AND THE COMPUTATIONAL ASPECTS

In the previous sections we applied the reduction method to obtain a sequence of approximating functions G_n

$$G_n = \sum_{R=0}^{n} g_{k,n} \quad \phi_k$$

and showed that the matrix coefficients $\{g_{k,n}\}_{k=0}^{n}$ satisfy a Toeplitz set of linear matrix equations, namely

$$T_n \underline{g_n} = \underline{A} \tag{6.1}$$

where $\underline{A} = \text{column } [-A_0, -A_1, \ldots, -A_n]$

$$\hat{\underline{g}} = \text{column } [g_{0,n}, g_{1,n}, \cdots, g_{n,n}]$$

$$A_\ell = \langle K(j\omega), \phi_\ell \rangle = \int_{-\infty}^{\infty} \frac{K(j\omega) - 1}{\sqrt{\pi}\ (j\omega-1)} \left(\frac{j\omega+1}{j\omega-1}\right)^{\ell} d\,\omega$$

T_n turns out to be a positive Toeplitz matrix $\{\mathbf{B}_{K-j}\}_{K,j=0}^{n}$ where $\mathbf{B}_k$ are related to the coefficients A_k's by the simple relationship

$$\mathbf{B}_0 = I + \frac{1}{2\sqrt{\pi}} (A_0 + A_0^*)$$

$$\mathbf{B}_k = \frac{1}{2\sqrt{\pi}} (A_k - A_{k-1})$$

$$\mathbf{B}_{-k} = \mathbf{B}_k^*$$

In other words the problem of factorization is reduced to solving a Toeplitz set of linear matrix equations. Fortunately, there has been a great deal of interest in developing fast algorithms for the factorization and inversion of Block Toeplitz matrices in the last few years.

The pioneer work in this context is due to Akaike [11], and Morf [9] and Rissanen [10] who where able to show that the Cholesky decomposition of an nxn block Toeplitz matrix requires only of order of $O(n^2 m^3)$ operations compared to $O(n^3 m^3)$ for the general matrix case. An elaborate discussion and generalization with applications have recently been carried out by Kailath et al [8]. Here we choose to illustrate the method using the matrix form of the Levinson's recursive formula for solutions of Toeplitz set of linear equations [11]. The Akaike formula [11] gives directly the approximating solution

$$\underline{g}_{n+1} = \text{column } [g_{0,n+1}, g_{1,n+1}, \cdots g_{n+1,\, n+1}]$$

in terms of the old approximating solution

$$\underline{g}_n = \text{col } [g_{0,n}, g_{1,n}, \ldots, g_{n,n}]$$

without resolving the system of equations (6.1) for n+1, thus enables us to monitor the change of the approximating solutions as n increases.

Another surprising observation is that, under the normal assumption that the resolvent R(S,A) of the operator A in (2.1) is analytic in the closed right half plane, the approximate feedback gains $\mathbb{K}_n$ converge to the optimal feedback gain $\mathbb{K} = PB$ faster than the approximate spectral factors G_n converge to G.

This phenomenon may be explained roughly as follows:

Denote by θ the first term under the integral (2.5), namely

$$\theta = \frac{1}{2\pi} (-j\omega - A^T)^{-1} C^T Q C (j\omega I - A)^{-1} B$$

The integral (2.5) can then be written as

$$K = \int_{-\infty}^{\infty} \theta \, d\omega + \sum_{\ell=0}^{\infty} \theta_{-\ell} \; g_{\ell}$$

where $\theta_{-\ell}$ is the ℓth coefficient of θ w.r.t. $\{\bar{\varphi}_{\ell}\}$ and g_{ℓ} is the ℓth Laguerre coefficient of G. Using an argument similar to that used in the proof of theorem 8, part I, it can be shown that the coefficients $\theta_{-\ell}$ decays exponentially with ℓ. Thus the coefficients $\theta_{-\ell}$ act as weighting patterns reducing further the effect of the errors arrising in approximating (I + G) in general and the truncation error of (I + G) in particular.

Example

Consider the stationary state estimation problem for the following system

$$dx = A\,x\,dt + B\,d\omega\,(t)$$

$$dy = C\,x\,dt + d\,V\,(t)$$

where

$$A = \begin{bmatrix} 0 & -3 & 0 & 0 \\ 1 & -4 & 0 & 0 \\ 0 & 0 & 0 & -35 \\ 0 & 0 & 1 & -12 \end{bmatrix}, \quad B = \begin{bmatrix} \sqrt{11} & 0 \\ \sqrt{12} & 0 \\ 0 & \sqrt{83} \\ 0 & \sqrt{2} \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & .5 & 0 & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & 0 & .5 \end{bmatrix}$$

W(t) and V(t) are Wiener processes with incremental covariances

$Q = \begin{bmatrix} 5 & 0 \\ 0 & 13 \end{bmatrix}$, and $I_{2,2}$ respectively.

According to [14], the steady state filter may be given by

$$d\hat{x} = A\hat{x}\,dt + \mathbf{K}\,[dy - C\hat{x}\,dt]$$

$$\mathbf{K} = \frac{1}{2\pi}\int_{-\infty}^{\infty} (j\omega - A)^{-1} BQ\,B^T (-j\omega - A^T)^{-1} C^T [I + G^-]^T d\omega$$

where $[I + G^-]$ satisfies the canonical factorization

$$[I + C(-j\omega - A)^{-1} BQ\,B^T (j\omega - A^T)^{-1} C^T]^{-1}$$

$$= [I + G^+]\,[I + G^-]$$

We applied the method proposed in this paper with respect to the Laguerre basis

$$\varphi_\ell(s) = \sqrt{\frac{2}{\pi}}\,\frac{1}{s+4}\left(\frac{s-4}{s+4}\right)^\ell$$

we come up with the following numerical results

$$\mathbf{K} = \begin{bmatrix} 2.5 & 4.3301 \\ 1.0 & 1.7321 \\ -11.258 & 6.5 \\ -1.7321 & 1.0 \end{bmatrix}$$

. The spectral factor is approximated using only one term of the Laguerre expansion

$$\mathbf{K} \cong \begin{bmatrix} 2.5357 & 4.3920 \\ 1.0089 & 1.7475 \\ -11.432 & 6.6004 \\ -1.7593 & 1.0157 \end{bmatrix}$$

Maximum error $\leq$ 1.6%

. The spectral factor is approximated using two terms of the Laguerre expansion

$$\mathbf{K} \cong \begin{bmatrix} 2.5040 & 4.3370 \\ 1.0010 & 1.7338 \\ -11.274 & 6.5093 \\ -1.7346 & 1.0015 \end{bmatrix}$$

Maximum error $\leqslant$ .15%.

VII. DISCUSSION AND CONCLUSION

The paper has addressed the factorization problem in the Hardy H^p spaces of matrix valued functions. The classical Gohberg and Krein factorization is re-examined within the framework developed here. The relation between the outer factorization and the canonical factorization has also been investigated. We developed a fast algorithm for implementing the spectral factorization, particularly suitable for the control and filtering applications. In part III of this study, see [37], we have generalized the Davis and Barry formula (2.5) to cover a wider range of control problems. The results for the discrete time case, together with other applications as rational functions and positive polynomial factorization, can be found as well in [37]. A summary of our results in the discrete time case is given in the Appendix A. A relatively close step along this line of thought has been attempted by Jonckeer and Silverman [32], who studied the analytic factorization and its connection with the Toeplitz operators, but only for the rational functions on the unit circle. Application of our results to distribute parameter systems is currently being investigated and will be reported elsewhere.

APPENDIX A

Consider the stationary filtering model given by

$$\underline{x}(k+1) = A\,\underline{x}(k) + B\,\underline{U}(k) \qquad E\{U_k U_j^*\} = Q\,\delta_{kj}$$

(A.1)

$$\underline{y}(k) = C\,\underline{x}(k) + V(k) \qquad E\{V(k)\,V(j)\} = I\,\delta_{kj}$$

Assume the system is minimal and A is stable, then the optimal linear state estimator is given in the steady state by the Kalman filter

$$\underline{x}(k+1) = A\,\underline{x}(k) + K\,[\underline{y}(k) - C\,\underline{x}(k)] \qquad \text{(A.2)}$$

$K = A\,P\,C^T (I + C\,P\,C^T)^{-1}$ is the Kalman gain matrix, and P satisfies the Riccati equation

$$P = A\,P\,A^T - A\,P\,C^T (I + C\,P\,C^T)^{-1} C\,P\,A^T + B\,Q\,B^T$$

Again here the solution of the Riccati equation can be avoided by using the formula [1]

$$K = \frac{1}{2\pi j} \oint_{|Z|=1} (Z\,I - A)^{-1} B\,Q\,B^T (Z^{-1} I - A^T)^{-1} C^T [I + G_0(Z)]\, dZ\, W^{-1} \qquad \text{(A.3)}$$

where W and $G_0(Z)$ satisfy the factorization

$$K(e^{i\theta}) \doteq I + C\,(I e^{i\theta} - A)^{-1} B\,Q\,B^T (I e^{-i\theta} - A^T)^{-1} C^T$$

(A.4)

$$= [I + G_0^*(e^{i\theta})]^{-1}\; W\, [I + G_0(e^{i\theta})]^{-1}$$

with $G_0(0) = 0$, and $[I + G_0(Z)]^{\pm 1}$ analytic inside the unit circle.

The formulation of the factorization (A.4) in the Hardy $H^P(T)$ spaces is a well established result, e.g. [17, 18 and 27]. Most of the results in the continuous time case can be developed here also with even simpler proofs. So to avoid cumbersome repatition we shall only bring up the summary of the results for the discrete time case. In the first place, it is not difficult to see that if $K(e^{i\theta}) \varepsilon L^2_{mxm}(T) \cap L^\infty_{mxm}(T)$ Then $[I + G_0(Z)]$ satisfy the Toeplitz equation

$$P[K(I + G_0)] = W \tag{A.5}$$

with K positive definite, the Toeplitz operator T_K is positive definite, and admits reduction relative to any basis in $H^{2+}_{mxm}(T)$. By applying the reduction method w.r.t. the natural basis $\{Z^k\}^\infty_{k=0}$, a sequence of approximating functions $\{G_n\}$ can be generated

$$G_n(Z) = \sum_{\ell=1}^{n} g_{\ell,n} Z^\ell \tag{A.6}$$

As in the continuous time case, $\{g_{\ell,n}\}^n_{\ell=1}$ turn to satisfy a Toeplitz set of linear equations

$$T_n \underline{g} = -\underline{h} \tag{A.7}$$

$$\underline{g} = \text{column } [g_{1,n}, g_{2,n}, \ldots, g_{n,n}]$$

$$\underline{h} = \text{column } [h_1, h_2, \ldots, h_n]$$

where h_k is the kth Fourier coefficient of $K(e^{i\theta})$

$$T_n = \begin{bmatrix} h_0 & h_1^T & h_2^T & \cdots & h_n^T \\ h_1 & h_0 & h_1^T & & \\ h_2 & & & & h_1 \\ h_n & & & h_1 & h_0 \end{bmatrix}$$

Finally the Akaike-Levinson algorithm may be used to solve (A.7) exactly as we explained with continuous time case. The approximate colution (A.6) is then substituted in (A.4) to find the Kalman gain. With a proof similar to that of theorem (7, part I), but much simpler, we maintain that, if $(I + G_0)$ is analytic in the closed unit disk, then G_n converge to G_0 exponentially.

APPENDIX B

Lemma 7

Let Ψ be a matrix valued function $\varepsilon\ L^{\infty}_{m\times m}(R)$, define the operators T_{Ψ} and $T^{\cup}_{\Psi}$ as in (4.4) and (4.5) respectively, then

i) If T_{Ψ} $(T^{\cup}_{\Psi})$ is invertible then $\Psi^{-1}\ \varepsilon\ L^{\infty}_{m\times m}(R)$

ii) Suppose Ψ is positive definite (a.e.w.) then T_{Ψ} $(T^{\cup}_{\Psi})$ is invertible.

Proof:

We first introduce the two operators V and U as follows

$$V\ g(e^{i\theta}) = g\left(\frac{i\omega-1}{j\omega+1}\right) \tag{B.1}$$

$$U\ g(e^{i\theta}) = \frac{1}{\sqrt{\pi}}\ \frac{g\left(\frac{i\omega-1}{j\omega+1}\right)}{j\omega+1} \tag{B.2}$$

It is not difficult to conceive that V is an isometric operator from $L^{\infty}_{m\times\ell}(T)$ onto $L^{\infty}_{m\times\ell}(R)$, and U is also isometric operator taking $H^{2+}_{m\times\ell}(T)$ onto $H^{2+}_{m\times\ell}(R)$.

Consider now the Toeplitz operator $T^{\cup}_{\Psi}$, the operator $U^{-1}\ T^{\cup}_{\Psi}\ U\ (g) = T_{\tilde{\Psi}}\ (g)$, $g\ \varepsilon\ H^{2+}_{m\times 1}(T)$, is a Toeplitz operator, where

$$\tilde{\Psi} = \Psi\left(\frac{1+e^{i\theta}}{1-e^{i\theta}}\right)$$

Clearly T^{ν}_{Ψ} is invertible iff $T_{\tilde{\Psi}}$ is invertible, then the above lemma follows directly from exactly similar statements on Toeplitz operators on $H^{2+}_{m\times 1}(T)$, (see e.g.) [18], (Corollary 2.7 and theorem 2.8) .

Lemma 8

Suppose that $I + K$, $K(j\omega) \in L^{2}_{mxm}(R) \cap L^{\infty}_{mxm}(R)$, admits the outer factorization (4.23). Then the necessary and sufficient conditions for T_{I+K} to be invertible is that there exists a matrix valued function $B(S) \in H^{\infty}_{mxm}(R)$, such that $B^{-1}(S) \in H^{\infty+}_{mxm}(R)$, and

$$\| B - (I + U) \|_{\infty} < 1 \tag{B.3}$$

where

$$\| A \|_{\infty} = \text{ess}_{w} \sup \max_{j} (\lambda_{j} (A A^{*}))^{\frac{1}{2}}$$

The proof relies on the following result of Rabindranathan which states that if U is a unitary function $\in L^{\infty}_{mxm}(T)$, then T_u is invertible on $H^{2+}_{mx1}(T)$ iff there exists a $B \in H^{\infty+}_{mxm}(T)$ such that $B^{-1}(S) \in H^{\infty+}_{mxm}(T)$ and $\| U - B \|_{\infty} < 1$.

By the argument used in the proof of lemma 8, one can see that the Rabindranathan lemma is applicable also to the Toeplitz operators on $H^{2+}_{mx1}(R)$. Now suppose that T_{I+K} is invertible, then T_{I+U} must be invertible. Then it follows from the above argument and the Rabindranathan lemma that there must be a matrix valued function $B(S) \in H^{\infty+}_{mxm}(R)$ and such that

$$B^{-1}(S) \in H^{\infty+}_{mxm}(R) \text{ and } \| B - (I + U) \|_{\infty} < 1$$

Conversely, suppose that there exists a function $B(S) \in H^{\infty+}_{mxm}(R)$ such that condition (B.3) is fulfilled. Then again from the Rabindranathan lemma and the above argument the operator T_{I+U} must be invertible. But

$T_{I+K} = T_{I+U} \cdot T_{I+K_2}$ and T_{I+K_2} is invertible, so we conclude that T_{I+K} must be invertible. Q.E.D.

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A FACTORIZATION ALGORITHM WITH APPLICATIONS TO THE LINEAR FILTERING AND CONTROL PROBLEMS; APPLICATIONS

PART III

I. INTRODUCTION

This short paper proceeds to the same objective of part I & part II of this study; namely, the objective of developing fast algorithms for spectral factorization of functions with application to some control and systems problems.

In the preceeding parts we have addressed the factorization of functions in the Hardy H^P spaces. In particular it is established that if $I + K(j\omega)$, $K(j\omega) \in L^2_{mxm}(R) \cap L^\infty_{mxm}(R)$ is positive definite Hermitian, it admits the factorization

$$[I + K]^{-1} = [I + G][I + G]^* \tag{1.1}$$

where

- $G \in H^2_{mxm}(R) \cap H^\infty_{mxm}(R)$
- $[I + G]^{-1} \in H^\infty_{mxm}(R)$

A sequence $\{G_n\}$ of approximate spectral factors is sought in the form

$$G_n(j\omega) = \sum_{\ell=0}^{n} g_{\ell,n} \, \phi_\ell(j\omega) \tag{1.2}$$

where $\{\phi_\ell\}$ is the Laguerre orthonormal basis in $H^2(R)$

$$\phi_\ell(j\omega) = \sqrt{\frac{P}{\pi}} \; \frac{1}{j\omega+P} \left(\frac{j\omega-P}{j\omega+P}\right)^\ell \qquad P > 0$$

It is shown that such sequence may be generated by solving the following Toeplitz set of linear matrix equations using, say, the Akaike-Levinson algorithm [2]

$$T_n \underline{g} = -\underline{A} \tag{1.3}$$

where

$$\underline{A} = \text{column } [A_0, A_1, \ldots, A_n]$$

$$A_\ell = \langle K(j\omega),\ \phi_\ell \rangle$$

$$\underline{g} = [g_{0,n},\ g_{1,n},\ \ldots,\ g_{n,n}]$$

and

$$T_n = \{B_{k-j}\}_{k,j=0}^{n}$$

$$B_0 = I + \frac{1}{2\sqrt{p\pi}} (A_0 + A_0')$$

$$B_\ell = \frac{1}{2\sqrt{p\pi}} (A_\ell - A_{\ell-1})$$

$$B_{-\ell} = B_\ell'$$

Applications of this method to the linear quadratic control and to the filtering problems have been demonstrated in [1] and [2] respectively. Here we shall expand the scope of this approximation method to cover a wider class of systems and control problems. In section 2 we generalize the Davis and Barry [3] integral formula for the optimal feedback gain in the LQR problem. The new setting not only covers a wider class of cost functions but also overcomes the difficulty of treating unstable systems. The new formula, which utilizes a factorization of type (1.1), together with the proposed method of spectral factorization, provides a fast and efficient way for solving many LQR problems. The new formula also enables us to prescale the eigenvalues of the system in such a way that accelerates the convergence of our algorithm.

In network synthesis, implementing the factorization (1.1) directly by the reduction method may not be appreciated since the rational structure of the spectral factor is destroyed. Nevertheless, in section 3 we shall show that our approximation method is still applicable in this case as well by employing an elegant trick to restore the rational structure of the spectral factors. Our approach proceeds by reducing the factorization problem to the solution of a quadratic matrix equation in a similar way as in [12]. Such quadratic matrix equation is transformed in turn to an integral formula involving another factorization which is performed using our method.

The discrete time case has been considered on section 4. A similar integral formula for the optimal feedback gain is derived. The new formula, as its continuous time mate, overcomes the difficulty in treating unstable systems, and provide a means of prescaling the eigen values of the system in such a way to accelerate the convergence of the approximation method. We have also given a summary of our results in the discrete time case together with many other interesting properties.

The potentialities of our approach are demonstrated once more in section 5 where we provide a fast algorithm for the factorization of positive polynomials. Finally, our results are illustrated by an example.

II. AN INTEGRAL REPRESENTATION OF THE OPTIMAL FEEDBACK GAIN

We shall generalize the formula of Davis et al [3,4 & 5] for the optimal feedback gain. The new setting not only covers a wider class of cost functions but overcomes the difficulty in treating unstable systems [5], or systems with poles on the imaginary axes. The new formula provides also a means for prescaling the eigenvalues of the system in such a way that accelerates the convergence of the numerical method proposed by the author [1,2] for the LQR problems.

Consider the finite dimensional continuous time linear system

$$\dot{\underline{x}} = A\,\underline{x} + B\,\underline{u} \tag{2.1}$$

where (A,B) is controllable, $\underline{x}(t) \in R^{n\times 1}$, and $\underline{U}(t) \in R^{m\times 1}$.

Let the cost function be

$$V = \int_0^\infty \{\underline{U}^T(t)\,U(t) + 2\,\underline{x}^T(t)\,J\,\underline{U}(t) + \underline{x}^T(t)\,Q\,\underline{x}(t)\}\,dt \tag{2.2}$$

and assume that

$$\phi(j\omega) \doteq I + B'(-j\omega I - A')^{-1}Q(j\omega I - A)^{-1}B + J'(j\omega I - A)^{-1}B + B'(-j\omega I - A')^{-1}J \tag{2.3}$$

is positive semi definite. By well known results [6-10], The optimal control which minimizes (2.2) subject to (2.1) is given by

$$\underline{U}(t) = -(B'P + J')\,\underline{x}(t) \tag{2.4}$$

where P^{o} is the so called stablizing solution of the ARE

$$PA + A'P - [PB + J][PB + J]' + Q = 0 \tag{2.5}$$

One advantage of the above minimization problem is that it is invariant under feedback in the sense that a feedback of the form

$\underline{U}(t) = L\,\underline{x}(t) + \underline{v}(t)$

transforms the data (A,B,J,Q) into (A + BL, B, L' + J, L'L + JL + L'J' + Q). It has been recognized in several papers [6] - [10] that this transformation does not change the problem. By the controllability assumption, it is always possible to stabilize the system. Hence, there is no loss of generality in assuming that the system has already been prestabilized by a suitable feedback. Thus, throughout the paper we shall assume that A is strictly stable.

Willems [6] and others [9] & [12] showed that the above minimization problem is closely related to the factorization of $\phi(s)$ in the form

$$\phi(S) = W'(-S)\,W(S) \tag{2.7}$$

with W(s) and $W^{-1}(s)$ analytic in the R.H.P. In fact, it turns out that

$$W(S) = I + \mathbf{K}(SI - A)^{-1}B \tag{2.8}$$

$$W^{-1}(S) = I - \mathbf{K}\,(SI - A + B\,\mathbf{K})^{-1}B \tag{2.9}$$

where $\mathbf{K} = B'P + J'$ is the optimal feedback gain.

The following lemma gives the inverse relation of (2.8), i.e., the optimal gain in terms of the spectral factor W(S).

Lemma 1

The feedback gain $\mathbf{K}$

$$K' = \frac{1}{2\pi j} \int_{\Gamma} (SI + A')^{-1} Q (SI - A)^{-1} B W^{-1} (S) \, ds$$

$$+ \frac{1}{2\pi j} \int_{\Gamma} (SI + A')^{-1} J W^{-1} (S) \, ds \qquad (2.9)$$

where Γ is a closed rectifiable contour enclosing $\sigma(-A)$.

Proof:

The ARE (5) can be written as

$$-P (SI - A) + (SI + A') P - K' K + Q = 0 \qquad (2.10)$$

multiplying (10) from the right by $(SI - A)^{-1} B$ we set

$$-P B + (SI + A') P (SI - A)^{-1} B - K' K (SI - A)^{-1} B$$
$$+ Q (SI - A)^{-1} B = 0 \qquad (2.11)$$

using the relations

$$P B = K' - J$$

and

$$I + K (SI - A)^{-1} B = W (S)$$

we have

$$J W^{-1} (S) + (SI + A') P (SI - A)^{-1} B W^{-1} (S) - K'$$
$$+ Q (SI - A)^{-1} B W^{-1} (S) = 0$$

finally multiplying by $(SI + A')^{-1}$ from the left

$$(SI + A')^{-1} J W^{-1}(S) \, d + P (SI - A)^{-1} B W^{-1} (S) - (SI + A') K'$$
$$+ (SI + A')^{-1} Q (SI - A)^{-1} B W^{-1} (S) = 0 \qquad (2.12)$$

Now integrating (12) over a closed rectifiable contour Γ enclosing

$\sigma(-A^{T})$ and using the relation (e.g. [31], pp. 225).

$$\frac{1}{2\pi j}\int_{\Gamma}(SI + A^{T})^{-1}\,d\,s = I$$

and

$$\frac{1}{2\pi j}\int_{\Gamma}(SI - A)^{-1}\,d\,s = 0$$

if the spectrum of A is not enclosed by Γ, we come up with (9). Q.E.D.

The difficulty of treating unstable systems is overcome here very smoothly by simply prestabilizing the system by a suitable feedback. In this case our results are simpler than the treatment proposed by Davis in [5]. It has been noticed that the convergence of the algorithm proposed by the author in [1,2] to evaluate the feedback gain **K** could be quite slow if applied to systems having poles close to the $j\omega$-axis or having their poles scattered over a large region in the complex plan. These problems are also tackled down here by chosen a proper prescaling feedback which reallocates the poles of the system in such a way to accelerate the convergence of the algorithm, for example, if the prescaling feedback is chosen to accumulate all the poles at one point on the negative real axis of the complex plane, then the above formula may be evaluated using only (n-1) terms of the Laguerre expansion as explained in the introduction.

III. FACTORIZATION OF RATIONAL MATRICES

The problem of giving a spectral factorization of a class of real rational matrices arising in Wiener-Hopf problems and network synthesis is tackled via the integral formula derived in the previous section.

Suppose there is given a real rational matrix $\phi(S)$ with the properties

- $\phi(S) = \phi'(-S) = Z(S) + Z'(-S) + R$ (3.1)
- $\phi(j\omega) \geqslant 0 \ \forall$ real ω (3.2)

The representation in the r.h.s. of (3.1) can always be implemented via partial fraction expansion of $\phi(s)$, and such that $Z(S)$ is analytic in Real $(S) > 0$.

A real rational matrix $W(S)$ is sought which satisfies

$$\phi(S) = W'(-S)\, W(S) \tag{3.3}$$

and is analytic in the O.R.H.P. and possesses an analytic inverse there.

The techniques of performing such factorization of rational matrices are legion [1] - [12]. A thorough examination of this case has appeared in [13]. The majority of these techniques, including those of [14] - [18], rely on frequency domain manipulations in which the problem of factoring a matrix of real rational functions is reduced to factoring an even polynomial or a self-inversive polynomial. If the factorization (3.3) is needed in other contexts than the filtering and control problems, Anderson et al suggested to reduce the factorization problem to the solution of a continuous [12] or a discrete type [11] matrix Riccati equation. Our approach here is quite similar to the one adapted by Anderson in [12], except that the solution of the Riccati equation is avoided by utilizing the integral formula derived in the previous section and the approximate factorization method proposed by the author in [1,2].

Without loss of generality, we shall consider the case when $\phi(\infty) = R > 0$. If $\phi(\infty)$ is singular, a procedure to reduce this case to factoring a $\phi_r(S)$ with det $\phi_r(\infty) \neq 0$ can be found in [12].

The algorithm proceeds as follows

1) Factorize $R = N'N$, $N' = N$

2) Find a minimal realization (A,B, J') for the transfer function $Z_0(S) = N^{-1} Z(S) N^{-1}$, then

$$\phi_1(S) \doteq N^{-1} \phi(S) N^{-1} = J' (SI - A)^{-1} B + B' (-SI - A')^{-1} J + I \tag{3.4}$$

Comparing $\phi_1(j\omega)$ with (2.3), we easily realize the similarity of this factorization problem with the minimization problem dealt with in the previous section (with $Q = 0$). Accordingly the solution to the factorization problem may be given by

$$W(S) = W_1(S)N$$

$$W_1(S) = I + K (SI-A)^{-1} B$$

$$K = B' P + J'$$

and P is the stabilizing solution of the ARE (2.5), with $Q = 0$.

We shall now apply our method to avoid the solution of the A.R.E. (2.5).

Assume further, only for the moment, that the strict inequality in (3.2) holds. Moreover, let $Z_0(S)$ has no poles on the $j\omega$-axis. Hence, ϕ_1 $(j\omega)$ admits the factorization (1.1). We can now proceed as follows:

4) Apply the reduction method, as explained in [2] and summarized in the introduction, to obtain an approximate spectral $[\hat{W}_1^{-1}(S)]$.

5) Find the optimal gain K using the integral formula (2.9) (with $Q = 0$). Namely

$$\hat{K} = \frac{1}{2\pi j} \int_{\Gamma} (SI + A')^{-1} J \, [\hat{W}^{-1}(S)] \, d\, s$$

6) Restore the rational structure of the spectral factor $W(S)$ using the formula

$$W(S) = (I + \hat{K}(SI - A)^{-1} B) N$$

If $Z(S)$ has some $j\omega$-axis poles and/or det $\phi(j\omega)$ vanishes at some isolated points, say $\xi_1, \xi_2, \ldots, \xi_x$, one can still perform the factorization (3.3) by considering instead the factorization of the matrix function

$$\phi_2(S) = \Lambda'(S)\, \phi_1(S)\, \Lambda(S)$$

where $\Lambda(S) = I - K_1 (SI - A + B K_1) B$,

and K_1 is chosen such that $(A - B K_1)$ possesses eigen values at ξ_1, ξ_2, and ξ_x. Clearly then $\phi_2(S)$ is positive definite and bounded and can be factored as before to say,

$$\phi_2(S) = W_2'(-S)\, W_2(S)$$

Then the required spectral factor $W(S)$ is obtained via the formula

$$W(S) = W_2(S)\, [I + K_1 (SI - A)^{-1} B]\, N.$$

IV. AN INTEGRAL REPRESENTATION OF THE OPTIMAL FEEDBACK GAIN (Discrete-Time Case)

The discrete version of the integral formula (2.9) will be derived. Unlike the integral formula of Davis and Dickinson [4], the one derived here covers a wider range of control problems, including unstable systems.

We consider the problem of giving a stable feedback which minimizes the cost function

$$V(U) = \sum_{t=0}^{\infty} \underline{x}^{*}(t)\, Q\, \underline{x}(t) + 2\,\underline{x}^{*}(t)\, J\, \underline{U}(t) + U'(t)\, R\, U'(t) \tag{4.1}$$

subject to the dynamical constraint

$$\underline{x}(t+1) = A\,\underline{x}(t) + B\,\underline{U}(t) \tag{4.2}$$

and (A,B) is a controllable pair.

By the invariance property of this problem under feedback [20] - [22], there will be no loss of generality in assuming that $|\lambda j\ (A)| < 1$ $j = 1,2,\ldots \dim(x)$.

It has been established [20,21] that the solution of this problem is associated with the factorization of the matrix function

$$\begin{aligned}\phi(Z) = \phi'(Z^{-1}) &= B'(Z^{-1} I - A')^{-1} Q\, (ZI - A)^{-1} B \\ &\quad + B'(Z^{-1} I - A')^{-1} J + J'(ZI - A)^{-1} B + R\end{aligned} \tag{4.3}$$

In the form

$$\phi(Z) = W_0'(Z^{-1})\, D\, W_0(Z) \tag{4.4}$$

Where $W_0(Z)$ is analytic and possesses an analytic inverse in $|Z| > 1$.

D is a normalizing constant such that $W_0(\infty) = I$. If $\phi(Z) \geq 0$

$\forall\ |Z| = 1$, and $D > 0$,

then the control law is governed by

$$\underline{U}(t) = -K\, \underline{x}(t)$$

$$K = (R + B'PB)^{-1}(J + A'PB)' \tag{4.5}$$

and P is the real symmetric solution of the ARE

$$P = A'PA - [J + A'PB](R + B'PB)^{-1}[J + A'PB]' + Q \tag{4.6}$$

Furthermore, the factorization (4.4) is given explicitly by

. $D = (R + B'PB)$

. $W_0(Z) = I + K(ZI - A)^{-1}B$

we shall now use the above results to derive an explicit integral formula for the optimal gain K in terms of the spectral factors of $\phi(Z)$.

<u>Lemma 2</u>

The optimal gain is given by

$$K = \frac{1}{2\pi j}\int_{|Z|=1} D^{-1}[W_0'(Z^{-1})]^{-1} J'(ZI - A)^{-1}\, dZ$$

$$+ \frac{1}{2\pi j} D^{-1}\int_{|Z|=1} [W_0'(Z^{-1})]^{-1} B'(Z^{-1} - A^T)^{-1} Q(ZI - A)^{-1}\, dZ \tag{4.7}$$

Proof:

Consider the A.R.E. (4.5), let us write it in the form

$$P = A' P A - K D K + Q \tag{4.8}$$

Adding Z^{-1} P A to both sides of (4.8), we get

$$P Z^{-1} (ZI - A) = -(Z^{-1}I - A') P A - K' D K + Q \tag{4.9}$$

multiplying (4.9) by $B' (Z^{-1}I - A')^{-1}$ on the left and using the relations

. $B' P A = D K - J'$

. $W_0' (Z^{-1}) = I + B' (Z^{-1}I - A')^{-1} K'$, we come up with

$$B' (Z^{-1}I - A')^{-1} P Z^{-1} (ZI - A)^{-1} = J' - W_0' (Z^{-1}) D K + B' (Z^{-1}I - A')^{-1} Q \tag{4.10}$$

Again multiplying on the left by $[W_0' (Z^{-1})]^{-1}$ and on the right by $Z (ZI - A)^{-1}$

$$[W_0' (Z^{-1})]^{-1} B' (Z^{-1}I - A')^{-1} P = [W_0' (Z^{-1})]^{-1} J' (ZI - A)^{-1} Z - DK (ZI - A)^{-1} Z + [W_0' (Z^{-1})]^{-1} B' (Z^{-1}I - A')^{-1} Q (ZI - A)^{-1} Z \tag{4.11}$$

Integrating both sides of (4.11) over the unit circle, the left handside will be zero since it is analytic inside the unit circle. So we get

$$DK = \frac{1}{2\pi j} \oint_{|Z|=1} [W_0' (Z^{-1})]^{-1} J' (ZI - A)^{-1} dZ + \frac{1}{2\pi j} \int_{|Z|=1} [W_0' (Z^{-1})]^{-1} B' (Z^{-1} - A^T)^{-1} Q (ZI - A)^{-1} dZ$$

Q.E.D.

The preceding treatment, as in the continuous time case, covers as well the open loop unstable case, simply by applying a suitable prestabilizing feedback. The problem of factorizing a given discrete rational matrix function will also be dealt with using the same procedure of the continuous time case, except for few obvious changes.

The factorization (4.4) in the Hardy $H^P(T)$ spaces, unlike the continuous time case, is a well established result [23] - [27]. The special case when $\phi(Z)$ is a rational function of Z has been considered recently by Jankheere and Silverman [19]. Moreover, most of our results in the continous time case can be developed here with even simpler proofs. So to avoid cumbersome repetitions we shall bring up the summary of the results for the discrete time case.

In the first place, it is not difficult to see that if $\phi(Z) \in L^2_{mxm}(T) \cap L^\infty(T)$, then $W_0(Z) = [I + G_0(Z)]^{-1}$ satisfies the Toeplitz equation

$$P\,[\phi(Z)\,(I + G_0)] = D \tag{4.12}$$

With $\phi(Z)$ positive definite, the Toeplitz operator T_ϕ is positive definite [24], and hence admits reduction relative to any basis in $H^2_{mxm}(T)$, (see e.g. [27], Theorem 2.1). By applying the reduction method w.r.t. the natural basis $[Z^{-\ell}]_{\ell=\infty}^{\infty}$, a sequence of approximating functions can be generated in the form

$$G_n(Z) = \sum_{\ell=1}^{n} g_{\ell,n}\, Z^{-\ell} \tag{4.13}$$

$$D_n = \sum_{\ell=0}^{n} g_{\ell,n} h'_\ell \qquad g_{o,n} = I$$

As in the continuous time case, $\{g_{\ell,n}\}$ turns out to be a Toeplitz set of linear equations $\ell=1$

$$T_n \underline{g} = -\underline{h} \tag{4.14}$$

$\underline{g}$ = column $[g_{1,n}, g_{2,n}, \ldots, g_{n,n}]$

$\underline{h}$ = column $[h_1, h_2, \ldots, h_n]$

where h_ℓ is the ℓth Fourier coefficient of $\phi(z)$, w.r.t. $\{Z^{-\ell}\}_{\ell=0}^{\infty}$.

$$T_n = \begin{bmatrix} h_0 & h'_1 & \cdots & h'_{n-1} \\ h_1 & h_0 & \cdots & \\ & & h_0 & \\ & & & h'_1 \\ h_n & & & h_0 \end{bmatrix}, \quad h_0 = R$$

and $T_n > 0$.

Finally, the Akaike-Levinson algorithm may be used to solve (4.14) exactly as we explained in the continuous time case [2]. The approximate solution $G_n(Z)$ is then substituted in the integral formula (4.7) to find the optimal feedback gain. With a proof similar to that of theorem 7, Part I, but much simpler, we maintain that, if $[I + G'_0(Z^{-1})]$ is analytic in some open desk containing the unit circle, then G_n converges to G_0 exponentially.

The result in the discrete time case is more interesting than the continuous time from the computation point of view. Consider the integral formula (4.7) and let us substitute

$$W_0'(Z^{-1}) \cong I + \sum_{\ell=1}^{n} g'_{\ell,n} Z^{\ell} \tag{4.15}$$

into it to get

$$DK \cong \frac{1}{2\pi j} \int (I + \sum_{\ell=1}^{n} g'_{\ell,n} Z^{\ell}) J' (ZI - A)^{-1} dz$$

$$+ \frac{1}{2\pi j} \int [I + \sum_{\ell=1}^{n} g'_{\ell,n} Z^{\ell}] B' (Z^{-1} - A')^{-1} Q (ZI - A)^{-1} dz$$

After some manipulation the integral turns out to be

$$K \cong D_n^{-1} \{J' + \sum_{\ell=1}^{n} g'_{\ell,n} J' A^{\ell}\} +$$

$$D_n^{-1} \{I_0 B' + \sum_{\ell=1}^{n} g'_{\ell,n} B' I_{\ell}\} \tag{4.16}$$

where

$$I_{\ell} = \oint_{|Z|=1} (Z^{-1}I - A')^{-1} Q (ZI - A)^{-1} Z^{\ell} dz \qquad \ell = 0,1,2,... \tag{4.17}$$

the integral (4.17) can be evaluated very efficiently using the FFT or the Astrom algorithm described in [28].

It should be noticed that the integrals (I_{ℓ}) are exactly those needed to evaluate the coefficients $\{h_{\ell}\}$ in the Toeplitz equation (4.14), namely

$$h_{\ell} = B' I_{\ell} B + J' A^{\ell-1} B \qquad \ell = 1,2,3,...n$$

A final remark here is that one can always make the algorithm to converge efficiently fast by applying a prescaling feedback shrinking the poles of the system sufficiently inside the unit circle.

V . FACTORIZATION OF POSITIVE POLYNOMIALS

In this section we shall discuss in brief the factorization of self-inversive scalar polynomials and positive polynomial matrices, and then the problem of rational matrix factorization.

Consider first the self-inversive polynomial

$$P(Z) = C_0 + C_1 Z^{-1} + \ldots C_n Z^{-n} + C_1 Z + \ldots C_n Z^n \qquad (5.1)$$

$$C_0 \neq 0$$

Let us assume that $P(Z) > 0 \;\; \forall \; |Z|=1$. To apply the methods of this paper towards finding a polynomial $W_0(Z)$ such that

$$P(Z) = W_0(Z^{-1}) \; D \; W_0(Z) \qquad (5.2)$$

$$W_0(Z) \neq 0 \;\; \forall \;\; |Z| \geq 1$$

We set

$$P(Z) = C_0 + J' \; (ZI - A) \; b + b' \; (ZI - A') \; J \qquad (5.3)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & & & \\ & & & & & 0 & 1 \\ 0 & 0 & 0 & & \ldots & & 0 \end{bmatrix}, \; \underline{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \; J' = [C_n, C_{n-1}, \ldots, C_1]$$

Comparing (5.3) with (4.3), we can see easily the correspondence between the factorization problem (5.2) and the minimization problem discussed in the previous section. In particular, it is not difficult to see that the coefficients of the factor $W_0(Z)$ turn out to be $K_1, K_2, \ldots K_n$, where

$$\mathbf{K} = [K_n, K_{n-1}, \ldots K_1]$$

is just the optimal feedback gain (4.5).

Inspired by the results of the previous section, we may proceed to find $\mathbb{K}$ as follows

1) We seek an approximate inverse polynomial $G_n(Z)$ to the factor $W_0(Z)$ using the approximation method discussed before, namely,

$$G_m(Z) = 1 + \sum_{\ell=0}^{m} g_{\ell,m} Z^{-\ell} \tag{5.5}$$

where $\{g_{m,\ell}\}$ are obtained by solving the Toeplitz equation

$$\begin{bmatrix} C_0 & C_1' & \cdots & C_{m-1}' \\ C_1 & C_0 & \cdots & \\ \cdots & & C_0 & \\ & & & C_o \\ C_{m-1} & \cdots & & C_0 \end{bmatrix} \begin{bmatrix} g_{1,m} \\ g_{2,m} \\ \vdots \\ \\ g_{m,m} \end{bmatrix} = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \\ 0 \\ 0 \\ \vdots \end{pmatrix} \tag{5.6}$$

using the fast algorithm of [29].

2) Restore the coefficients of $W_0(Z)$ by substituting $G_m(Z)$ into the integral formula (4.7), explicitly

$$D_m \equiv \sum_{\ell=0}^{m} C_\ell' \, g_{\ell,m} \quad , \quad g_{0,m} = I \tag{5.7}$$

$$\mathbb{K}_m = D_m^{-1} \frac{1}{2\pi j} \int_{|Z|=1} [1 + \sum_{\ell=1}^{m} g_{\ell,m}' Z^P] \, J' \, (ZI - A)^{-1} \, dz \tag{5.8}$$

The last integral (5.8) may be simplified to the following explicit expression

$$k_{\ell,m} = D_m^{-1} \{ \sum_{r=\ell}^{n} g'_{r-\ell,m} \; C_r \}, \; g'_{0,m} = I \tag{5.9}$$

$$\ell = 1,2, \ldots, m$$

$$W_0(Z) \cong 1 + \sum_{\ell=1}^{n} k_{\ell,m} \; Z^{-\ell} \tag{5.10}$$

Our algorithm has certain computational advantage over [29] and [30], which depend on Cholesky factorization of Toeplitz matrices, while solution of (5.6) can be performed directly with computational complexity of at most $O(m \log^2 m)$!. When we turn to the rxr matrix polynomial

$$\Gamma(Z) = C_0 + C_1' Z + \ldots C_n' Z^n + C_1 Z^{-1} + \ldots C_n Z^{-n} \tag{5.11}$$

with $\Gamma(Z) > 0$, $|Z|=1$, a similar approach works.
One takes A as the direct sum of r compies of the A matrix used for the scalar polynomial A, B as $B' = [0 \; 0 \; \ldots \; I]$ and finally, $J' = [C_n', \ldots C_1']$.

Even scalar polynomials and the corresponding matrix polynomials can be handled by conversion through the bi-linear transformation [12], [18]. Considering the problem of factoring a power spectrum matrix $\phi(Z) > 0$, one can write obviously, $\phi(Z)$ as

$$\phi(Z) = \frac{\Gamma(Z)}{P(Z)}$$

where $P(Z)$ is a scalar self inversive polynomial with $P(0) > 0$, and $\Gamma(Z)$ is a matrix of the form (5.11). One can then proceed into two ways; either one can factor $\Gamma(Z)$ and $P(Z)$ separately as described previously, or one can write

$$\phi(Z) = R + Z(z) + Z'(z^{-1})$$

and initiate a procedure similar to the one implemented to factorize rational matrices in section 3.

Example

Let $\Gamma(Z) = C_0 + C_1 Z^{-1} + C_1' Z$,

$$C_0 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 6 & .5 \\ 0 & .5 & 1 \end{bmatrix} \quad \& \quad C_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & -.5 \\ 0 & 0 & 0 \end{bmatrix}$$

we applied our method to implement the factorization

$$\Gamma(Z) = (I + K_1' Z)\, D\, (I + K_1 Z^{-1})$$

After only 2 iterations in the Akaike-Levenson algorithm, we get

$$K_1 = \begin{bmatrix} .2264919 & -.2264919 & -.113255 \\ .2113245 & -.2113245 & -.1056623 \\ -.113255 & .113255 & .0566275 \end{bmatrix}$$

$$D = \begin{bmatrix} .7886755 & -.7886755 & .1056623 \\ -.7886755 & 5.7886755 & .3943377 \\ .1056623 & .3943377 & .9471689 \end{bmatrix}$$

which are correct results to within at least 5 decimal places.

VI. CONCLUSION

The paper has discussed in brief some important applications of the spectral factorization scheme proposed in the preceding papers. In particular, the generalized Davis and Barry formula, equipped with the factorization scheme provides a simple and fast procedure for solving many control problems. The potentialities of our approach has been further demonstrated by presenting an algorithm for the spectral factorization of a class of rational matrices arising in Wiener-Hopf problems and network synthesis. The parallel results in the discrete time case has been given in brief with stress on some computational aspects. Although this paper has considered the lumped parameter systems only, most of the results are directly applicable or extenddble to the distributed parameter systems as well, and will be reported soon in a separate paper.

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CONCLUSION

Although the LQR and the stationary filtering problems are considered to be well established results in systems theory, the Riccati equation, which indispensably appears in their solution, remains a stumbling stone in many applications such as large scale systems, distributed parameter systems, and adaptive systems.

In this study we have introduced fast and simple algorithms for solving such LQR and filtering problems, without the Riccati equation, in continuous and discrete time, and for lumped and distributed parameter systems. The modified formula of Davis and Barry enables the treatment of unstable systems as well, and provides a prescaling technique for the eigenvalues of the systems in such a way as to accelerate the convergence of the algorithm. Moreover, the flexibility of the approach has been demonstrated by providing subalgorithms for the factorization of rational matrices and positive polynomials arising in other contexts than control. It is hoped that this approach, because of its speed and simplicity, will prove to be a significant contribution in narrowing the gap between control theory and control applications.

We have presented also a novel characterization of the factorization problem in the Hardy H^p spaces. A formulation sufficiently general to encompass practically all such engineering problems, lumped or distributed parameter. This formulation may prove to be useful, as a rigorous methodology, to address many related open issues in control theory, e.g., the spectral theory of the linear quadratic regulator problems in the continuous time case, some distributed filtering problems, Wiener-Hopf equations with unsummable kernels, and many others. Other interesting and useful extensions may be initiated, for example, by constructing other bases in $H^2(T)$ and $H^2(R)$ which preserve the Toeplitz structure, or by investigating those models and realizations which reduce further the computations of the algorithm, e.g., modeling systems in terms of polynomials in shift operators.

Finally, it should be realized that although the formulation of the factorization problem is carried out in the subalgebras $H^{2\pm} \cap H^{\infty\pm}$, the analysis itself is valid equally well, after few amendments, in the subalgebras $H^{p\pm} \cap H^{\infty\pm}$ $(1 \leq p < \infty)$. In this case, one may still develop the algorithm by considering the related Toeplitz equations in the Banach spaces H^p $(1 \leq p < \infty)$ instead of the Hilbert space H^2.

All in all, we hope that this study has not only paved the way towards a better understanding of the factorization problem as one of the most vital issues in modern systems theory, but provided the control engineer with fast and simple algorithms for implementing some of the most important control schemes as well.