Monte Carlo renormalization-group study of self-organized criticality

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We have studied a class of dynamical models exhibiting self-organized criticality, which have recently been introduced by Bak, Tang, and Wiesenfeld [Phys. Rev. Lett. **59**, 381 (1987)] by the Monte Carlo renormalization-group (MCRG) technique. In particular, we estimate critical exponents for the "sandpile" model in dimensions $d=2$ and 3 by MCRG, and present numerical simulations for $d=4$, which is thought to be the upper critical dimension. Our results are to higher precision than, although consistent with, the original numerical work on the problem. We further compare our results to those of a recent conjecture.

I. INTRODUCTION

Recently Bak, Tang, and Wiesenfeld[1–3] have introduced the concept of self-organized criticality. They have devised dynamical models that evolve automatically into a critical state without tuning any parameter. Such a self-organized critical state is characterized by the absence of length and time scales and is argued to be responsible for long-range temporal correlations with "$1/f$" power spectrum, where f is frequency, in many dissipative dynamical systems. This new critical phenomena is fundamentally different from that near a second-order phase transition. For that case, the critical point can only be reached by tuning parameters in the phase diagram, such as temperature, to reach criticality: Gibbs's phase rule forbids an inhomogeneous state in thermal equilibrium, except on, say, lines and points in a two-dimensional phase diagram. Self-organized criticality can also be thought of as an attractor of a dynamical system reached by starting far from equilibrium. In this regard, there is a close connection to irreversible cellular automata, which satisfy similar properties.[4] However, like standard critical phenomena, a few critical exponents describe the self-organized critical state, which involves scaling and universal behavior.

While it has been argued that many phenomena in nature show self-organized critical behavior, a particular physical system which has served to focus research is avalanches in sandpiles, although it should be noted that the experimental situation for $1/f$ noise in avalanches is at present unresolved.[5] Many other experimental representations are presumably possible, and some novel applications of the ideas of self-organized criticality have been suggested.[6] For example, scaling in first-order phase transitions may also be thought of as self-organized in that the scaling regime is reached without tuning, since first-order transitions are controlled by trivial stable fixed points; indeed, a recent study[7] indicates the presence of $1/f$ noise in spinodal decomposition. In this paper, we will explicitly consider Bak, Tang, and Wiesenfeld's model of avalanches in sandpiles.

The most powerful concepts in phase transitions are universality and scaling. These allow a unified description of the critical behavior of vastly different physical systems. It is natural to expect, for self-organized critical systems, that there are also universality classes in which systems share critical exponents and scaling functions. In a recent article, Kadanoff *et al.*[8] showed that this is the case. Again, as in second-order phase transitions, one would like to accurately determine the critical exponents in the self-organized system. This is a nontrivial matter. Tang and Bak[3] have calculated the mean-field exponents, which one would expect to be valid at and above the upper critical dimension; Obukhov[9] has argued that upper critical dimension is $d_c=4$. Hwa and Kardar[10] applied field-theoretical renormalization group to a dynamical equation which they conjecture shares a universality class with the sandpile model. Their model has $d_c=4$, with the same mean-field exponents as previously obtained by Tang and Bak. Furthermore, they obtained results for dimension $d<d_c$. Several exponents found by them are consistent with the original simulation of sandpiles by Bak, Tang, and Wiesenfeld,[1,2] although considerable numerical discrepancy still persists.

A well-established method for computing critical exponents is the Monte Carlo renormalization group[11–15] (MCRG). This method can give more precise numerical values for critical exponents than direct calculations because it efficiently iterates the effect of irrelevant variables away. Usually, it is difficult to find a point on the critical surface to begin the renormalization-group analysis. Thus, in applications of MCRG to second-order transitions, the success of the method requires knowledge of the critical temperature which is, in general, unknown. However, for *self-organized* critical phenomena, one needs only a convenient starting configuration. Hence for those systems, the MCRG method is, in principle, easier to use than its counterpart in second-order phase transitions.

In this paper, we apply the MCRG method[16] to the sandpile model to calculate the critical exponents. Two- and three-dimensional systems are considered. Since the upper critical dimension is believed to be $d_c = 4$, we also directly simulated the system there. We estimate two exponents, and use scaling relations to obtain others. In Sec. II we introduce the model and method, and give results for the critical exponents, while in Sec. III we close the article with a short discussion.

II. METHOD AND RESULTS

A. Two-dimensional model

In two dimensions, the sandpile model consists of a square array of heights $z(\mathbf{x})$, which represent the slope of a sandpile at position $\mathbf{x}=(x,y)$. To reach the critical state, one adds units of slope one by one. If the slope at a particular site exceeds a critical value z_c, an avalanche occurs which allows the system to locally relax. The process conserves the total slope of the system, except at the boundaries. Once the excess of slope has been distributed among the nearest neighbors, the relaxation process is applied to them, and so on, until all the slopes of the system are less than or equal to the critical slope. The model for $z_c = 4$ is as follows:

$$z(\mathbf{x}) \to z(\mathbf{x}) + 1 , \tag{1}$$

unless $z(\mathbf{x}) > 4$, then

$$\begin{aligned} &z(\mathbf{x}) \to z(\mathbf{x}) - 4 , \\ &z(x\pm 1,y) \to z(x\pm 1,y) + 1 , \\ &z(x,y\pm 1) \to z(x,y\pm 1) + 1 . \end{aligned} \tag{2}$$

The particular value of z_c only shifts the value of the average slope. The boundary conditions are

$$z(0,y) = z(x,0) = z(L+1,y) = z(x,L+1) = 0 ,$$

where L is the edge length of the lattice.

To estimate critical exponents, one infinitesimally perturbs the system and monitors its response. One unit of slope is added to random sites, after which the system is allowed to relax. See Fig. 1. Once the relaxation process has ended, another unit of slope is dropped, without resetting the critical state to its original configuration. The relaxation takes place through avalanches of size s and duration t. The size of the avalanche is the number of sites that have been affected during the process. Critical exponents characterize the size $s(t)$ of an avalanche of duration t, and the size distribution function $D(s)$ in the critical state. They are defined by

$$s \sim t^{\alpha} \tag{3}$$

and

$$D(s) \sim s^{1-\tau} . \tag{4}$$

These are the two independent exponents we have estimated. Note that the asymptotic behavior of other quantities can be determined[2] from τ and α. For example, the power spectrum $P(f)$ as a function of frequency f satisfies $P(f) \sim 1/f^{\phi}$, where $\phi = \alpha(3-\tau)$. Furthermore, once weighted by the average response s/t, the time distribution $D(t)$ obeys $D(t) \sim 1/t^{b}$, where $b = 2 - \alpha(3-\tau)$.

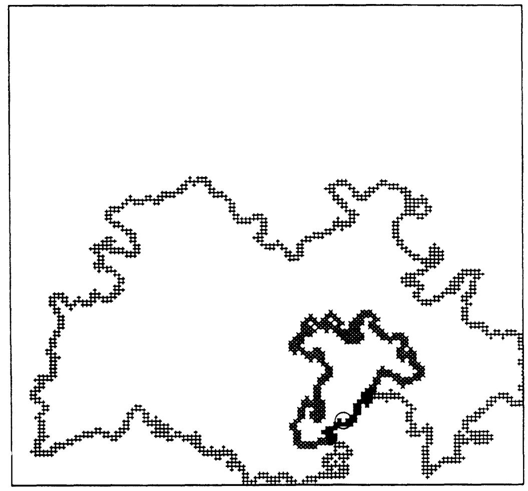

FIG. 1. Sites on a 128^2 system whose value changed at two times during a large avalanche. Circle shows initial perturbation site, $\times$'s the sites changed after 50 time steps, and $+$'s the sites changed after 301 time steps, at the end of the avalanche. Size of avalanche is essentially region within $+$'s.

Our simulations were performed on a system of size 256^2. We averaged our data over 100 independent configurations, and each configuration was perturbed 100 000 times. The duration of the avalanche was calculated in a similar fashion to Bak and Tang.[2] The time t was incremented every time a new nearest-neighbor generation is reached, so that t could be thought of as the linear dimension of the avalanche. The size s was measured by counting the number of sites affected by an avalanche. No matter how many times a site has been touched, it was given a weight of one in the calculation of the size of the avalanche. There is thus a subtle difference between the definition of our sizes and the sizes defined elsewhere.[2] The dynamics of the model have led us to believe that this is the correct quantity to consider, especially when using MCRG methods: occasionally sites increase and decrease their values many times during an avalanche. Indeed, after an avalanche has taken place, it leaves the critical state almost unchanged; comparing the initial and the final configurations, one often finds appreciable changes only at the boundaries of the avalanche. We suspect this is due to the combination of the criticality of the system and the conservation law which is present. Figure 1 shows the effect.

To investigate the degree of universality in the self-organized critical state, different lattice structures, impurities, and varying initial conditions were studied. For a hexagonal lattice, there were no detectable differences in the exponents, as one would expect. The effects of impurity were incorporated by breaking bonds randomly in the lattice such that no redistribution of slope occurs across those bonds. Again, the critical exponents did not seem to change for up to $\frac{1}{3}$ of the bonds being broken, confirming previous results.[2] We varied initial conditions by considering a state with the same slopes at the critical state (with, say, 44% sites at the critical value), but randomly dispersed. These random configurations were probed as before, but now after each avalanche we restored the system to its initial configuration to prevent the system from relaxing to the true critical state. Surprisingly enough, we obtained roughly the same effective τ for the size distribution, although the effective α for the growth law was 10% too small, as compared to the values from the critical state. This indicates that the differences between the random configurations and the critical state are rather subtle, and that there is little spatial organization in the critical state. It would be interesting to probe this spatial organization in more detail.

We performed MCRG calculations using the size of the avalanches as the renormalized quantity. After perturbing the critical state, we studied the subsequent relaxation of the critical state by applying a majority-rule transformation to groups of $b^d=2^d$ sites, letting the sites vote whether they participated in the avalanche. In the case of a tie, the choice was made at random. Then the renormalized size of the avalanche was calculated on the new lattice. We analyzed the data by simply fitting to find exponents in the renormalized configurations and particularly by using a matching criterion to find α.[11–15] After the irrelevant variables have been iterated away, the probability distribution function will remain invariant under further renormalization-group transformations. We expect that, after a finite number of iterations, contributions from the irrelevant variables will be negligible, so that any quantity determined after m blockings of a system of size N should be identical to those determined after, say, $m+1$ blockings of a system of size Nb^d. However, since the larger lattice has been renormalized once more, quantities will be at different times t and t'. Hence, close to the fixed point, one can expect a matching condition to hold, $s(N,m,t)=s(Nb^d,m+1,t')$, from which the time rescaling factor t'/t can be calculated, and the exponent α obtained, since $t'/t=b^{d/\alpha}$. Thus

$$\alpha=\frac{d\log_{10}2}{\log_{10}t/t'} . \tag{5}$$

Simply fitting to the distribution of sizes gives

$$\tau=2.06\pm0.05 , \tag{6}$$

independent of the level m of renormalization. We applied the matching criterion to a system of size 128^2 renormalized m times and one of size 256^2 renormalized $m+1$ times. The results are shown in Fig. 2. The value obtained for the time dependence of s is

$$\alpha=1.54\pm0.02 . \tag{7}$$

The growth exponent α is relatively insensitive to m, which confirms that we are close to the fixed point.

Our results are reasonably close to those of Ref. 2. As mentioned above, Hwa and Kardar[10] have proposed a continuum model they conjecture shares the same universality class as that of the sandpile model. Their model, which they solve by a dynamic renormalization group, has an upper critical dimension $d_c=4$. In particular, for $d\le4$, they give $\tau=(11+d)/6$ and $\alpha=2$. While their value for τ is close to ours, we do not find agreement with their α. Presumably, this is because their results apply to a different universality class.

B. Three- and four-dimensional models

The algorithm for the three-dimensional system is a straightforward generalization of Eq. (1) above. The typical size of our system was 64^3. Because the number of possible values for a site increases with dimension, which in turn decreases the chance of obtaining large avalanches, the number of steps per independent configuration was increased to 500 000.

The same MCRG method discussed above was used. The renormalized data for $s(t)$ are shown in Fig. 3, which are not as good as those for the two-dimensional system, due to longer transients. Indeed, even when the system is close to its asymptotic regime, the convergence toward the limiting value is still slower than in $d=2$. In any case, we estimate

$$\tau=2.33\pm0.06 \tag{8}$$

and

$$\alpha=1.78\pm0.06 \tag{9}$$

for $d=3$.

Our study of the four-dimensional sandpile model was done as above. The typical size of our system was 50^4 and each independent configuration was perturbed 1 000 000 times. Because of limited computer memory, we could not use the MCRG method to obtain the value of the growth coefficient. Nevertheless, we report direct estimates of τ and α. For $d=4$, we find

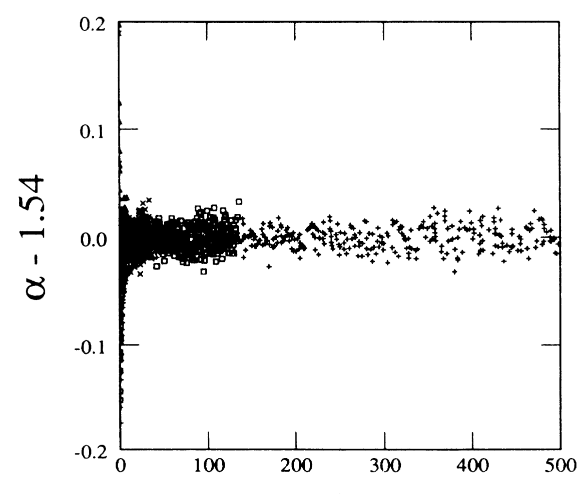

FIG. 2. Matching for α, plotted vs size s for 256^2 and 128^2 systems. Large (small) system blocked 1 (2), 2 (3), 3 (4), and 4 (5) times, shown by +, □, ×, and △, respectively. Smallest systems compared are 8^2. Results shown cannot be distinguished, though 0 (1) matching, which is not shown, gives different results.

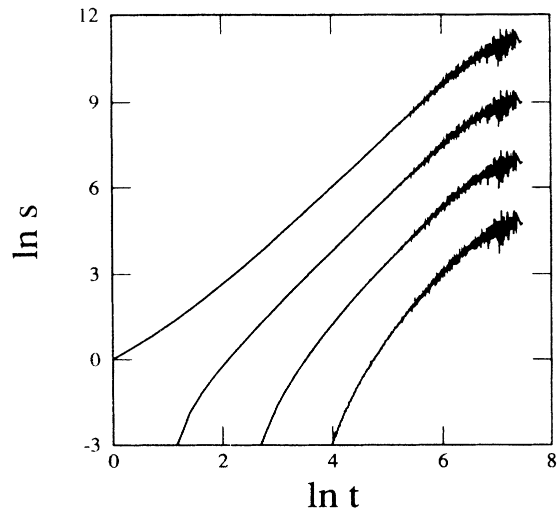

FIG. 3. Results for renormalized $s(t)$ for 64^3 system, blocked 0, 1, 2, and 3 times, from top to bottom.

$$\tau = 2.5 \pm 0.15 \tag{10}$$

and

$$\alpha = 1.9 \pm 0.15 \; . \tag{11}$$

These are consistent with Tang and Bak's[3] mean-field results, which Hwa and Kardar[10] recover at the upper critical dimension ($d_c = 4$) of their model. Although our results are to relatively poor accuracy, they suggest, in agreement with Obukhov,[9] that this is the upper critical dimension of the sandpile model. To provide a convincing demonstration of $d_c = 4$, however, requires higher accuracy in $d = 4$, as well as study of $d > 4$.

III. CONCLUSIONS

In conclusion, we have presented the first extensive MCRG calculations for the sandpile model. The fact that the renormalization group quickly converges to a growth exponent is quite consistent with the existence of a self-organized critical state in the model, and shows the power of the dynamic MCRG method. The numerical results of Bak, Tang, and Wiesenfeld[1–3] are consistent with ours, although ours are to higher accuracy. However, those of the conjecture of Ref. 10 are not, as discussed above. Nevertheless, as the dimensionality of the system increases towards $d = 4$, the exponents converge to those of the model considered by Hwa and Kardar, indicating the possibility that the models share an upper critical dimension. In the future, we shall study the criteria which determine universality classes in self-organized critical phenomena.

ACKNOWLEDGMENTS

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