THREE-BODY FORCES

IN

HYPERNUCLEI

by

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TABLE of CONTENTS

TABLE of CONTENTS (cont'd)

SUMMARY

An investigation is made to ascertain the relative importance of three and two-body Λ-nucleon potentials which have the same order of coupling constant. The three-body potential, which is derived using perturbation theory from a pseudoscalar interaction, is found to be weakly singular and hence much smaller than the two-body potential. In addition, the largest terms are non-central.

In the hypertriton where correlations are large, it is found that three-body forces are repulsive for S states and may amount to 4%. D state admixture tends to make the three-body force less repulsive.

In heavy hypernuclei, three-body forces are found to have a relative importance of about .5% if correlations are neglected, and may be as large as 5.5% if correlations are as strong as in the hypertriton. In either case, they are repulsive.

It is also found that, by omitting the Feynman graphs with bare lines, the three-body contributions are negligible.

The material in this thesis is the original work of the author unless otherwise explicitly stated.

Section I

INTRODUCTION

A. Introduction

It is now well known that the unstable fragments which are observed in many types of nuclear disintegrations may be interpreted by assuming that they are due to a Λ particle which remains bound in the nuclear fragment. Such nuclei are called hypernuclei. The Λ particle decays by two modes

$$\Lambda \rightarrow \begin{cases} p + \pi^- + 36.9 \text{ Mev} \\ n + \pi^0 \end{cases} \qquad 1.1$$

with a mean life $\sim 3 \times 10^{-10}$ seconds. This time is long compared to atomic times and very long compared to characteristic nuclear times.

Hypernuclei are observed in cosmic-ray emulsions. A typical event taken from Schneps et al (1) is shown in Fig. 1.1.

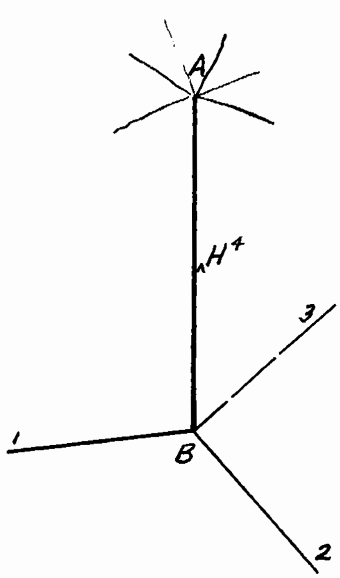


Fig. 1.1

Here the hyperfragment ${}_\Lambda H^4$ is observed as a result of the cosmic-ray star at A. The ${}_\Lambda H^4$ decays at B with a proton 1, triton 2 and a π^- meson 3. One can therefore write this reaction as

$${}_\Lambda H^4 \rightarrow p + t + \pi^- + Q \qquad 1.2$$

Q is measured to be $35.1 \pm .3$ Mev. The particles and their energies are obtained from the length of the tracks, the grain densities, and by a momentum balance.

From equation 1.1 and 1.2 we deduce that the separation energy of the Λ particle is 1.8 Mev. The binding energy and separation energy of the Λ particle are taken throughout this thesis to mean the same thing. The average binding energies (separation energies) of the Λ particle in some hypernuclei are shown in Table 1.1 .

The situation becomes very complicated for $Z > 5$ since then the decay products become very complex. Also, the binding energy of the Λ particle becomes comparable to 37 Mev, so that there is no characteristic energy to look for and the π meson emitted may be absorbed by another nucleon so that it leaves no track. In addition, the heavier Σ particle no doubt plays an important role in the heavier hypernuclei, since the mass difference between the Σ and Λ (80 Mev) becomes more comparable to the binding energy. However, for very light hypernuclei the Λ particle will play a more important role, except for hyperfragments such as the Σ^{+}- proton (2).

Hyperfragment	B_Λ
${}_\Lambda H^3$	-0.3 ± 0.4
${}_\Lambda H^4$	1.8 ± 0.4
${}_\Lambda He^4$	1.9 ± 0.4
${}_\Lambda He^5$	1.6 ± 0.6
${}_\Lambda Li^6$	6.8 ± 3.0
${}_\Lambda Be^8$	5.1 ± 4.0
${}_\Lambda Be^9$	6.3 ± 0.6
${}_\Lambda C^{11}$	13 ± 6

Taken from Schneps et al (1)

Table 1.1

Various authors (3, 4) have tried to calculate the binding energies of the Λ particle for a few specific cases, but in all of the early attempts the binding was attributed to the exchange force between the Λ particle and the nucleon through the K meson, that is,

$$N \rightarrow K + \Lambda \quad \text{or} \quad \Lambda \rightarrow \bar{K} + N \qquad 1.3$$

Dalitz (5) pointed out the possibility that the hypernuclear force could be brought about through the π meson. This would mean that the lowest order process would be a double pion exchange. The possibility of a single pion exchange

$$\Lambda \rightarrow \Lambda + \pi \qquad 1.4$$

may be ruled out since the isotopic spins of the Λ and nucleon are 0 and $\frac{1}{2}$ and the force resulting from 1.4 would not be charge independent. In fact, the Λ-proton and Λ-neutron forces would be opposite in sign in violation with experimental evidence (see Table 1.1). Therefore, one must consider the Λ-nucleon force as being due to the reaction

$$\Lambda \rightarrow \Lambda + \pi^{\pm}_{0} + \pi^{\mp}_{0} \qquad 1.5$$

in the lowest order process.

Calculations of the Λ-nucleon potential for the double pion exchange have been carried out by Dallaporta and Ferrari (6) (hereon this paper will be referred to by the letters DF) and by Lichtenberg and Ross (7). The elementary reactions are taken as

$$\begin{aligned} N &\rightarrow N + \pi \\ \Lambda &\rightarrow \Sigma + \pi \\ \Sigma &\rightarrow \Lambda + \pi \end{aligned} \qquad 1.6$$

The interaction terms of the Hamiltonian may be written (8) as

$$\begin{aligned} H_{int} &= ig \int \bar{\Psi}_N \gamma_5 \tau_i \Psi_N \phi_i \, d\tau \\ &+ \left[ig_\Lambda \int \bar{\Sigma}_i \gamma_5 \Lambda \phi_i \, d\tau + \text{Herm. Conjugate} \right] \end{aligned} \qquad 1.7$$

if one assumes pseudoscalar coupling.

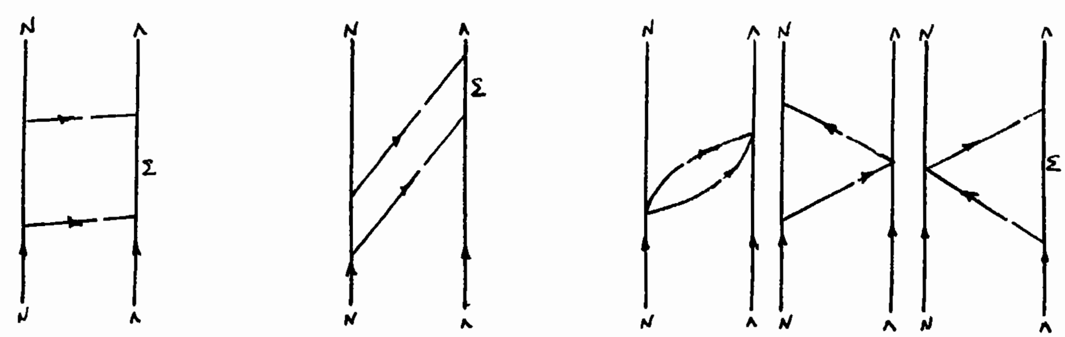
where the repeated index i denotes summation and ψ_N, Σ and Λ are the baryon wave functions and ϕ is the wave function of the π meson.

The work of Ruderman and Karplus (9) indicates that the spin of the Λ particle is $\leq 3/2$. We assume that the spins of the Λ and Σ are $\frac{1}{2}$ and that they have the same relative parity according to the "global" theory (8). Further, we also investigate 1.7 in the non-relativistic limit and setting $M_\Sigma = M_\Lambda$ the lowest order terms of the Dyson (10) transformation give

$$H_{int} = \frac{g}{2m}\int d^3x\, \bar{\psi}_N \tau_i \bar{\sigma} \psi_N \cdot \bar{\nabla}\phi_i + \left[\frac{g_\Lambda}{2m_\Lambda}\int d^3x\, \bar{\Sigma}_i \cdot \bar{\sigma}\Lambda \cdot \bar{\nabla}\phi_i + \text{conjugate}\right]$$

$$+ \frac{g^2}{2m}\int d^3x\, \bar{\psi}_N \psi_N \phi^2 + \frac{g_\Lambda^2}{2m_\Lambda}\int d^3x\, \bar{\Lambda}\Lambda\phi^2 \qquad 1.8$$

This interaction may be schematically represented by means of Feynman graphs as shown in Fig. 1.2, where a and b show the first two terms of 1.8 , and c shows the pair terms.



(a) (b) (c)

Fig. 1.2

In the analyses of Lichtenberg and Ross (7) only graphs with bare nucleon lines were considered, that is, graphs corresponding to (a) in Fig. 1.2 . Their calculation led to the spin triplet state as lower than the singlet for the bound Λ-nucleon problem. The recent work of Dalitz and Downs (11) shows that the singlet spin state is more attractive. Dallaporta and Ferrari considered graphs corresponding to Fig. 1.2 (b) and (c), as well as exchanges with the K meson. They found that the potential from graphs (b) dominated the pair terms and the K meson exchange terms, and that graphs (b) led to a potential more attractive in the Λ-nucleon singlet spin state, which indicates that it is probably a more appropriate potential. They also assumed that the pair terms were damped by a factor λ according to Brueckner and Watson (12), and neglected them entirely in the subsequent analysis. They also did not consider bare nucleon lines. The potential resulting from (b) was

$$V_{DF} = -\mu \frac{g^2}{4\pi}\frac{\mu^2}{4m^2}\frac{g_\Lambda^2}{4\pi}\frac{\mu^2}{4m_\Lambda^2}\cdot\frac{2}{\pi x^3}\Big\{3\Big((4+4x+x^2)\frac{e^{-x}}{x}K_1(x)+(2+2x+x^2)e^{-x}K_0(x)\Big)$$

$$-2\vec{\sigma}_1\cdot\vec{\sigma}_\Lambda\Big(6K_0(2x)+(6+4x^2)\frac{K_1(2x)}{x}-(1+x)e^{-x}K_0(x)-(2+2x+x^2)\frac{e^{-x}}{x}K_1(x)\Big) \qquad 1.9$$

$$+\frac{1}{3}S_{1\Lambda}\Big(36K_0(2x)+(45+12x^2)\frac{K_1(2x)}{x}-3\big[(1+x)e^{-x}K_0(x)+(5+5x+x^2)\frac{e^{-x}}{x}K_1(x)\big]\Big)\Big\}$$

where μ is the meson mass which equals 140 Mev, and x equals $\mu r_{1\Lambda}$, the distance between the nucleon and the Λ-particle in units of a meson compton wavelength. Here $\hbar$ = c = 1 . The K functions are Hankel functions of an imaginary argument where $K_1(x) \rightarrow \frac{1}{x}$ for small x and $K_0(x) \rightarrow \ln\frac{2}{\gamma x}$ where γ is Euler's constant. (See Jahnke and Emde.)

Lichtenberg and Ross obtained

$$V_{LR} = -\mu\frac{g^2}{4\pi}\frac{\mu^2}{4m^2}\frac{g_\Lambda^2}{4\pi}\frac{\mu^2}{4m_\Lambda^2}\cdot\frac{3\mu}{\Delta}\frac{e^{-2x}}{x^2}\Big\{\Big(1+\frac{4}{x}+\frac{10}{x^2}+\frac{12}{x^3}+\frac{6}{x^4}\Big)$$

$$+\frac{2}{3}\vec{\sigma}_1\cdot\vec{\sigma}_\Lambda\Big(\frac{2}{x}+\frac{5}{x^2}+\frac{6}{x^3}+\frac{3}{x^4}\Big)-\frac{1}{3}S_{1\Lambda}\Big(\frac{2}{x}+\frac{8}{x^2}+\frac{12}{x^3}+\frac{6}{x^4}\Big)\Big\} \qquad 1.10$$

where $\Delta \equiv M_{\Sigma} - M_{\Lambda} = 80$ Mev .

It is interesting to notice that V_{LR} given by 1.10 behaves like $\frac{1}{x^6}$ for small x , while V_{DF} behaves like $\frac{1}{x^5}$.

In both of these calculations, the baryon recoil was neglected, and their wave functions assumed to be delta functions. In the intermediate states, the mass difference Δ was neglected in comparison with ω_k, the meson energy, if a meson was present. The factor $1/\Delta$ appears in V_{LR} since in the bare-nucleon intermediate state, the energy denominator is Δ (no meson is present) .

The calculation performed by DF was carried out further than that of Lichtenberg and Ross. After deriving the potential V_{DF} given by 1.9 , they constructed an approximate model for the hypernucleus as consisting of a nuclear core together with a rather loosely bound Λ-particle. The hypertriton, for example, was assumed to consist of a deuteron with a Λ-particle where the deuteron part of the hypernucleus retains its characteristic shape. Actually, the presence of the Λ-particle will deform the deuteron. Estimates of the distortion have been made by Brown and Peshkin (13) using a phenomenological Λ-nucleon potential and it was found that the linear distortion of the deuteron is about 10% .

DF then assumed a spherically-symmetric density function for the nuclear core $\rho(|\bar{x}|)$ and then calculated the effective nuclear core potential

$$V_{eff}(R) = \int \bar{V}(|\bar{R}-\bar{x}|)\,\rho(|\bar{x}|)\,d^3x \qquad 1.11$$

where $\bar{R}$ is the radius vector from the Λ to the centre of mass of the nuclear core. $\bar{V}(|\bar{R} - \bar{r}|)$ is the potential given by equation 1.9 plus the contributions from the K meson graphs after spin integrations. The K meson contribution was only singular to first order here and much smaller than V_{DF} since the tensor terms vanished.

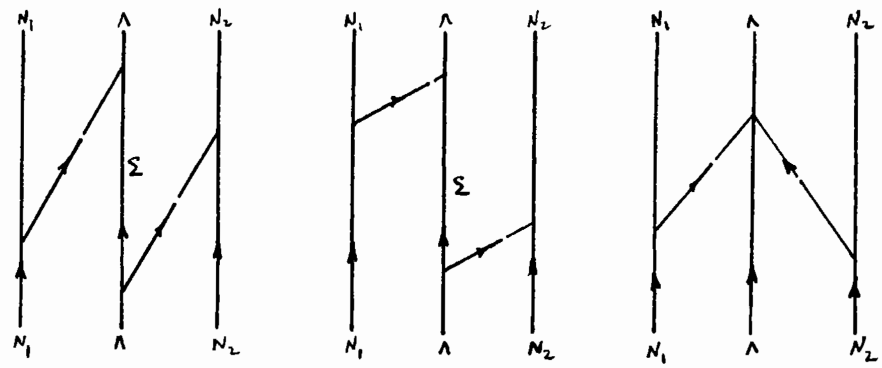
In the evaluation of V_{eff} one cannot integrate over all space since V_{DF} has a singularity of order 5. They therefore assumed that V_{DF} was cut off at $.33 \times 10^{-13}$ cm. or $x = .234$ meson units. By cut off it was meant that $V_{DF} = 0$, $0 \leq x \leq 0.234$. This may be considered as having a compensating effect for the high singularity incurred by neglecting the nucleon recoil. They also assumed that $g = g_{\Lambda} = g_{K}$.

A trial wave function $\psi = e^{-aR}$ was then chosen for the system and the parameter a was varied for minimum binding by the Ritz variational method. The results for the binding energies of the hypernuclei $_{\Lambda}H^2$, $_{\Lambda}H^3$, $_{\Lambda}H^4$, $_{\Lambda}He^4$ and $_{\Lambda}He^5$ gave remarkably good agreement with experiment. One might, however, view the results with a certain distrust since the entire procedure depended very critically on the value of the core radius of V_{DF}. In addition, no attempt was made to correlate the spins and space parts of the Λ-nucleons. Either of these two factors would seriously change the relative magnitudes of the binding energies of the hypernuclei considered by DF .

B. Three-Body Force Considerations

From equation 1.5 , it seems likely that three-body forces may play an important role in hypernuclei. In ordinary nuclei, many-body forces are believed to be small. Estimates have been made by Drell and Huang (14) who find that the many-body forces may be important for nuclear saturation. However, a later calculation by Brueckner et al (15) has shown that three-body nuclear forces are probably negligible. One reason for the smallness of many-body forces in ordinary nuclei may be thought of in terms of perturbation theory. Three-body forces require at least fourth-order perturbation theory, which is the next order above the lowest, that is, second order perturbation theory or single pion exchange.

In hypernuclei the situation is different. The lowest order two-body process requires two pions exchanged, or fourth-order perturbation theory, but this is just the order required for a three-body force.



(a) (b) (c)

Fig. 1.3

The diagrams shown in Fig. 1.3 all lead to three-body forces. In ordinary nuclei one could consider a graph such as (b) in Fig. 1.3 as an iteration of two-body processes. However, for Λ-nucleon forces, since the lowest order process is a double-pion exchange, no combination of two-body force diagrams leads to any of the graphs in Fig. 1.3 . We therefore call these graphs three-body force diagrams. From this we see that it is possible that three-body forces could be large enough in hypernuclei to effect the binding energy of the Λ-particle, and, if we accept the "global" concept $g_\Lambda = g$, it may also effect the value of the core in the potential. For the heavier hypernuclei, one might also expect that its effect on saturation would be important.

C. Outline of the Work

In Section II the three-body potential is derived considering all three graphs in Fig. 1.3 . It was found that the potential due to (a) was only singular to the order $\frac{1}{x^3}$ for small x and that with no correlations the expectation value was negligible. The potential resulting from the bare lines (b) was found to be singular to the same order as the graphs (a) , but was larger in magnitude. The potential resulting from the pair terms was only singular to $\frac{1}{x^2}$ for small x . In addition, with no correlations the expectation value of the latter potential was zero, and since it was assumed to be multiplied by the damping factor λ (12) it was neglected in all of the subsequent analyses.

Without correlations in the wave functions the expectation value of the three-body potential was small. It was decided, therefore, to make a quantitative calculation on the hypertriton with correlations included so that a reasonable estimate could be made of the relative importance of the three-body to the two-body hypernuclear force. The procedure adopted for this calculation was as follows:

(1) A trial wave function

$$\psi = \chi_S e^{-\frac{\alpha}{2}(x+y) - \frac{\beta}{2}\rho} \qquad 1.12$$

was chosen where α is the variational parameter, χ_S is the spin wave function

$$\chi_S = -\sqrt{\tfrac{2}{3}}\,\alpha(1)\alpha(2)\beta(\Lambda) + \sqrt{\tfrac{1}{6}}\left(\alpha(1)\beta(2) + \beta(1)\alpha(2)\right)\alpha(\Lambda)$$

which is a spin wave function for $J = \frac{1}{2}$, $M_J = \frac{1}{2}$ and x, y and ρ are the relative co-ordinates in meson units. This particular spin wave function was chosen so that the two nucleons were in a mutual triplet spin state. The constant β was chosen from other theoretical work (17 and 18).

(2) In Section III, using the trial wave function 1.12, the separation energy of the Λ-particle was calculated considering only two-body forces. The neutron-proton phenomenological potential was taken from Gammel and Thaler (16), and the two-body Λ-nucleon potential was assumed to be given by V_{DF} (equation 1.9). The expectation value of V_{DF} depends on the parameters α and β and also on the core ϵ. With β, the deuteron part of the wave function fixed, a variation of α will not necessarily give the correct separation energy B_Λ of the Λ-particle at the minimum, which is experimentally known to be near zero. It was found, in fact, that when using $\epsilon = 0.234$, which is the value used by DF, a variation in α yielded a value for $|B_\Lambda|$ which was greater than 60 Mev. This clearly showed the inadequacy of the treatment by DF. The difference in these two results can be explained by noting that DF introduced no correlations, and in the case of ${}_\Lambda H^3$, for example, they assumed that the Λ-particle interacted in a pure singlet state with each nucleon, which is certainly not true for the spin wave function χ_s (see equation 1.12).

Therefore, the core ϵ was varied until the minimum value of $B_\Lambda(\epsilon, \alpha)$ with respect to α occurred at $B_\Lambda = 0$.

(3) The values of α, β and ϵ, which were calculated in Section III, were then used to calculate the expectation value of the three-body potential in Section IV. The same core ϵ, as was determined for the Λ-nucleon two-body potential V_{DF}, was used for the three-body potential. When calculating the three-body expectation values, an admixture of D state was included in the hypertriton wave function to determine the sensitivity of the three-body potential to amounts of D state. This was done by writing the total wave function as

$$\psi = \cos\phi\,\psi_S + \sin\phi\,\psi_D \qquad 1.13$$

where ψ_S and ψ_D are normalized and the percentage D state is given by $100 \times \sin^2\phi$. ψ_S was taken to be

$$\frac{\chi_s}{\sqrt{N_S}}\, e^{-\frac{\alpha}{2}(x+y)-\frac{\beta}{2}\rho} \qquad \text{(see 1.12)}$$

where N_S is the normalizing constant and ψ_D was taken as

$$\psi_D = \frac{1}{\sqrt{N_D}}\left(x^2 S_{1\Lambda,x} + y^2 S_{2\Lambda,y}\right)\chi_s\, e^{-\frac{\alpha}{2}(x+y)-\frac{\beta}{2}\rho} \qquad 1.14$$

Here N_D is the normalizing constant and

$$S_{1\Lambda,x} = \frac{3(\bar{\sigma}_1\cdot\bar{x})(\bar{\sigma}_\Lambda\cdot\bar{x})}{x^2} - \bar{\sigma}_1\cdot\bar{\sigma}_\Lambda$$

$$S_{2\Lambda,y} = \frac{3(\bar{\sigma}_2\cdot\bar{y})(\bar{\sigma}_\Lambda\cdot\bar{y})}{y^2} - \bar{\sigma}_2\cdot\bar{\sigma}_\Lambda \qquad 1.15$$

Other D state wave functions are possible (19) but it was assumed that the symmetric one between the nucleons would give the largest contribution. The ranges α and β were chosen the same for the D state and S state wave functions. α was taken from the result of the variational calculation using only two-body potentials.

In Section V, the ratios of the expectation value of the three-body potentials $\langle V(3) \rangle$ with arbitrary admixtures of D state was made with the expectation value of the two-body potential $\langle V_{DF}(2) \rangle$ where the value of $\langle V_{DF}(2) \rangle$ was taken from the results of Section III, that is, with only the S state wave function given by 1.12. These ratios were considered for both the three-body potential derived from diagrams with no bare lines (Fig. 1.3 (a)) and also when including the bare line diagrams (Fig. 1.3 (a) and (b)). The pair terms (Fig. 1.3 (c)) were not included. It was found that for small D admixtures, the ratio considering only Fig. 1.3 (a) was only about .2%. When Fig. 1.3 (a) and (b) were considered, the ratio became about 4%. In both cases, the three-body contribution was repulsive. A 4% contribution would be significant in any precise calculation, since the value of the core would be changed and this in turn would affect the relative binding of hyperfragments significantly.

In Section VI an investigation was made into the variation of the values of $\langle V(\text{3-body}) \rangle$ with small changes in the constants α, β and ϵ. This was done in the event that the value chosen for β did not correspond to the best value or that the form of the wave function was deficient.

Finally in Section VII a semi-quantitative calculation was made on a heavy hypernucleus to determine the relative importance of three-body forces. This was done since in the hypertriton the Λ-particle is quite loosely bound and three-body forces would be expected to be quite small. It was found, however, that with no correlations (one expects correlations to decrease as A increases), the ratio of the expectation values of the three-body to the two-body Λ-nucleon potentials was only about -.005. The method used for this calculation followed Drell and Huang (14). The reason for the smallness of three-body forces in hypernuclei can be seen if one examines the potentials given in Section II. The angular dependent parts of the potential are more strongly singular than the central parts. In the case of the hypertriton, correlations lead to contributions from these more singular parts of the potential, and in heavy hypernuclei the Λ- particle interacts with many pairs and the effect of correlations may be expected to be smeared out. If strong correlations are assumed for heavy hypernuclei, then repulsive three-body contributions in excess of 5% can be expected.

Section II

DERIVATION OF THE THREE-BODY POTENTIAL

A. The Perturbation Terms

Graphs of the type in Fig. 1.3 (a) (no bare lines) were first considered. There are sixteen possible time orderings that can occur which are shown in Fig. 2.1 . One therefore writes

$$V(3\text{-body}) = \sum_{I,I',I''} \frac{\langle F|H_i|I\rangle\langle I|H_i|I'\rangle\langle I'|H_i|I''\rangle\langle I''|H_i|0\rangle}{(E_0-E_I)(E_0-E_{I'})(E_0-E_{I''})} \qquad 2.1$$

where E_0 is the total energy of the system, i.e. $2M + M_\Lambda$, 0, I'', I', I and F are the original, three intermediate and final states. The sums are taken over all intermediate spins, isotopic spins and momenta. Actually, since the baryon recoil is neglected the state F and 0 are the same.

We shall now compute V(3-body) in detail for the diagram in Fig. 2.2 . The interaction Hamiltonian is given by 1.8 . The meson function ϕ can be written as

$$\phi = \sum_k \frac{1}{\sqrt{2V\omega_k}} \left(a_{k,\lambda} e^{i\bar{k}\cdot\bar{r}} + a^{*}_{k,\lambda} e^{-i\bar{k}\cdot\bar{r}}\right) \qquad 2.2$$

where the a's are the usual creation and destruction operators for mesons of momentum $\bar{k}$ and isotopic spin component λ , and ω_k is the meson energy $\sqrt{\mu^2 + k^2}$. Here, of course, $\hbar = c = 1$ and

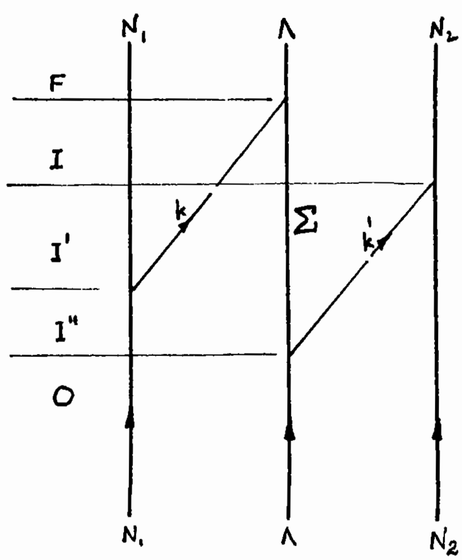
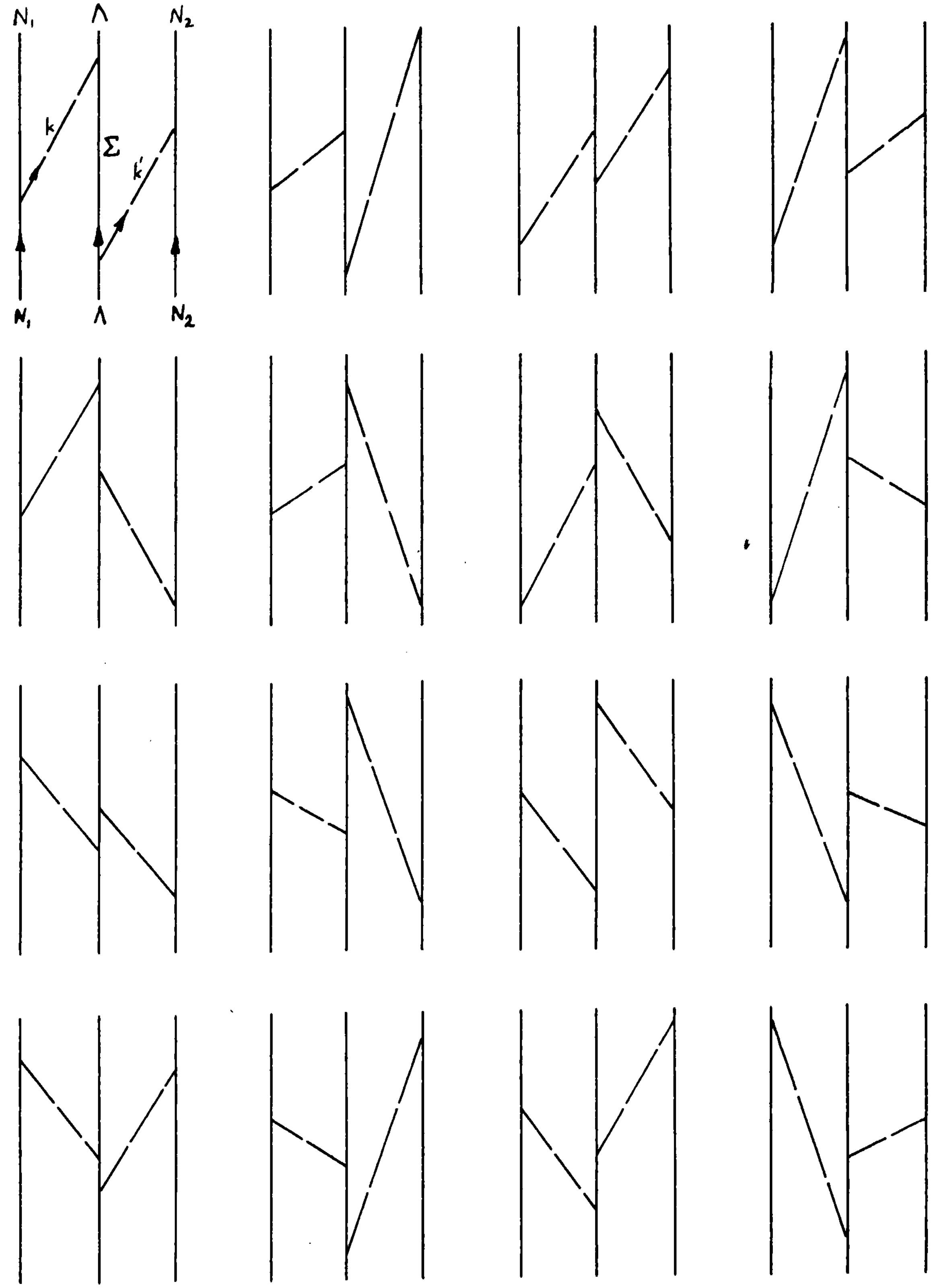


Fig. 2.2



The Three-Body Graphs With No Bare Lines

FIG. 2.1

is the meson mass.

If the baryon wave functions are assumed to be delta functions, then the gradient coupling part of 1.8 becomes

$$H_i = \sum_{k,\lambda} \frac{ig}{2m} \tau_\lambda \frac{\bar{\sigma}\cdot\bar{k}}{\sqrt{2\omega_k V}} \left(a_{k\lambda} e^{i\bar{k}\cdot\bar{r}} - a^*_{k,\lambda} e^{-i\bar{k}\cdot\bar{r}}\right)$$

$$+ \frac{g_\Lambda}{2m_\Lambda}\left[\sum_{k,\lambda} i\,\Sigma^*_\lambda \frac{\bar{\sigma}\cdot\bar{k}}{\sqrt{2\omega_k V}} \left(a_{k,\lambda} e^{i\bar{k}\cdot\bar{r}} - a^*_{k,\lambda} e^{-i\bar{k}\cdot\bar{r}}\right) + H.c.\right] \qquad 2.3$$

where Σ_λ denotes the isotopic spin part of the Σ particle.

For Fig. 2.2, $E_o - E_{I''} = -\omega_{k'}$, $E_o - E_{I'} = -(\omega_k + \omega_{k'})$ and $E_o - E_I = -\omega_k$. Note that $E_o - E_{I''} = -\omega_{k'} - \Delta$ where $\Delta = M_\Sigma - M_\Lambda$. We assume that $\omega_{k'} \gg \Delta = 80$ Mev and neglect it in all of the diagrams of Fig. 2.1 . Then we get

$$\langle I'' | H_i | 0 \rangle = \frac{-ig_\Lambda}{2m_\Lambda} \Sigma^*_\lambda e^{-i\bar{k}'\cdot\bar{r}_\Lambda} \frac{(\bar{\sigma}_\Lambda\cdot\bar{k}')}{\sqrt{2\omega_{k'} V}}$$

$$\langle I' | H_i | I'' \rangle = \frac{-ig}{2m} \tau_{1\kappa} e^{-i\bar{k}\cdot\bar{r}_1} \frac{(\bar{\sigma}_1\cdot\bar{k})}{\sqrt{2\omega_k V}} \qquad 2.4$$

$$\langle I | H_i | I' \rangle = \frac{ig}{2m} \tau_{2\lambda} e^{i\bar{k}'\cdot\bar{r}_2} \frac{(\bar{\sigma}_2\cdot\bar{k}')}{\sqrt{2\omega_{k'} V}}$$

$$\langle F | H_i | I \rangle = \frac{ig_\Lambda}{2m_\Lambda} \Sigma_\kappa e^{i\bar{k}\cdot\bar{r}_\Lambda} \frac{(\bar{\sigma}_\Lambda\cdot\bar{k})}{\sqrt{2\omega_k V}}$$

We now multiply these equations together, divide by the energy denominators and sum over all intermediate state quantum numbers. The result is

$$V(\text{3-body}) = -\sum_{k,k'} \frac{g^2}{4M^2}\frac{g_\Lambda^2}{4M_\Lambda^2} \frac{(\bar{\Sigma}\cdot\bar{\tau}_1)(\bar{\Sigma}'\cdot\bar{\tau}_2)(\bar{\sigma}_\Lambda\cdot\bar{k})(\bar{\sigma}_2\cdot\bar{k}')(\bar{\sigma}_1\cdot\bar{k})(\bar{\sigma}_\Lambda\cdot\bar{k}')}{4\omega_k\omega_{k'}V^2} \times$$

$$\times \frac{e^{i[\bar{k}\cdot(\bar{r}_\Lambda-\bar{r}_1)+\bar{k}'\cdot(\bar{r}_2-\bar{r}_\Lambda)]}}{\omega_k\omega_{k'}(\omega_k+\omega_{k'})}$$

$$= -\frac{g^2}{4M^2}\frac{g_\Lambda^2}{4M_\Lambda^2}\frac{1}{4V^2}\sum_{k,k'} \frac{(\bar{\tau}_1\cdot\bar{\tau}_2)(\bar{\sigma}_\Lambda\cdot\bar{k})(\bar{\sigma}_2\cdot\bar{k}')(\bar{\sigma}_1\cdot\bar{k})(\bar{\sigma}_\Lambda\cdot\bar{k}')}{\omega_k^2\omega_{k'}^2(\omega_k+\omega_{k'})} \times$$

$$\times e^{i[\bar{k}\cdot(\bar{r}_\Lambda-\bar{r}_1)+\bar{k}'\cdot(\bar{r}_2-\bar{r}_\Lambda)]}$$

The $\sum_\lambda$ parts of the isotopic spin portion reduce to unity in the same manner as the meson isotopic spin parts. (See Bethe and de Hoffman Vol. II, p. 58) We now put $\bar{r}_1 - \bar{r}_\Lambda = \bar{r}$ and $\bar{r}_2 - \bar{r}_\Lambda = \bar{r}'$ and replace $\frac{1}{V}\sum_k$ by $\frac{1}{(2\pi)^3}\int d^3\bar{k}$
so that

$$V(\text{3-body}) = -\frac{g^2}{4M^2}\frac{g_\Lambda^2}{4M_\Lambda^2}(\bar{\tau}_1\cdot\bar{\tau}_2)\int(\bar{\sigma}_\Lambda\cdot\bar{k})(\bar{\sigma}_2\cdot\bar{k}')(\bar{\sigma}_1\cdot\bar{k})(\bar{\sigma}_\Lambda\cdot\bar{k}') \times$$

$$\times \frac{e^{i\bar{k}\cdot\bar{r}+i\bar{k}'\cdot\bar{r}'}}{\omega_k^2\omega_{k'}^2(\omega_k+\omega_{k'})} d^3\bar{k}\, d^3\bar{k}' \qquad 2.5$$

We notice that the integrand of 2.5 is invariant if we put $\bar{k} \to -\bar{k}$ or if we put $\bar{r} \to -\bar{r}$ which is why the phase factor was put into the form as shown.

Evaluation of the fifteen other diagrams in Fig. 2.1 leads to the same phase factor but different energy denominators and different ordering of the two Λ spin parts. The orderings of $\bar{\sigma}_1$, $\bar{\sigma}_2$ and $\bar{\sigma}_\Lambda$ do not matter since they operate in different spaces. Therefore, apart from the phase factor, the integrand for the diagrams in

Fig. 2.1 is

$$2(\bar{\sigma}_1 \cdot \bar{k})(\bar{\sigma}_2 \cdot \bar{k}')\left[(\bar{\sigma}_\Lambda \cdot \bar{k})(\bar{\sigma}_\Lambda \cdot \bar{k}') + (\bar{\sigma}_\Lambda \cdot \bar{k}')(\bar{\sigma}_\Lambda \cdot \bar{k})\right]\left(\frac{1}{\omega_k^3 \omega_{k'}^2} + \frac{1}{\omega_k^2 \omega_{k'}^3}\right) \qquad 2.6$$

The three-body potential with no bare nucleon lines is therefore given by

$$V_{NB} = -\frac{g^2}{4m^2} \cdot \frac{g_\Lambda^2}{4m_\Lambda^2} \frac{(\bar{\tau}_1 \cdot \bar{\tau}_2) \cdot 2}{4(2\pi)^6} \int (\bar{\sigma}_1 \cdot \bar{k})(\bar{\sigma}_2 \cdot \bar{k}')\left[(\bar{\sigma}_\Lambda \cdot \bar{k})(\bar{\sigma}_\Lambda \cdot \bar{k}') + \right.$$

$$\left. + (\bar{\sigma}_\Lambda \cdot \bar{k}')(\bar{\sigma}_\Lambda \cdot \bar{k})\right]\left(\frac{1}{\omega_k^3 \omega_{k'}^2} + \frac{1}{\omega_k^2 \omega_{k'}^3}\right) e^{i\bar{k}\cdot\bar{r} + i\bar{k}'\cdot\bar{r}'} d^3\bar{k}\, d^3\bar{k}' \qquad 2.7$$

The obvious relation $(\bar{\sigma}_\Lambda \cdot \bar{k})(\bar{\sigma}_\Lambda \cdot \bar{k}') + (\bar{\sigma}_\Lambda \cdot \bar{k}')(\bar{\sigma}_\Lambda \cdot \bar{k}) = 2\bar{k} \cdot \bar{k}'$ was not utilized since the resulting integration becomes more complex. It was found that the form given by 2.7 was more easily handled.

B. Evaluation of the Integrals

First consider

$$\int (\bar{\sigma}_1 \cdot \bar{k})(\bar{\sigma}_\Lambda \cdot \bar{k})(\bar{\sigma}_2 \cdot \bar{k}')(\bar{\sigma}_\Lambda \cdot \bar{k}')\left[\frac{1}{\omega_k^3 \omega_{k'}^2} + \frac{1}{\omega_k^2 \omega_{k'}^3}\right] e^{i\bar{k}\cdot\bar{r} + i\bar{k}'\cdot\bar{r}'} d^3\bar{k}\, d^3\bar{k}'$$

Then take $\bar{r}$ and $\bar{r}'$ as the polar axes of $\bar{k}$ and $\bar{k}'$ and examine the angular part of the $\bar{k}$ integration.

Then

$$(\bar{\sigma}_1 \cdot \bar{k})(\bar{\sigma}_\Lambda \cdot \bar{k}) = |\bar{\sigma}_1||\bar{\sigma}_\Lambda| k^2 (\sin\theta_{\sigma_1}\cos\phi_{\sigma_1}\sin\theta\cos\phi + \sin\theta_{\sigma_1}\sin\phi_{\sigma_1}\sin\theta\sin\phi + \cos\theta_{\sigma_1}\cos\theta) \times$$

$$\times (\sin\theta_{\sigma_{1\Lambda}}\cos\phi_{\sigma_{1\Lambda}}\sin\theta\cos\phi + \sin\theta_{\sigma_{1\Lambda}}\sin\phi_{\sigma_{1\Lambda}}\sin\theta\sin\phi + \cos\theta_{\sigma_{1\Lambda}}\cos\theta)$$

where θ, ϕ ; $\theta_{\sigma_1}, \phi_{\sigma_1}$ and $\theta_{\sigma_{1\Lambda}}, \phi_{\sigma_{1\Lambda}}$ are the polar angles of $\bar{k}$, $\bar{\sigma}_1$ and $\bar{\sigma}_\Lambda$ referred to the direction $\bar{r}$. The integrand does not depend on ϕ so that integration with respect to ϕ gives

$$\pi |\bar{\sigma}_1||\bar{\sigma}_\Lambda| k^2 \Big(\sin^2\theta \sin\theta_{\sigma_1} \cos\phi_{\sigma_1} \sin\theta_{\sigma_{1\Lambda}} \cos\phi_{\sigma_{1\Lambda}}$$

$$+ \sin^2\theta \sin\theta_{\sigma_1} \sin\phi_{\sigma_1} \sin\theta_{\sigma_{1\Lambda}} \sin\phi_{\sigma_{1\Lambda}} + 2\cos^2\theta \cos\theta_{\sigma_1} \cos\theta_{\sigma_{1\Lambda}} \Big)$$

$$= \frac{\pi}{3} k^2 \left(S_{1\Lambda,r} (2 - 3\sin^2\theta) + 2\bar{\sigma}_1 \cdot \bar{\sigma}_\Lambda \right)$$

where

$$S_{1\Lambda,r} = \frac{3(\bar{\sigma}_1 \cdot \bar{r})(\bar{\sigma}_\Lambda \cdot \bar{r})}{r^2} - \bar{\sigma}_1 \cdot \bar{\sigma}_\Lambda$$

The angular integration (ϕ' part) of the $\bar{k}'$ momentum yields a similar term.

Omitting the factor $-\frac{g^2}{4m^2} \cdot \frac{g_\Lambda^2}{4m_\Lambda^2} \frac{(\bar{\tau}_1 \cdot \bar{\tau}_2)}{4(2\pi)^6}$, equation 2.7 becomes

$$\frac{2\pi^2}{9} \iint \Big[\left(S_{1\Lambda,r} (2 - 3\sin^2\theta) + 2\bar{\sigma}_1 \cdot \bar{\sigma}_\Lambda \right) \times$$

$$\times \left(S_{2\Lambda,r'} (2 - 3\sin^2\theta') + 2\bar{\sigma}_2 \cdot \bar{\sigma}_\Lambda \right) + \left(S_{2\Lambda,r'} (2 - 3\sin^2\theta') + 2\bar{\sigma}_2 \cdot \bar{\sigma}_\Lambda \right) \times \tag{2.8}$$

$$\times \left(S_{1\Lambda,r} (2 - 3\sin^2\theta) + 2\bar{\sigma}_1 \cdot \bar{\sigma}_\Lambda \right) \Big] e^{ikr\cos\theta + ik'r'\cos\theta'} \sin\theta \, d\theta \, \sin\theta' \, d\theta' \times$$

$$\times \left(\frac{1}{\omega_k^3 \omega_{k'}^2} + \frac{1}{\omega_k^2 \omega_{k'}^3} \right) k^4 dk \, k'^4 dk'$$

We next make use of the relations

$$
\begin{aligned}
(\bar{\sigma}_1\cdot\bar{\sigma}_n)(\bar{\sigma}_2\cdot\bar{\sigma}_n) &+ (\bar{\sigma}_2\cdot\bar{\sigma}_n)(\bar{\sigma}_1\cdot\bar{\sigma}_n) &= 2\bar{\sigma}_1\cdot\bar{\sigma}_2 \\
(\bar{\sigma}_1\cdot\bar{\sigma}_n)S_{2n,r'} &+ S_{2n,r'}(\bar{\sigma}_1\cdot\bar{\sigma}_n) &= 2S_{12,r'} \\
(\bar{\sigma}_2\cdot\bar{\sigma}_n)S_{1n,r} &+ S_{1n,r}(\bar{\sigma}_2\cdot\bar{\sigma}_n) &= 2S_{12,r} \\
S_{2n,r'}S_{1n,r} &+ S_{1n,r}S_{2n,r'} &= 2S_{12,rr'}-2S_{12,r}-2S_{12,r'}
\end{aligned}
\qquad 2.9
$$

where $S_{12,\rho} = \dfrac{3(\bar{\sigma}_1\cdot\bar{\rho})(\bar{\sigma}_2\cdot\bar{\rho})}{\rho^2} - (\bar{\sigma}_1\cdot\bar{\sigma}_2)$

and $S_{12,rr'} = 9(\bar{r}\cdot\bar{r}')\dfrac{(\bar{\sigma}_1\cdot\bar{r})(\bar{\sigma}_2\cdot\bar{r}')}{r^2r'^2} - (\bar{\sigma}_1\cdot\bar{\sigma}_2)$

Also $\displaystyle\int_0^\pi e^{ikr\cos\theta}\sin\theta\, d\theta = \frac{2\sin kr}{kr}$

and $\displaystyle\int_0^\pi e^{ikr\cos\theta}\sin^3\theta\, d\theta = 4\left(\frac{\sin kr}{(kr)^3} - \frac{\cos kr}{(kr)^2}\right)$

From these relations, equation 2.8 becomes

$$
\begin{aligned}
\frac{192\pi^2}{9}\Bigg[& S_{12,rr'}\left(\frac{3I_{33}}{r^3r'^3} + \frac{3I_{22}}{r^2r'^2} + \frac{I_{11}}{3rr'} - 3\left(\frac{I_{32}}{r^3r'^2} + \frac{I_{23}}{r^2r'^3}\right)\right. \\
& \left. -3\left(\frac{I_{31}}{r^3r'} + \frac{I_{13}}{rr'^3}\right) + \frac{I_{21}}{r^2r'} + \frac{I_{12}}{rr'^2}\right) + S_{12,r}\left(-\frac{3I_{33}}{r^3r'^3} - 3\frac{I_{22}}{r^2r'^2}\right. \\
& \left. +3\left(\frac{I_{32}}{r^3r'^2} + \frac{I_{23}}{r^2r'^3}\right) + \frac{I_{13}}{rr'^3} + \frac{I_{12}}{rr'^2}\right) + S_{12,r'}\left(-\frac{3I_{33}}{r^3r'^3} - \frac{3I_{22}}{r^2r'^2}\right. \\
& \left. +3\left(\frac{I_{32}}{r^3r'^2} + \frac{I_{23}}{r^2r'^3}\right) + \frac{I_{31}}{r^3r'} + \frac{I_{21}}{r^2r'}\right) + (\bar{\sigma}_1\cdot\bar{\sigma}_2)\frac{I_{11}}{3rr'}\Bigg]
\end{aligned}
\qquad 2.10
$$

where

$$\begin{bmatrix} I_{33} & I_{23} & I_{13} \\ I_{32} & I_{22} & I_{12} \\ I_{31} & I_{21} & I_{11} \end{bmatrix} = \begin{bmatrix} 1 & \frac{\partial}{\partial r} & -\frac{\partial^2}{\partial r^2} \\ \frac{\partial}{\partial r'}, & \frac{\partial^2}{\partial r \partial r'}, & -\frac{\partial^3}{\partial r^2 \partial r'} \\ -\frac{\partial^2}{\partial r'^2} & -\frac{\partial^3}{\partial r \partial r'^2} & \frac{\partial^4}{\partial r^2 \partial r'^2} \end{bmatrix} I_{33}$$

and $I_{33} = \displaystyle\int_0^\infty \int_0^\infty \sin kr \sin k'r' \left(\frac{1}{\omega_k^2 \omega_{k'}^3} + \frac{1}{\omega_k^3 \omega_{k'}^2} \right) k\,dk\, k'\,dk'$

$$= \frac{\pi}{2} \left(r' e^{-\mu r} K_0(\mu r') + r e^{-\mu r'} K_0(\mu r) \right)$$

From the relations

$$K_0'(Z) = -K_1(Z)$$

and

$$K_1'(Z) = -K_0(Z) - \frac{K_1(Z)}{Z}$$

and by putting $\mu r = x$ and $\mu r' = y$ the three-body potential for the graphs corresponding to Fig. 1.3 (a) (no bare lines) become

$$V_{NB} = -\mu \frac{g^2}{4\pi}\frac{\mu^2}{4M^2}\frac{g_\Lambda^2}{4\pi}\frac{\mu^2}{4M_\Lambda^2}\cdot\frac{2}{\pi}\frac{(\bar{\tau}_1\cdot\bar{\tau}_2)}{3}\Big\{ S_{12,xy}\Big(\frac{e^{-x}}{3x^3y}(2K_1(y)+yK_0(y))(x^2+3x+3)$$

$$+\frac{e^{-y}}{3y^3x}(2K_1(x)+xK_0(x))(y^2+3y+3)\Big) - S_{12,x}\Big(\frac{e^{-x}}{x^3y}K_1(y)(x^2+3x+3)$$

$$+\frac{e^{-y}}{xy^3}(2K_1(x)+xK_0(x))(y+1)\Big) - S_{12,y}\Big(\frac{e^{-y}}{xy^3}K_1(x)(y^2+3y+3) \qquad 2.11$$

$$+\frac{e^{-x}}{x^3y}(2K_1(y)+yK_0(y))(x+1)\Big) - \frac{\bar{\sigma}_1\cdot\bar{\sigma}_2}{3xy}\Big(e^{-x}(K_1(y)-yK_0(y))$$

$$+e^{-y}(K_1(x)-xK_0(x))\Big\}$$

In a similar manner the bare nucleon line graphs, or the types shown in Fig. 1.3 (b), give

$$V_B = -\mu \frac{g^2}{4\pi}\frac{\mu^2}{4M^2}\frac{g_\Lambda^2}{4\pi}\frac{\mu^2}{4M_\Lambda^2}\cdot\frac{\mu}{\Delta}\cdot\frac{2(\bar{\tau}_1\cdot\bar{\tau}_2)}{3}\frac{e^{-x-y}}{x^3y^3}\Big[3S_{12,xy}(1+x+\frac{x^2}{3})(1+y+\frac{y^2}{3})$$

$$-3S_{12,x}(1+x+\frac{x^2}{3})(1+y) - 3S_{12,y}(1+y+\frac{y^2}{3})(1+x) \qquad 2.12$$

$$+\bar{\sigma}_1\cdot\bar{\sigma}_2\frac{x^2y^2}{3}\Big]$$

The pair terms, Fig. 1.3 (c), give

$$V_P = \lambda \mu \frac{g^2}{4\pi}\frac{\mu^2}{4M^2}\frac{g_\Lambda^2}{4\pi}\frac{\mu^2}{4M_\Lambda^2}(\bar{\sigma}_1\cdot\bar{x})(\bar{\sigma}_2\cdot\bar{y})(\bar{\tau}_1\cdot\bar{\tau}_2)(1+x)(1+y)\frac{e^{-x-y}}{x^3y^3} \qquad 2.13$$

where λ is the damping coefficient introduced by Brueckner and Watson (11). It is interesting to notice that the three-body forces depend on the total isotopic spin due to the factor $(\bar{\tau}_1 \cdot \bar{\tau}_2)$. They are independent of the spin of the Λ-particle.

We also see that with no correlations in the direction $\bar{x}$ and $\bar{y}$ or with these directions and the nucleon spins, the terms involving $S_{12,xy}$ $S_{12,x}$ and $S_{12,y}$ become zero and that one is only left with a small $\bar{\sigma}_1\cdot\bar{\sigma}_2$ part. This part is much smaller since it has a weaker singularity than the angular dependent terms. The term V_P becomes exactly zero with no correlations. It is also seen that even with correlations, V_P behaves like $\frac{1}{x^2\ y^2}$ for small x and y, whereas V_B and V_{NB} are more strongly singular. The damping coefficient $\lambda = (1 + g^2/4\pi)^{-1}$ then further reduces the importance of this potential. It is therefore neglected in all of the subsequent analysis.

Comparing V_{NB} and V_B, we see that all things being equal V_B is larger than V_{NB} by a factor of approximately $\pi\mu/\Delta \simeq 6$. To get orders of magnitude from these potentials, then, we examine V_B with no correlations. It becomes

$$V_B \text{ (no corr.)} = -\mu\frac{g^2}{4\pi}\frac{\mu^2}{4M^2}\frac{g_\Lambda^2}{4\pi}\frac{\mu^2}{4M_\Lambda^2}\frac{\mu}{\Delta}\cdot\frac{2}{3}(\bar{\tau}_1\cdot\bar{\tau}_2)(\bar{\sigma}_1\cdot\bar{\sigma}_2)\frac{e^{-x-y}}{3xy} \qquad 2.14$$

If we consider the two nucleons in a spatially symmetric state, the expectation value of $(\bar{\tau}_1 \cdot \bar{\tau}_2)(\bar{\sigma}_1 \cdot \bar{\sigma}_2) = -3$

Putting $\frac{g^2}{4\pi} \cdot \frac{\mu^2}{4m^2} = 0.08$, $g_\Lambda = g$, $\mu = 140$ Mev, $\Delta = 80$ Mev and $M_\Lambda = 2182$ electron masses we get

$$V_B(\text{no corr.}) = 0.7402 \, \frac{e^{-x}}{x} \, \frac{e^{-y}}{y} \quad \text{Mev} \qquad 2.15$$

which is very small and can certainly be neglected in comparison with the two-body potential.

It therefore is essential to have correlations in the wave function in order that three-body forces be important.

Section III

VARIATIONAL CALCULATION ON THE HYPERTRITON CONSIDERING ONLY TWO-BODY FORCES

A. Preliminary Considerations

The form of the three-body potentials derived in Section II show that without correlations in the wave function for the hypernucleus, one would expect that they would not be very important. However, since the angular dependent parts are more singular, it is worth while to consider a particular hypernucleus in some detail to see if correlations can make three-body forces important. The simplest hyperfragment, of course, is the hypertriton. A complete calculation on any heavier hyperfragment would become much more complex.

To calculate the expectation value of the three-body Λ-nucleon potential for the hypertriton, a wave function with correlations included is required as well as a knowledge of the core ϵ of the Λ-nucleon three-body potential. If one were to choose some arbitrary correlated wave function and calculate $\langle V(\text{3-body}) \rangle / \langle V(\text{2-body}) \rangle$ using the core $\epsilon = 0.234$ meson units for both the 2- and 3-body potentials, which was the value used by DF, the ratio would not mean very much since if the value of B_Λ, the separation energy, were calculated using $\epsilon = 0.234$, there is no guarantee that B_Λ would be anywhere near the true value of ~ 0. Therefore a variational trial function

$$\psi_s = \chi_s e^{-\frac{\alpha}{2}(x+y) - \frac{\beta}{2}\rho} \tag{3.1}$$

where

$$\chi_s = -\sqrt{\tfrac{2}{3}}\,\alpha(1)\alpha(2)\beta(\Lambda) + \sqrt{\tfrac{1}{6}}\left(\alpha(1)\beta(2) + \beta(1)\alpha(2)\right)\alpha(\Lambda) \qquad 3.2$$

was chosen where α is the variational parameter. In the calculation α was varied for a minimum of B_Λ, the Λ-separation energy, for various values of ϵ, the Λ-nucleon core radius. The core ϵ for the Λ-nucleon potential was then chosen so that the minimum B_Λ for a variation of α occurred at $B_\Lambda = 0$. This calculation then determined an ϵ which would give the proper value of B_Λ. It also determined an α which gave the best wave function of this type in the potential determined by ϵ.

As was stated in Section I, the spin part χ_s was chosen so that the two nucleons were in a mutual triplet state. This spin wave function has $J = \frac{1}{2}$ and $M_J = \frac{1}{2}$ in accord with evidence that the Λ-nucleon interaction is more attractive in the singlet state (11).

The parameter β was chosen to be 1.4 inverse meson units. This value was fixed under the following considerations: Downs (17) is at present carrying out a calculation on the hypertriton using a six-parameter trial function of the form

$$\psi = (e^{-ax} + c_1 e^{-bx})(e^{-ay} + c_1 e^{-by})(e^{-a'\rho} + c_2 e^{-b'\rho}) \qquad 3.3$$

where the variational parameters are a, b, a', b', c_1 and c_2. His best values for the parameter of the deuteron part of the wave function are

$a' = .635 \qquad b' = 1.835 \qquad$ and $\quad c_2 = 1.33$

If this is approximated by a simple exponential of the form $e^{-\frac{\beta\rho}{2}}$ where the approximation is best for the small values of ρ, one obtains $\beta \sim 1.5$. Also Feshbach and Rarita (18) chose $\beta = 1.52$ for the deuteron part when they carried out a calculation on the ordinary triton. One would expect that in the ordinary triton, β would be somewhat larger for the deuteron part than it would be for the deuteron part of the hypertriton, since the hypertriton is a more loosely-bound system. From these considerations, the value for β was chosen to be 1.40 inverse meson units.

B. The Hypertriton Hamiltonian

Excluding three-body forces, which we assume are small compared to two-body forces, the total energy of the hypertriton can be written as

$$H = T_{tot} + V_D + V_{DF}(\Lambda N_1) + V_{DF}(\Lambda N_2) \qquad 3.4$$

where T_{tot} is the total kinetic energy of the system excluding the motion of the centre of mass, V_D is the two-body nucleon potential, and $V_{DF}(\Lambda N_1)$, $V_{DF}(\Lambda N_2)$ are the Λ-nucleon two-body potentials. If one calculates the expectation value of 3.4 and adds the absolute value of the binding energy of the deuteron (2.226 Mev), one obtains B_Λ, the separation energy of the Λ particle.

V_D, the deuteron potential, was taken (16) as

$$V_D = -V_c \frac{e^{-\gamma\rho}}{\gamma\rho} - S_{12} V_T \frac{e^{-\gamma'\rho}}{\gamma'\rho}, \quad \rho > .283 \qquad 3.5$$

$$= 0 \qquad , \quad \rho < .283$$

where $V_c = 100.7$ Mev, $V_T = 257$ Mev, and γ and γ' are 1.735 and 1.696 inverse meson units. We see that the potential given by 3.5 differs from that mentioned in reference 16 in that the latter has a hard core at $\rho = .283$. It was modified by putting $V_D = 0$ at $\rho = .283$ so that it would be more suitable for the wave function 3. 1 . The most suitable wave function would be zero at the hard core. Therefore, use of the wave function 3.1 incurred an error since it is too large immediately outside the core, and too smooth near the core. This had the effect of making the value of the kinetic energy too small, since the wave function used was too smooth, and of making the value of $|\langle V_D \rangle|$ too large since the wave function was quite large at the core boundary. These errors, which are in the same direction, are discussed in the conclusions at Section VIII.

V_{DF}, the Λ nucleon potential, was taken to be given by equation 1.9 , that is, the potential with no bare nucleon lines derived by Dallaporta and Ferrari (5), as it is more attractive in the singlet state.

C. The Variational Calculation

The expectation value of the kinetic energy T_{tot} of equation 3.4 was done as follows:

$$T_{tot} = -\frac{\mu^2}{2m}\left(\nabla_1^2 + \nabla_2^2\right) - \frac{\mu^2}{2M_\Lambda}\nabla_\Lambda^2$$

where

$$\bar{x} = \bar{x}_1 - \bar{x}_\Lambda$$

$$\bar{y} = \bar{x}_2 - \bar{x}_\Lambda$$

$$\bar{\rho} = \bar{x}_2 - \bar{x}_1$$

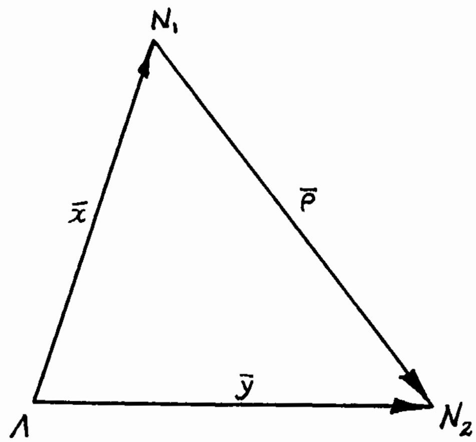


Fig. 3.1

Therefore using the wave function 3.1

$$\langle T_{tot}\rangle = \frac{-\mu^2}{2MN_S}\int e^{-\alpha(x+y)-\beta\rho}\, d\tau \Big[\frac{\alpha^2+\beta^2}{2} + \frac{\alpha^2}{2}\left(\frac{m}{m_\Lambda}\right) - \frac{2\alpha}{x}\left(1+\frac{m}{m_\Lambda}\right) - \frac{2\beta}{\rho} + \frac{1}{4}\left(\frac{2m}{m_\Lambda}\frac{\alpha^2 x}{y} - \frac{m}{m_\Lambda}\frac{\alpha^2\rho^2}{xy} + 2\alpha\beta\frac{x}{\rho} + 2\alpha\beta\frac{\rho}{x} - 2\alpha\beta\frac{x^2}{y\rho}\right)\Big] \qquad 3.6$$

where the spin integrations yield unity. Here, N_S is the normalization constant of 3.1 and $d\tau$ is the volume element of the product space $d^3x\, d^3y$ or $d\Omega_x x^2 dx\, d\Omega_y y^2 dy$. In all integrals of this type, if $\bar{y}$ is chosen as the polar axis for $\bar{x}$, then the only angular part which appears in the integrand is in ρ since $\rho^2 = x^2 + y^2 - 2xy\cos\theta_x$. Integration over $d\Omega_y$ and $d\phi_x$ yields a factor of $8\pi^2$ so that we are left with an integration of Fig. 3.1 over all configurations in a plane. $d\tau$ therefore reduces to

$$d\tau = 8\pi^2 \sin\theta\, d\theta\, x^2 dx\, y^2 dy$$

where the subscript x has been dropped on the θ. The factor $8\pi^2$ may be omitted providing it is also omitted in the normalizing constant, N_S yielding by a simple transformation

$$d\tau = x\, dx\, y\, dy\, \rho\, d\rho \qquad 3.7$$

Therefore one can write integrals of the type

$$\int_0^\infty\int_0^\infty\int_0^\pi F(x,y,\rho)\,\sin\theta\, d\theta\, x^2 dx\, y^2 dy$$

as $$\int_0^\infty\int_0^\infty\int_{|x-y|}^{x+y} F(x,y,\rho)\, x\, dx\, y\, dy\, \rho\, d\rho$$

which involves a split range integration after the ρ integration.

It frequently occurs that it may be more convenient to integrate x first, say, rather than ρ . We can then write our integral as

$$\int_0^\infty\int_0^\infty\int_{|\rho-y|}^{\rho+y} F(x,y,\rho)\, x\, dx\, y\, dy\, \rho\, d\rho$$

and get a split range integration of y and ρ after integrating with respect to x first. In Appendix C, the normalization integral is carried out. The result is

$$N_S = \frac{8\alpha^2+5\alpha\beta+\beta^2}{\alpha^3(\alpha+\beta)^5} \qquad 3.8$$

The expectation value of T_{tot} becomes

$$\langle T_{tot}\rangle = \frac{\mu^2}{2m}\,\frac{8\alpha^4+15\alpha^3\beta+11\alpha^2\beta^2+5\alpha\beta^3+\beta^4}{8\alpha^2+5\alpha\beta+\beta^2}$$

$$+\frac{\mu^2\alpha^2}{2m_\Lambda}\,\frac{8\alpha^2+10\alpha\beta+2\beta^2}{8\alpha^2+5\alpha\beta+\beta^2} \qquad 3.9$$

The spin integration over the tensor part of the phenomenological potential V_D gave zero so that

$$V_D = -\frac{V_c}{\gamma} e^{-.283(\alpha+\beta+\gamma)} \frac{\alpha^3(\alpha+\beta)^5}{8\alpha^2+5\alpha\beta+\beta^2} \left\{ \frac{.283^3}{6\alpha(\alpha+\beta+\gamma)} \right.$$

$$+ \frac{.283^2(2\alpha+\beta+\gamma)}{2\alpha^2(\alpha+\beta+\gamma)^2}$$

$$\left. + \frac{(1+.283(\alpha+\beta+\gamma))(2\alpha^2+2\alpha(\alpha+\beta+\gamma)+(\alpha+\beta+\gamma)^2)}{2\alpha^3(\alpha+\beta+\gamma)^4} \right\} \qquad 3.10$$

The spin integrations of V_{DF} are given in Appendix A. The results are

$$(\chi_s | S_{1\Lambda} | \chi_s) = 0$$

$$(\chi_s | \bar{\sigma}_1 \cdot \bar{\sigma}_\Lambda | \chi_s) = -2$$

The expectation value of the Λ-nucleon two-body potential is

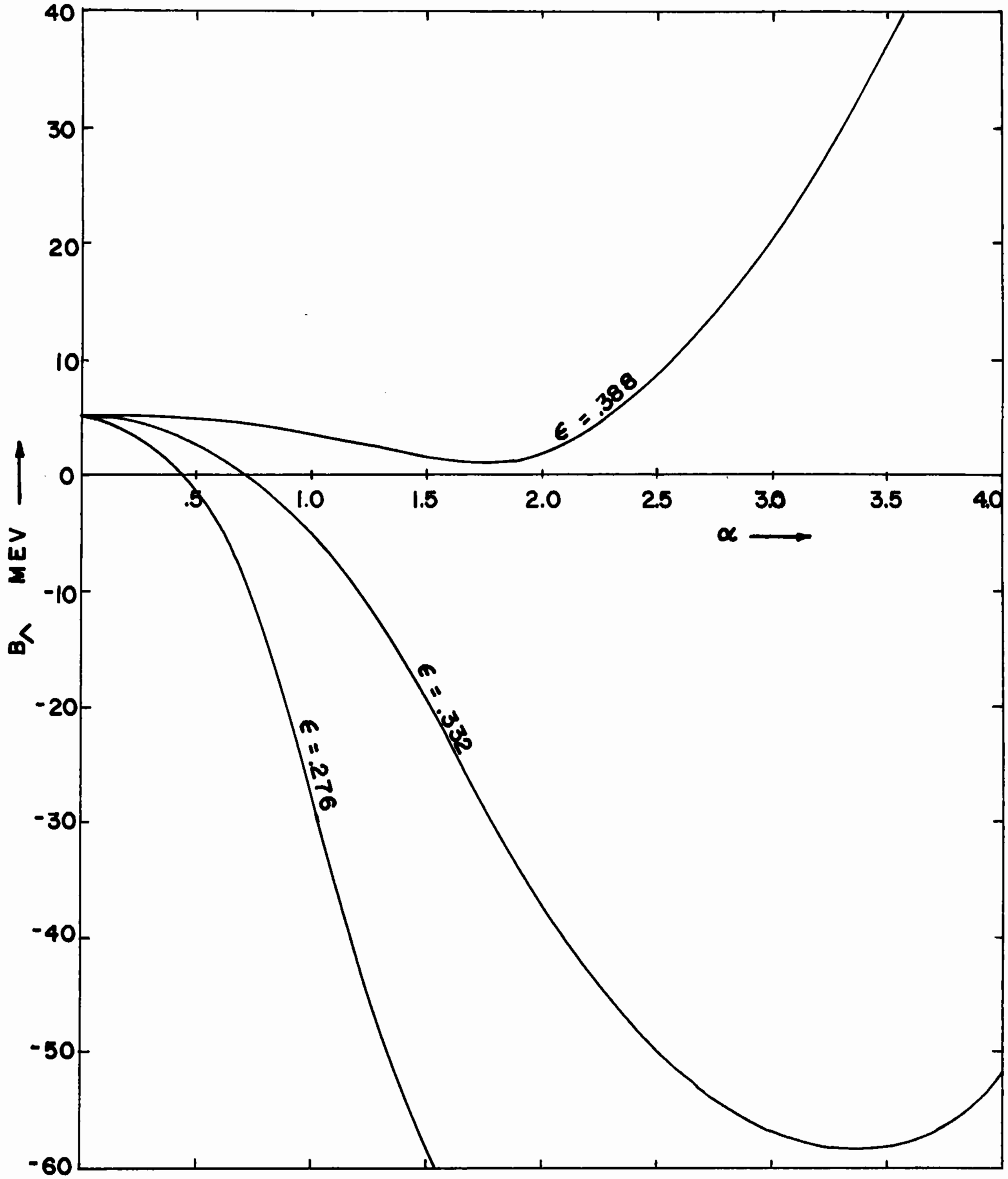
$$\langle V_{DF}(\Lambda N_1) + V_{DF}(\Lambda N_2) \rangle$$

$$= -8\mu \frac{g^2}{4\pi}\frac{\mu^2}{4m^2} \frac{g_\Lambda^2}{4\pi}\frac{\mu^2}{4m_\Lambda^2} \cdot \frac{2}{\pi} \frac{\alpha^3(\alpha+\beta)^2}{(\alpha-\beta)^2(8\alpha^2+5\alpha\beta+\beta^2)} \int_\epsilon^\infty \frac{dx}{x^3} \left(e^{-(\alpha+\beta+1)x} \left[\alpha(\alpha+\beta)x^2 \right.\right.$$

$$\left. - \frac{4\alpha\beta x}{\alpha-\beta} \right] + e^{-(2\alpha+1)x} \left[\beta(\alpha+\beta)x^2 + \frac{4\alpha\beta x}{\alpha-\beta} \right] \Bigg) \left(\frac{K_1(x)}{x} (4+4x-x^2) \right.$$

$$\left. + K_0(x)(2+2x+x^2) + 8\left[3K_0(2x) + (3+2x^2)\frac{K_1(2x)}{x} \right] \right) \qquad 3.11$$

The integration 3.11 was performed numerically for various values of α



The Separation Energy, B_Λ, of the Λ-Particle as a Function of α and ϵ where ϵ is the Core Radius of the Λ-Nucleon Two-Body Potential V_{DF}.

Units of Meson Compton Wavelengths Are Used

FIG. 3.2

and ϵ , the core radius.

To equations 3.9, 3.10 and 3.11 the absolute value of the binding energy of the deuteron, 2.226 Mev, was added. The result gave B_Λ , the separation energy of the Λ particle. The results are plotted in Fig. 3.2. We see that the value $\epsilon = .388$ yields the minimum B_Λ very close to zero. From this we took

$$\epsilon = .386 \quad \text{meson units}$$

$$\alpha = 1.8 \quad \text{inverse meson units} \qquad 3.12$$

Using these values of the parameters α and ϵ , we get

$$\langle T_{tot} \rangle = 58 \quad \text{Mev}$$

$$\langle V_D \rangle = -14.2 \quad \text{"}$$

$$\langle V_{DF}(\Lambda N_1) + V_{DF}(\Lambda N\ ) \rangle = -46 \quad \text{"} \qquad 3.13$$

It was stated in part B of this Section that an error was introduced by using the trial wave function 3.1 with the phenomenological neutron-proton potential 3.5 .

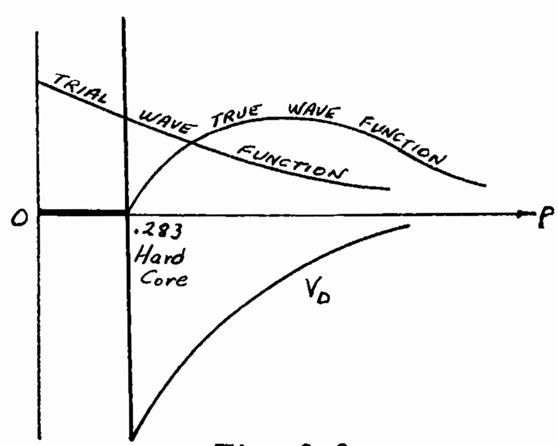


Fig. 3.3

If we examine Fig. 3.3 we see that in our calculation, the value of $\langle T_{tot} \rangle$ was probably too small since the true wave function has more curvature, and also $|\langle V_D \rangle|$ was too large, since the trial wave function was large in the region of the large potential.

To estimate the error, we first return to the expression for $\langle T_{tot} \rangle$ which is given by 3.9. The error would be closely connected with the value of β. However, the expression is more dependent on α than β so we infer that the error would not be too large. The value of $|\langle V_D \rangle|$ is also seen to be fairly small compared to the absolute value of the other quantities (see equation 3.13). In the conclusions at Section VIII a consideration of these errors is made. It is found that the combined error is probably no more than about 4.5 Mev. An examination of Fig. 3.2 shows that this would not effect α very much since the minima are quite shallow. ϵ, the core radius for the Λ-nucleon potential, could be decreased from 0.386 to 0.380 meson compton wavelengths. This small decrease in ϵ will not appreciably effect our ultimate result for

$$\langle V(3\text{-body}) \rangle / \langle V(2\text{-body}) \rangle$$

One item of interest at this point is the size of the parameter $\alpha = 1.8$, which has turned out to be larger than the deuteron parameter $(\beta = 1.4)$. This may seem objectionable since it infers that the Λ-N part of the wave function is more concentrated at the origin than is the deuteron part, which appears to contradict the assumption that the Λ-particle is loosely bound. To explain this we note that both the true Λ-nucleon and deuteron wave functions are composed of two parts: one being in the region of the potential, and the other in the tail region. The latter is the part which depends on how loosely-bound the particles are. The trial wave function chosen for this calculation corresponds more to the part in the region of the potential, since it was there that the parameters were determined. The value obtained for α thus reflects the fact that the effective range of the Λ-nucleon potential is smaller than the nucleon-nucleon potential.

A better trial function for the hypertriton would have been one similar to equation 3. 3, where the tail part is included. In view of the fact, however, that what was desired from those calculations was a measure of the ratio of the three-body forces to the two-body forces and not a precise value of their magnitudes, the use of the trial function 3. 1 was justified.

Section IV

THREE-BODY EXPECTATION VALUES

A. The Wave Function

In the calculation of the expectation value of the three-body potential, an admixture of D state was included and the wave function put into the form

$$\psi = \cos \phi \, \psi_S + \sin \phi \, \psi_D \qquad 4.1$$

where

$$\psi_S = \frac{\chi_S}{\sqrt{N_S}} e^{-\frac{\alpha}{2}(x+y) - \frac{\beta}{2}\rho} \qquad 4.2$$

ψ_D, the D state admixture, was taken to be

$$\psi_D = \frac{\chi_D}{\sqrt{N_D}} e^{-\frac{\alpha}{2}(x+y) - \frac{\beta}{2}\rho} \qquad 4.3$$

where χ_D, the spin part of this wave function, was assumed (16) to be

$$\chi_D = \left(x^2 S_{1A,x} + y^2 S_{2A,y} \right) \chi_S \qquad 4.4$$

N_S and N_D are normalizing constants and ϕ is a parameter which determines the percentage of S or D state. As was stated in Section I, the D state part was assumed as shown since it was thought that the one spatially symmetric between the two nucleons would give the largest contribution to the binding energy.

The expectation value of the three-body potential was calculated and compared to the value of $\langle V_{DF}(2) \rangle$ = -46 Mev (see equation 3.13). The wave function was not consistent in the evaluation of the ratio $\langle V(3\text{-body}) \rangle / \langle V(2\text{-body}) \rangle$ since $\langle V(2\text{-body}) \rangle$ was only calculated using the S state wave function 4.2 . For small values of D state admixture, it is expected that the ratio would be quite meaningful as $\langle V(2\text{-body}) \rangle$ does not change too rapidly with percentage D state. This can be seen by noticing that central and tensor parts of the Λ-nucleon two-body potential, V_{DF}, given by equation 1.9 , are singular to the same order $(\frac{1}{x^5})$. For large D state admixture the ratio would be expected to be somewhat poorer.

B. The Spin Isotopic-spin Integrations

The expectation value of $(\bar{\tau}_1 \cdot \bar{\tau}_2)$ is -3 . This is because the wave function ψ , given by 4.1, is symmetric in space and spin exchange of the two nucleons.

The spin integrations yield

$$(\chi_S|\chi_S) = 1$$

$$(\chi_S|\chi_D) = 0$$

$$(\chi_D|\chi_D) = 2(y^4 + x^2y^2(3\cos^2\theta - 1) + x^4)$$

$$(\chi_S|\bar{\sigma}_1 \cdot \bar{\sigma}_2|\chi_S) = 1 \qquad 4.5$$

$$
\begin{aligned}
(\chi_S \mid S_{12,x} \mid \chi_S) &= 0 \\
(\chi_S \mid S_{12,y} \mid \chi_S) &= 0 \\
(\chi_S \mid S_{12,xy} \mid \chi_S) &= 3\cos^2\theta - 1 \\
(\chi_D \mid \bar{\sigma}_1 \cdot \bar{\sigma}_2 \mid \chi_S) &= 0 \qquad\qquad 4.5 \\
(\chi_D \mid S_{12,x} \mid \chi_S) &= -4y^2 - 2x^2(3\cos^2\theta - 1) \\
(\chi_D \mid S_{12,y} \mid \chi_S) &= -2y^2(3\cos^2\theta - 1) - 4x^2 \\
(\chi_D \mid S_{12,xy} \mid \chi_D) &= -12\cos^2\theta\,(x^2 + y^2) \\
(\chi_D \mid \bar{\sigma}_1 \cdot \bar{\sigma}_2 \mid \chi_D) &= 2(y^4 + x^2y^2(3\cos^2\theta - 1) + x^4) \\
(\chi_D \mid S_{12,x} \mid \chi_D) &= -2y^4(3\cos^2\theta - 1) - 4x^2y^2(3\cos^2\theta - 1) - 4x^4 \\
(\chi_D \mid S_{12,y} \mid \chi_D) &= -4y^4 - 4x^2y^2(3\cos^2\theta - 1) - 2x^4(3\cos^2\theta - 1) \\
(\chi_D \mid S_{12,xy} \mid \chi_D) &= -2y^4(3\cos^2\theta + 1) - 2x^2y^2(9\cos^2\theta - 1) \\
&\quad -2x^4(3\cos^2\theta + 1)
\end{aligned}
$$

Sample spin integrations are explicitly carried out in Appendices A and B.

The spatial parts of the integration resulted in three types of terms since the expectation value of the three-body potential with the wave function 4.1 gave

$$\langle V(\text{3-body}) \rangle = \cos^2\phi\, \alpha_S + \sin\phi \cos\phi\, \alpha_{SD} + \sin^2\phi\, \alpha_D \tag{4.6}$$

where α_S, α_{SD} and α_D are the contributions from the $(\chi_S|\quad|\chi_S)$, $(\chi_D|\quad|\chi_S)$ and $(\chi_D||\chi_D)$ terms respectively, given in equation 4.6 . $\langle V(\text{3-body}) \rangle$ was separately calculated for the potential with no bare lines, or equation 2.11 , and also for the sum of both potentials 2.11 and 2.12 .

C. Calculation of the S State Part (α_S)

The calculation of the expectation value of equation 2.11 , considering only α_S, is carried out below.

We get

$$\alpha_s = 12.985 \int_\epsilon^\infty \int_\epsilon^\infty \int_{-1}^{1} e^{-\alpha(x+y)-\beta\sqrt{x^2+y^2-2xy\mu}} (3\mu^2-1) \frac{e^{-x}}{x^3 y} \cdot$$

$$\cdot (2K_1(y) + yK_0(y))(x^2+3x+3) x^2 dx\, y^2 dy\, d\mu$$

$$-12.985 \int_\epsilon^\infty \int_\epsilon^\infty \int_{-1}^{1} e^{-\alpha(x+y)-\beta\sqrt{x^2+y^2-2xy\mu}} \frac{e^{-x}}{xy} (K_1(y) - yK_0(y)) \cdot \tag{4.7}$$

$$\cdot x^2 dx\, y^2 dy\, d\mu$$

We first consider the y part of the integration of the first term of 4.7. We have

$$I = \int_{\epsilon}^{\infty} e^{-\alpha y - \beta\sqrt{x^2+y^2-2xy\mu}} \left(2k_1(y) + yk_0(y)\right) y\,dy$$

$$= \int_{\epsilon}^{\infty} e^{-\alpha y - \beta\sqrt{x^2+y^2-2xy\mu}} \left(y^2\left(\frac{k_1(y)}{y} + k_0(y)\right) + yk_1(y)\right) dy$$

Integrating by parts we get

$$I = \left(-y^2 k_1(y) - yk_0(y)\right) e^{-\alpha y - \beta\sqrt{x^2+y^2-2xy\mu}} \Big|_{\epsilon}^{\infty} + I'$$

$$\text{where} \quad I' = \int_{\epsilon}^{\infty} e^{-\alpha y - \beta\sqrt{x^2+y^2-2xy\mu}} \left\{ k_1(y)\left[2y - y^2\left(\alpha + \beta\frac{(y - x\mu)}{\sqrt{x^2+y^2-2xy\mu}}\right)\right] + k_0(y)\left[1 - y\left(\alpha + \beta\frac{(y - x\mu)}{\sqrt{x^2+y^2-2xy\mu}}\right)\right]\right\} dy$$

Returning to our original integral given by equation 4.7, we see that most of the contribution to the integral is from the region of small x and y and also from the region $\mu = +1$, since then the factor in the exponential $\beta\sqrt{x^2 + y^2 - 2xy\mu}$ is smaller.

We therefore compare the values of I and I' for x equal to ϵ and $\mu = \pm 1$

$$I = e^{\pm\beta\epsilon}\int_{\epsilon}^{\infty} e^{-(\alpha+\beta)y}\left(2k_1(y) + y k_0(y)\right) y\,dy$$

$$I' = e^{\pm\beta\epsilon}\int_{\epsilon}^{\infty} e^{-(\alpha+\beta)y}\left[y k_1(y)\left(2-(\alpha+\beta)y\right) + k_0(y)\left(1-(\alpha+\beta)y\right)\right] dy$$

Calculating I and I' numerically with α, β and ϵ equal to 1.8, 1.4 and 0.386 gives

$$I'/I = -0.543$$

and since most of the contribution to the integral I comes from the region of small x, we take

$$I \simeq \frac{e^{-\alpha\epsilon - \beta\sqrt{x^2+\epsilon^2-2x\epsilon\mu}}}{1.543}\left(\epsilon^2 k_1(\epsilon) + \epsilon k_0(\epsilon)\right)$$

We would not expect the ratio I'/I to remain at $-.543$ for large values of x, but since the contributions to the integral of equation 4.7 will be much smaller from the larger values of x, we take our ratio $I'/I = -.543$, the ratio at $x = \epsilon$. In addition, this ratio is seen to be the same ($-.543$) for $\mu = \pm 1$ when $x = \epsilon$, so that we expect that it would not change much for other values of μ. In part D of this Section, an estimate of the error involved in evaluating integrals of the above type is made. It is found that for a less singular integral than the first term in equation 4.7, the error is about 10%. (The less singular the integral, the poorer the approximation.) The first integral of 4.7 then becomes

$$\frac{12.985}{1.543}\, e^{-\alpha\epsilon}\left(\epsilon^2 k_1(\epsilon) + \epsilon k_0(\epsilon)\right)\int_{\epsilon}^{\infty}\int_{-1}^{1} e^{-\alpha x - \beta\sqrt{x^2+\epsilon^2-2x\epsilon\mu}}\, \frac{e^{-x}}{x}(x^2+3x+3)(3\mu^2-1)\, dx\, d\mu$$

The same procedure was carried out for the x integration.

Put

$$J = \int_{\epsilon}^{\infty} e^{-(\alpha+1)x} e^{-\beta\sqrt{x^2+\epsilon^2-2\epsilon x\mu}}\,(x^2+3x+3)\frac{dx}{x}$$

Integrating by parts, we obtain

$$J = -\mathrm{Ei}\left[-(\alpha+1)\epsilon\right] e^{-2\beta\epsilon\sqrt{\frac{1-\mu}{2}}}(\epsilon^2+3\epsilon+3) \quad + J'$$

where $$J' = \int_{\epsilon}^{\infty} e^{-\beta\sqrt{x^2+\epsilon^2-2x\epsilon\mu}}\left[-Ei(-(\alpha+1)x)\right]\left(3+2x - (x^2+3x+3)\frac{\beta(x-\epsilon\mu)}{\sqrt{x^2+\epsilon^2-2x\epsilon\mu}}\right) dx$$

at $\mu = \pm 1$, omitting a common factor

$$J = \int_{\epsilon}^{\infty} e^{-(\alpha+\beta+1)\epsilon}(x^2+3x+3)\frac{dx}{x}$$

$$J' = \int_{\epsilon}^{\infty} e^{-\beta x} Ei\left(-(\alpha+1)x\right)\left(\beta x^2 + x(3\beta-2) + 3(\beta-1)\right) dx$$

$$J = e^{-(\alpha+\beta+1)\epsilon}\left(\frac{\epsilon}{1+\alpha+\beta} + \frac{1}{(1+\alpha+\beta)^2} + \frac{3}{1+\alpha+\beta}\right) - 3Ei\left(-(\alpha+\beta+1)\epsilon\right)$$

$$= .422$$

The exponential function Ei(x) is tabulated in Jahnke and Emde. By numerical integration

$$J' = -0.0596$$

$$\text{so that} \quad J/J' = -0.14$$

The first term in equation 4.7 therefore becomes

$$\frac{12.985}{1.543} e^{-\alpha\epsilon}\left(\epsilon^2 K_1(\epsilon) + \epsilon K_0(\epsilon)\right)\underbrace{\left(-Ei[-(\alpha+1)\epsilon]\right)}_{1.14}(\epsilon^2+3\epsilon+3) \times$$

$$\times \int_{-1}^{1} e^{-2\beta\epsilon\sqrt{\frac{1-\mu^2}{2}}}(3\mu^2-1)\,d\mu$$

$$= 2.236\int_{-1}^{1} e^{-2\beta\epsilon\sqrt{\frac{1-\mu^2}{2}}}(3\mu^2-1)\,d\mu$$

$$= 2.236\left\{\frac{8}{a^2} - \frac{8e^{-a}}{a^2}(1+a) - 48\left[\left(1+\frac{20}{a^2}\right)\left(\frac{1}{a^2} + \frac{2e^{-a}}{a^2} + \frac{6e^{-a}}{a^3}\right.\right.\right.$$

$$\left.\left.\left. + \frac{6e^{-a}}{a^4} - \frac{6}{a^4}\right) - \frac{1}{a^2} - \frac{8}{a^2}\left(\frac{1}{a^2} - \frac{e^{-a}}{a^2}(1+a)\right)\right]\right\}$$

$$= 2.236 \times .07214$$

$$= 0.161 \text{ Mev}$$

where $a = 2\beta\epsilon$

The errors involved in the above approximations are estimated in part D of this Section and are found to be about 10%.

The second term of equation 4.7 was evaluated exactly. Omitting the factor -12.985, and putting $K_1(y) - yK_0(y) = K$, we obtain by the transformation given by equation 3.7

$$\int_\epsilon^\infty e^{-\alpha y} k\,dy \int_\epsilon^\infty e^{-(\alpha+1)x}\,dx \int_{|x-y|}^{x+y} \rho e^{-\beta\rho}\,d\rho$$

$$= -\int_\epsilon^\infty e^{-\alpha y} k\,dy \int_\epsilon^\infty e^{-(\alpha+1)x} e^{-\beta(x+y)} \left(\frac{x+y}{\beta} + \frac{1}{\beta^2} \right) dx$$

$$+\int_\epsilon^\infty e^{-\alpha y} k\,dy \int_\epsilon^y e^{-(\alpha+1)x} e^{-\beta(y-x)} \left(\frac{y-x}{\beta} + \frac{1}{\beta^2} \right) dx$$

$$+\int_\epsilon^\infty e^{-\alpha y} k\,dy \int_y^\infty e^{-(\alpha+1)x} e^{-\beta(x-y)} \left(\frac{x-y}{\beta} + \frac{1}{\beta^2} \right) dx$$

$$= \int_\epsilon^\infty k\,dy\, e^{-(2\alpha+1)y} \left[\frac{1}{\beta(\alpha+\beta+1)^2} - \frac{1}{\beta^2(\alpha-\beta+1)} + \frac{1}{\beta(\alpha-\beta+1)^2} + \frac{1}{\beta^2(\alpha+\beta+1)} \right]$$

$$+\int_\epsilon^\infty k\,dy\, e^{-(\alpha+\beta)y} \left[y \left(\frac{e^{-(\alpha-\beta+1)\epsilon}}{\beta(\alpha-\beta+1)} - \frac{e^{-(\alpha+\beta+1)\epsilon}}{\beta(\alpha+\beta+1)} \right) + e^{-(\alpha-\beta+1)\epsilon} \left(\frac{1}{\beta^2(\alpha-\beta+1)} \right. \right.$$

$$\left. \left. - \frac{\epsilon}{\beta(\alpha-\beta+1)} - \frac{1}{\beta(\alpha-\beta+1)^2} \right) - e^{-(\alpha+\beta+1)\epsilon} \left(\frac{1}{\beta^2(\alpha+\beta+1)} + \frac{\epsilon}{\beta(\alpha+\beta+1)} + \frac{1}{\beta(\alpha+\beta+1)^2} \right) \right]$$

$$= \int_\epsilon^\infty k\,dy\, F(y)$$

where $F(y) = .16197e^{-4.6y} + e^{-3.2y}(.26360\,y - .15970)$ 4.9

Performing this integration numerically and including the factor -12.985 gave the result -0.0530 Mev.

Combining both results, we see that with no bare lines,

$$\alpha_S = 0.161 - 0.0530$$
$$= 0.108 \text{ Mev} \qquad 4.10$$

which is repulsive.

Since $\langle V(2\text{-body}) \rangle$ from Section III was found to be -46 Mev, our ratio with just S state in the wave function is therefore

$$\frac{\langle V(3\text{-body}) \rangle}{\langle V(2\text{-body}) \rangle} = .00235$$

which is quite small.

A similar calculation for the expectation value of V_B(3-body) given by equation 2.12 for the S state part of the wave function gave

1.892 Mev

which is also repulsive and much larger than the preceding value of 0.108 Mev.

Therefore the total expectation value of the ratio

$$\frac{\langle V_{NB}(3) + V_B(3) \rangle}{\langle V(2\text{-body}) \rangle}$$

is

$$-2/46 = -0.0435$$

so that with an S state wave function the relative importance may be as large as 4.35% and repulsive.

D. Calculations of the D State Part (α_{SD} and α_D)

The D state part of the calculations involves the cross terms resulting from $(\chi_S| \quad |\chi_D)$ and $(\chi_D| \quad |\chi_D)$, or the evaluation of α_{SD} and α_D. Many of the integrals contained the factor $(3\mu^2 - 1)$. If $\bar{x}$ and $\bar{y}$ were not correlated by the μ in the factor $e^{-\beta\sqrt{x^2+y^2-2xy\mu}}$, or if β were zero, the integral of μ^2 would be $1/3$, or all integrals with a factor $(3\mu^2 - 1)$ would be zero. A measure of the correlation is given by

$$\xi = \frac{\int_{-1}^{1} e^{-2\beta\epsilon\sqrt{\frac{1-\mu}{2}}}(3\mu^2-1)\,d\mu}{\int_{-1}^{1} e^{-2\beta\epsilon\sqrt{\frac{1-\mu}{2}}}\,d\mu}$$

$$= 0.07171 \qquad 4.11$$

which we see is small compared to unity. Therefore in all of the D state integrations μ^2 was put equal to $1/3$. In the evaluation of α_S, the S state part of the expectation value, this device would have led to very erroneous results since the major part of the expectation value came from the term which has a $(3\mu^2 - 1)$ in it (see equation 4.7). In the D state calculations, however, the more strongly singular parts of the potential still were present even after putting $\mu^2 = 1/3$.

The normalization constant N_D is evaluated in Appendix C. It is 0.64766. The cross terms α_{SD} for no bare nucleon lines give

$$\alpha_{SD} = 6.97 \int_{\epsilon}^{\infty}\int_{\epsilon}^{\infty}\int_{-1}^{1} 2e^{-\alpha(x+y)-\beta\rho}\Big\{(4y^2+2x^2(3\mu^2-1))\Big[\frac{e^{-x}}{x^3y}k_1(y)(x^2+3x+3)$$

$$+\frac{e^{-y}}{xy^3}(2k_1(x)+xk_0(x))(y+1)\Big] - 12\mu^2(x^2+y^2)\cdot\frac{e^{-x}}{3x^3y}\cdot$$

$$\cdot(2k_1(y)+y\,k_0(y))(x^2+3x+3)\Big\}\;x^2dx\,y^2dy\,d\mu$$

which, after interchanging y for x to get x in the exponential throughout, becomes

$$27.88\int_{\epsilon}^{\infty}\int_{\epsilon}^{\infty}\int_{-1}^{1} e^{-\alpha(x+y)-\beta\rho}e^{-x}\Big[(2y^2+x^2(3\mu^2-1))\frac{k_1(y)}{x^3y}(x^2+3x+3)$$

$$+(2x^2+y^2(3\mu^2-1))(2k_1(y)+yk_0(y))\frac{(x+1)}{x^3y}$$

$$-2\mu^2\frac{(x^2+y^2)}{x^3y}(2k_1(y)+y\,k_0(y))(x^2+3x+3)\Big]\,x^2dx\,y^2dy\,d\mu$$

This was then split up into integrals with various powers of x. Neglecting the factor 27.88, we get 5 integrations

$$A = \int_{\epsilon}^{\infty}\int_{\epsilon}^{\infty}\int_{-1}^{1} e^{-\alpha(x+y)-\beta\rho}\frac{e^{-x}}{x^3}\Big[6y\,k_1(y)$$

$$-y(2k_1(y)+yk_0(y))(3\mu^2+1)\Big]\,x^2dx\,y^2dy\,d\mu$$

$$B = \int_{\epsilon}^{\infty}\int_{\epsilon}^{\infty}\int_{-1}^{1} e^{-\alpha(x+y)-\beta\rho} \, y\frac{e^{-y}}{x^2}\left[6k_1(y) - (3\mu^2+1)(2k_1(y)+yk_0(y))\right] x^2dx\,y^2dy\,d\mu$$

$$C = \int_{\epsilon}^{\infty}\int_{\epsilon}^{\infty}\int_{-1}^{1} e^{-\alpha(x+y)-\beta\rho} \frac{e^{-x}}{x}\left[y\left(2k_1(y) - 2\mu^2\left(2k_1(y)+yk_0(y)\right)\right) - \frac{1}{y}\,(3\mu^2-1)\left(k_1(y)+2yk_0(y)\right)\right] \cdot$$

$$\cdot\; x^2dx\,y^2dy\,d\mu \qquad 4.13$$

$$D = -\int_{\epsilon}^{\infty}\int_{\epsilon}^{\infty}\int_{-1}^{1} e^{-\alpha(x+y)-\beta\rho} \frac{e^{-x}}{y}(3\mu^2-1)\left(k_1(y)+2yk_0(y)\right) x^2dx\,y^2dy\,d\mu$$

$$E = \int_{\epsilon}^{\infty}\int_{\epsilon}^{\infty}\int_{-1}^{1} e^{-\alpha(x+y)-\beta\rho} \, x\frac{e^{-x}}{y}\left[(3\mu^2-1)k_1(y) - 2\mu^2(2k_1(y)+yk_0(y))\right] x^2dx\,y^2dy\,d\mu$$

At this point we put $\mu^2 = 1/3$ and obtain

$$A = 2\int_{\epsilon}^{\infty}\int_{\epsilon}^{\infty}\int_{-1}^{1} e^{-\alpha(x+y)-\beta\rho} \frac{e^{-x}}{x^3}\, y\left(k_1(y) - yk_0(y)\right) x^2dx\,y^2dy\,d\mu$$

$$B = 2\int_{\epsilon}^{\infty}\int_{\epsilon}^{\infty}\int_{-1}^{1} e^{-\alpha(x+y)-\beta\rho} \frac{e^{-x}}{x^2}\, y\left(k_1(y) - yk_0(y)\right) x^2dx\,y^2dy\,d\mu$$

$$C = \int_{\epsilon}^{\infty}\int_{\epsilon}^{\infty}\int_{-1}^{1} e^{-\alpha(x+y)-\beta\rho} \frac{e^{-x}}{x}\, y\left[2k_1(y) - \frac{2}{3}(2k_1(y)+yk_0(y))\right] x^2dx\,y^2dy\,d\mu \qquad 4.14$$

$$D = 0$$

$$E = -\frac{2}{3}\int_\epsilon^\infty\int_\epsilon^\infty\int_{-1}^{1} e^{-\alpha(x+y)-\beta\rho}\, \frac{x e^{-x}}{y}\left(2K_1(y) + yK_0(y)\right) x^2 dx\, y^2 dy\, d\mu$$

The integrals 4. 14 are now done one at a time

$$A = 2\int_\epsilon^\infty \frac{e^{-(\alpha+1)x}}{x}dx \int_\epsilon^\infty e^{-\alpha y} y^3 (K_1(y)\, yK_0(y))dy \int_{-1}^{1} e^{-\beta\rho} d\mu$$

Carrying out the x integration by parts we get

$$I'' = \int_\epsilon^\infty \frac{e^{-(\alpha+1)x}}{x}\, e^{-\beta\sqrt{x^2+y^2-2xy\mu}}\, dx$$

$$= -\,\mathrm{Ei}\left[-(\alpha+1)\epsilon\right] e^{-\beta\sqrt{x^2+y^2-2xy\mu}} - I'''$$

where $I''' = -\int_\epsilon^\infty Ei\left[-(\alpha+1)x\right] e^{-\beta\sqrt{x^2+y^2-2xy\mu}}\, \frac{\beta(x-y\mu)}{\sqrt{x^2+y^2-2xy\mu}}\, dx$

Again, as in part C of this Section, we compare I'' and I''' for $\mu = \pm 1$, and y which equals ϵ.

$$I'' = e^{\pm\beta\epsilon}\int_\epsilon^\infty \frac{e^{-(\alpha+\beta+1)x}}{x}dx$$

$$I''' = -\beta e^{\pm\beta\epsilon}\int_\epsilon^\infty e^{-\beta x} Ei(-(1+\alpha)x)\, dx$$

and

$$\frac{I'''}{I''} = 0.337$$

Therefore

$$A = \frac{2}{1.337}\left[-Ei\left(-(1+\alpha)\epsilon\right)\right]\int_{\epsilon}^{\infty} e^{-\alpha y-\beta\sqrt{y^2+\epsilon^2-2y\epsilon\mu}}\, y^3(k_1(y)-yk_0(y))\,dy\int_{-1}^{1} d\mu$$

$$= \frac{2}{1.337}\left[-Ei\left(-(1+\alpha)\epsilon\right)\right]\cdot\frac{1}{\epsilon}\int_{\epsilon}^{\infty}\int_{y-\epsilon}^{y+\epsilon} e^{-\alpha y-\beta\rho}\, y^2 dy\,(k_1(y)-yk_0(y))\,\rho\, d\rho$$

$$= 1.679\int_{\epsilon}^{\infty} y^2(k_1(y)-yk_0(y))\,e^{-(\alpha+\beta)y}\,G(y)\,dy$$

where $G(y) = .40506\,y - .02763$ 4.15

and $A = .00803$ 4.16

$$B = 2\int_{\epsilon}^{\infty} e^{-(1+\alpha)x}dx\int_{\epsilon}^{\infty} e^{-\alpha y}\, y^3(k_1(y)-yk_0(y))\,dy\int_{-1}^{1} e^{-\beta\rho}d\mu$$

Again, performing the x integration using the same method as for A, we get

$$B = 0.8372\int_{\epsilon}^{\infty} y^2(k_1(y)-yk_0(y))\,e^{-(\alpha+\beta)y}\,G(y)\,dy$$

where $G(y)$ is given by 4.15,

so that $B = .00401$ 4.17

C was carried out using two methods. The first method involved the use of the approximations of integrating by parts as we did for A and B, and for some of the integrations in part C of this Section. The second method was exact.

(i) Approximate Method

$$C = \frac{2}{3}\int_{\epsilon}^{\infty} e^{-(1+\alpha)x} x\,dx \int_{\epsilon}^{\infty} e^{-\alpha y} y^3 (K_1(y) - yK_0(y))\,dy \int_{-1}^{1} e^{-\beta\rho} d\mu$$

We write the x part of the integration as

$$J'' = \int_{\epsilon}^{\infty} e^{-(1+\alpha)x - \beta\sqrt{x^2+y^2-2xy\mu}}\, x\,dx$$

$$= \epsilon \frac{e^{-(1+\alpha)\epsilon - \beta\sqrt{\epsilon^2+y^2-2\epsilon y\mu}}}{1+\alpha} + J'''$$

where $$J''' = \frac{1}{1+\alpha}\int_{\epsilon}^{\infty} e^{-(1+\alpha)x - \beta\sqrt{x^2+y^2-2xy\mu}} \left(1 - \beta x \frac{(x-\mu y)}{\sqrt{x^2+y^2-2xy\mu}} \right) dx$$

Again for $y = \epsilon$ and $\mu = \pm 1$

$$J'' \propto \int_{\epsilon}^{\infty} x e^{-(1+\alpha+\beta)x} dx$$

$$J''' \propto \int_{\epsilon}^{\infty} e^{-(1+\alpha+\beta)x} \left(\frac{1-\beta x}{1+\alpha}\right) dx$$

so that

$$\frac{J'''}{J''} = \frac{1+\alpha-\beta\epsilon(1+\alpha+\beta)}{(1+\alpha)[1+\epsilon(1+\alpha+\beta)]} = .072$$

Therefore

$$C = .164\int_{\epsilon}^{\infty} y^2(k_1(y)-yk_0(y))\,G(y)\,dy\,e^{-(\alpha+\beta)y}$$

where G(y) is given by 4.15

$$C = .00079 \tag{4.18}$$

(ii) <u>Exact Method</u>

$$C = \frac{2}{3}\int_{\epsilon}^{\infty} e^{-(1+\alpha)x}dx\int_{\epsilon}^{\infty} y^2e^{-\alpha y}(k_1(y)-yk_0(y))\,dy\int_{|x-y|}^{x+y}\rho e^{-\beta\rho}d\rho$$

$$= \frac{2}{3}\int_{\epsilon}^{\infty} y^2(k_1(y)-yk_0(y))\,F(y)\,dy$$

where F(y) is given by 4.9 .

Therefore

$$C = .00087 \tag{4.19}$$

Comparing 4.18 and 4.19 we see that the error involved in using these approximations is about 10%.

Continuing,

$$E = -\frac{2}{3}\int_{\epsilon}^{\infty} e^{-(1+\alpha)x} x^2 dx \int_{\epsilon}^{\infty} (2k_1(y) + yk_0(y)) e^{-\alpha y} dy \int_{|x-y|}^{x+y} \rho e^{-\beta\rho} d\rho$$

$$= -\frac{2}{3\beta}\int_{\epsilon}^{\infty} (2k_1(y) + yk_0(y)) H(y)\, dy$$

where $H(y) = (.69501y - 1.0795)e^{-3.2y} + (.22676y^2 + .86384y + 1.0798)e^{-4.6y}$

4.20

$$E = -.00864 \qquad 4.21$$

Therefore, adding the equations A, B, C and E we obtain

$$.00803 + .00401 + .00087 - .00864$$
$$= .00427$$

Therefore the term α_{SD} with no bare lines becomes

$$.00427 \times 27.88$$
$$= .119 \text{ Mev}$$

We see that the terms nearly cancelled out. This result is therefore a poor estimate for α_{SD}. However, if .119 were in error by a large amount (even 100%) the term would still give an unimportant contribution to the three-body force (see equation 4.25).

The value of α_{SD} calculated for the bare line potential $V_B(3)$, after putting $3\mu^2 = 1$, gave

$$-306.6 \int_{\epsilon}^{\infty} x^2 e^{-(1+\alpha)x} dx \int_{\epsilon}^{\infty} \frac{e^{-(1+\alpha)y}}{y^2} (1+y+y^2/3) dy \int_{|x-y|}^{x+y} \rho e^{-\beta\rho} d\rho$$

$$= -219 \int_{\epsilon}^{\infty} \frac{e^{-y}}{y^2} (1+y+y^2/3) H(y) dy$$

$$= -3.21 \text{ Mev} \qquad 4.23$$

where $H(y)$ is given by equation 4.20.

The calculations for α_D were carried out in a similar manner. The results were

$$\alpha_D \text{ (no bare lines)} = -.106 \text{ Mev}$$
$$\alpha_D \text{ (bare lines included)} = -.719 \text{ Mev} \qquad 4.24$$

A summary of these calculations is given below.

Expectation values with no bare lines:

$$\alpha_S = +.108 \text{ Mev}$$
$$\alpha_{SD} = +.119 \text{ Mev} \qquad 4.25$$
$$\alpha_D = -.106 \text{ Mev}$$

Expectation values with bare lines included:

$$\begin{aligned} \alpha_S &= +\ 2.00 \ \text{Mev} \\ \alpha_{SD} &= -\ 3.09 \ \text{Mev} \\ \alpha_S &= -\ 0.719 \ \text{Mev} \end{aligned} \qquad 4.26$$

The comparison of $\langle V(3\text{-body})\rangle$ with $\langle V(2\text{-body})\rangle$ is made in the next Section.

Section V

COMPARISON OF THE THREE-BODY AND TWO-BODY POTENTIAL IN THE HYPERTRITON

As was stated at the beginning of Section IV, the wave function employed for the calculation of three-body expectation values was

$$\psi = \cos\phi\, \psi_S + \sin\phi\, \psi_D$$

We therefore get for our expectation value of V(3-body)

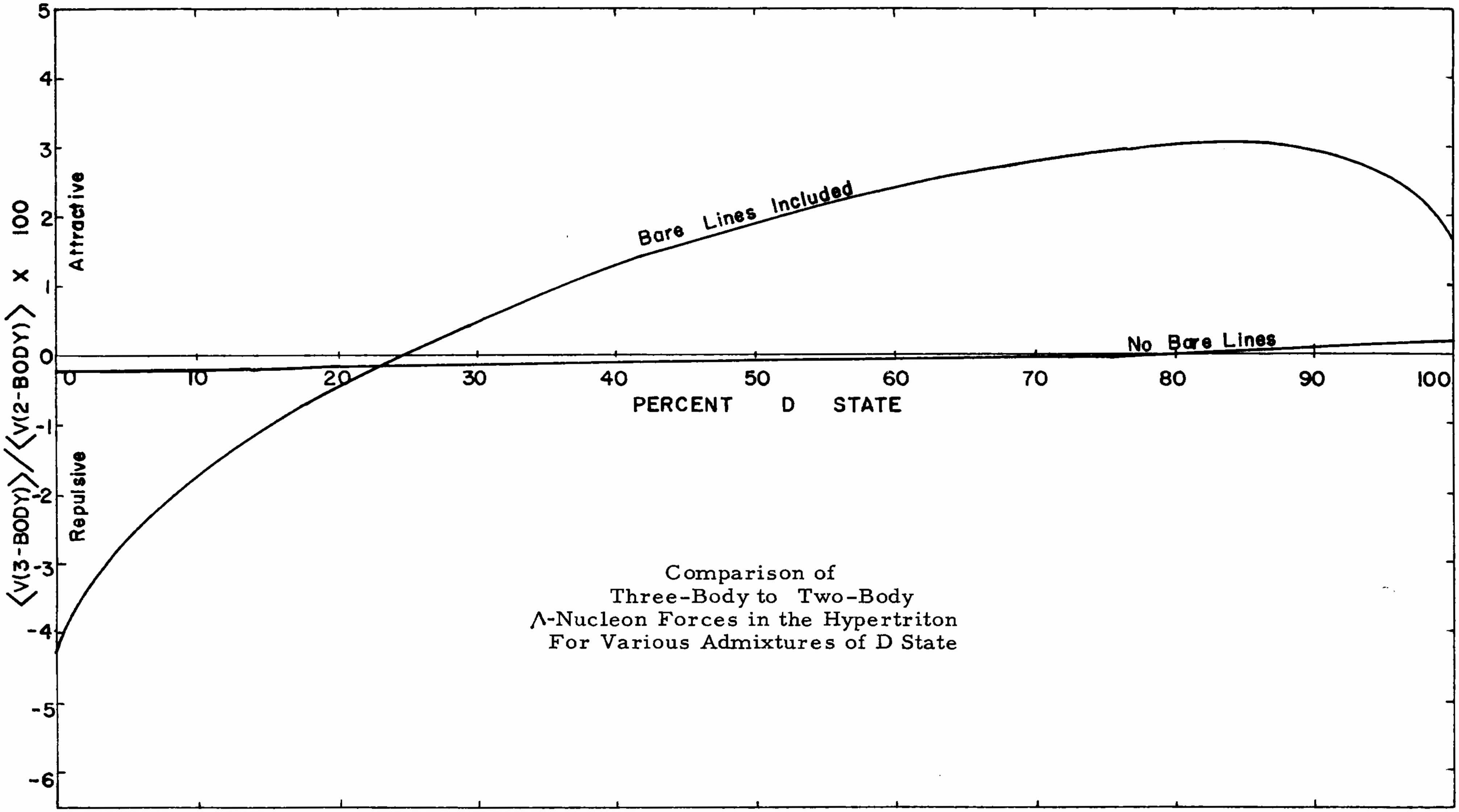
$$\langle V(\text{3-body}) \rangle = \cos^2\phi\, \alpha_S + \sin\phi \cos\phi\, \alpha_{SD} + \sin^2\phi\, \alpha_D \qquad 5.1$$

Table 5.1 shows the variation of $\langle V(\text{3-body})\rangle$ for both cases, that is, with the α's given by equation 4.25 and 4.26

The value of $\langle V(\text{2-body})\rangle$ was taken to be -46 Mev, as was calculated in Section III (equation 3.13). In Fig. 5.1, the ratio $\langle V(\text{3-body})\rangle$ / $\langle V(\text{2-body})\rangle$ is plotted in per cent against percentage D state. We see that if the three-body potential is given by the terms with no bare lines its contribution is small, whereas if the bare line three-body potential is used, it may contribute about 4% for small D admixtures, and will be repulsive. We also see that for large D admixtures the contribution just exceeds 3%. However, since $\langle V(\text{2-body})\rangle$ was calculated using the S state wave function given by equation 3.1, the curve in Fig. 5.1 shows only the effect of three-body forces for large D state admixtures in the wave function.

ϕ Degrees	Percent D State 100xsin ϕ	$\langle V(3\text{-body})\rangle$ Mev no bare lines	$\langle V(3\text{-body})\rangle$ Mev bare plus no bare lines	$[\langle V(3\text{-body})\rangle / \langle V(2\text{-body})\rangle] \times 100$ bare lines	$[\langle V(3\text{-body})\rangle / \langle V(2\text{-body})\rangle] \times 100$ bare plus no bare lines
0	0	.108	2.00	-.24	-4.35
10	3.01	.122	1.39	-.26	-3.02
20	11.7	.111	.70	-.24	-1.52
30	25	.107	-.02	-.23	.04
40	41.4	.055	-.65	-.12	1.41
50	58.7	.042	-1.11	-.09	2.41
60	75	0	-1.38	0	3.00
70	88.3	-.043	-1.39	.09	3.02
90	100	-.106	- .72	.23	1.56

<u>Table 5.1</u>



Comparison of
Three-Body to Two-Body
Λ-Nucleon Forces in the Hypertriton
For Various Admixtures of D State

FIG. 5.1

Section VI

VARIATION IN THE THREE-BODY EXPECTATION VALUES WITH TRIAL WAVE FUNCTION PARAMETERS

The foregoing calculations were made with the trial wave function parameters α and β equal to 1.8 and 1.4, and the core size of the Λ-nucleon potential ϵ equal to .386. The quantities α and ϵ were chosen in a consistent manner but if β, the deuteron parameter, were actually different from 1.4, α and ϵ would also change in order that zero binding for the Λ-particle would be obtained. There is also the possibility that the Λ-nucleon potential, which is taken to be given by V_{DF}, may not be a very satisfactory potential. The entire calculation from that standpoint was rather artificial since to be consistent one should also include the Λ-nucleon two-body force with bare lines V_{LR}. This, however, was not taken into consideration. Nevertheless, from a phenomenological point of view V_{DF} probably has all the required features of the Λ-nucleon two-body potential. Qualitative changes in the Λ-nucleon potential would be reflected in the wave function parameters α and β, and in the cut-off parameter ϵ.

It is therefore of interest to investigate changes of $\langle V(3\text{-body})\rangle$ with small variations in the quantities α, β and ϵ. Writing $E_\Lambda(\alpha, \beta, \epsilon)$ for $\langle V(3\text{-body})\rangle$ we have

$$E_\Lambda(\alpha, \beta, \epsilon) = E_\Lambda(\alpha_0, \beta_0, \epsilon_0) + \frac{\partial E_\Lambda}{\partial \alpha}\delta\alpha + \frac{\partial E_\Lambda}{\partial \beta}\delta\beta + \frac{\partial E_\Lambda}{\partial \epsilon}\delta\epsilon \qquad 6.1$$

where the derivatives are to be evaluated at α, β, ϵ = 1.8, 1.4 and .386.

For this calculation only the S part of the wave function was considered. Again, E_Λ was evaluated for the case of no bare lines and for the sum of bare and no bare line graphs. The central parts of the three-body potential were neglected also since they are small compared to the $S_{12,xy}$ part. We therefore write

$$E_{\Lambda NB} = \frac{.26927}{N_S} \int_\epsilon^\infty \int_\epsilon^\infty \int_{-1}^{1} e^{-\alpha(x+y)-\beta\rho} \frac{e^{-x}}{x^3 y} \left(2k_1(y) + yk_0(y)\right)(x^2+3x+3) \cdot (3\mu^2-1)\, x^2 dx\, y^2 dy\, d\mu \qquad 6.2$$

and
$$E_{\Lambda B} = \frac{6.6618}{N_S} \int_\epsilon^\infty \int_\epsilon^\infty \int_{-1}^{1} e^{-\alpha(x+y)-\beta\rho} \frac{e^{-x-y}}{x^3 y^3} \left(1+x+\frac{x^2}{3}\right)\left(1+y+\frac{y^2}{3}\right)(3\mu^2-1) \cdot x^2 dx\, y^2 dy\, d\mu \qquad 6.3$$

Consider 6.2 first

$$\frac{\partial E_{\Lambda NB}}{\partial \alpha} = -\frac{E_{\Lambda NB}}{N_S}\frac{\partial N_S}{\partial \alpha} - \frac{.26927}{N_S} \int_\epsilon^\infty \int_\epsilon^\infty \int_{-1}^{1} e^{-\alpha(x+y)-\beta\rho}(x+y)\frac{e^{-x}}{x^3 y}\left(2k_1(y)+yk_0(y)\right) \cdot (x^2+3x+3)(3\mu^2-1)\, x^2 dx\, y^2 dy\, d\mu \qquad 6.4$$

$$\frac{\partial E_{\Lambda NB}}{\partial \beta} = -\frac{E_{\Lambda NB}}{N_S}\frac{\partial N_S}{\partial \beta} - \frac{.26927}{N_S} \int_\epsilon^\infty \int_\epsilon^\infty \int_{-1}^{1} e^{-\alpha(x+y)-\beta\rho}\frac{e^{-x}}{x^3 y}\left(2k_1(y)+yk_0(y)\right)(x^2+3x+3) \cdot \sqrt{x^2+y^2-2xy\mu}\,(3\mu^2-1)\, x^2 dx\, y^2 dy\, d\mu \qquad 6.5$$

$$\frac{\partial E_{\Lambda NB}}{\partial \epsilon} = -\frac{.26927}{N_S}\left\{ \int_\epsilon^\infty \int_{-1}^{1} e^{-\alpha(y+\epsilon)-\beta\sqrt{\epsilon^2+y^2-2\epsilon y\mu}} \frac{e^{-\epsilon}}{\epsilon y}\left(2k_1(y)+yk_0(y)\right) \cdot (\epsilon^2+3\epsilon+3)(3\mu^2-1)\, y^2 dy\, d\mu \right.$$
$$\left. + \int_\epsilon^\infty \int_{-1}^{1} e^{-\alpha(x+\epsilon)-\beta\sqrt{\epsilon^2+x^2-2\epsilon x\mu}} \frac{e^{-x}}{x}\epsilon\left(2k_1(\epsilon)+\epsilon k_0(\epsilon)\right) \cdot (x^2+3x+3)(3\mu^2-1)\, dx\, d\mu \right\} \qquad 6.6$$

In these integrals $3\mu^2 - 1$ was replaced by $\zeta = .07171$ as given by equation 4.11.

$$\text{Since } N_S = \frac{8\alpha^2 + 5\alpha\beta + \beta^2}{\alpha^3(\alpha+\beta)^5} = .020737$$

$$\frac{\partial N_S}{\partial \alpha} = -.0485$$

$$\frac{\partial N_S}{\partial \beta} = -.0263$$

The integral part of 6.4 was performed in the same manner as in Section IV. The result gave

$$\frac{\partial E_{\Lambda NB}}{\partial \alpha} = .205$$

When calculating 6.4 it was noticed that the ratio of the second term to the first term on the right-hand side was about $-\frac{1}{4}$, and it was therefore assumed small when calculating 6.5. Using this we obtained

$$\frac{\partial E_{\Lambda NB}}{\partial \beta} = .096$$

Equation 6.6 is a much simpler expression since only a double integration was involved. The result is

$$\frac{\partial E_{\Lambda NB}}{\partial \epsilon} = -.977$$

Collecting terms we get

$$E_{\Lambda NB}(\alpha, \beta, \epsilon) = .108 + .205\,\delta\alpha + .096\,\delta\beta - .977\,\delta\epsilon \text{ Mev} \qquad 6.7$$

Similarly, the inclusion of the bare lines gives

$$E_{\Lambda B}(\alpha, \beta, \epsilon) + E_{\Lambda NB}(\alpha, \beta, \epsilon) = 2 + 3.61\delta\alpha + 2.49\delta\beta - 8.46\delta\epsilon \text{ Mev} \tag{6.8}$$

This result is discussed in the conclusions at Section VIII.

Section VII

THREE-BODY FORCES IN HEAVY HYPERNUCLEI

A. General Considerations

The results of Section V lead us to expect that three-body forces may be as large as 4% and repulsive when compared to the two-body Λ-nucleon potential for the case of the hypertriton. In the hypertriton, the Λ-particle is quite loosely bound and therefore one might expect that in heavy hypernuclei, three-body hypernuclear forces may be more important. Furthermore, in heavy hypernuclei, the Λ-particle is not governed by the Pauli principle, so that it could exist in an S state and be in the vicinity of a large number of nucleons.

If we again look at the heavy hypernucleus problem from another point of view, we can argue that three-body forces may not be very important. To illustrate this, we notice that the largest contribution to three-body forces comes from the angular parts of the potential, since they are more singular than the central part. In the case of the hypertriton, the angular parts gave almost all of the contributions, because the wave function introduced correlations. In heavy hypernuclei, one would expect that the effect of correlations would be small since they would tend to average out coherently when the Λ- particle interacted with many pairs of neighbouring nucleons.

The following consideration of the three-body problem follows in many respects the work reported by Drell and Huang (14). We consider a large hypernucleus consisting of A nucleons and one Λ-particle contained in a volume v. We write the volume v as

$$v = \frac{4}{3}\pi\left(\frac{\eta}{\mu}\right)^3 A \qquad 7.1$$

where $\frac{1}{\mu} = 1.42 \times 10^{-13}$ cm, a meson compton wave length, and η is a parameter which determines the nuclear density.

We then imagine that A is large and the hypernucleus is composed of an equal number of spin up-and-down protons and neutrons, and that the total spin of the nuclear part is zero. With this assumption, the total spin will be given by $\bar{\sigma}_\Lambda$. Following Drell and Huang, we further assume that no correlations are present. This will mean that the tensor and $(\bar{\sigma}_1 \cdot \bar{\sigma}_\Lambda)$ terms in the Λ-nucleon two-body potential, and all of the angular parts of the three-body Λ-nucleon potentials will average to zero when the expectation value is calculated. Also, the three-body pair term given by 2.13 will vanish The surviving terms are the spin independent central part of the two body potential and the $(\bar{\tau}_1 \cdot \bar{\tau}_2)(\bar{\sigma}_1 \cdot \bar{\sigma}_2)$ terms of the three-body potentials.

B. The Wave Function

Following Drell and Huang, we further assume that the wave function of the A nucleons is given by

$$\psi = \frac{1}{\sqrt{A!}}\begin{vmatrix} \phi_1(1) & \cdots & \phi_1(A) \\ \cdots & \cdots & \cdots \\ \phi_A(1) & \cdots & \phi_A(A) \end{vmatrix} \qquad 7.2$$

where $$\phi_i(j) = \frac{1}{\sqrt{V}} e^{i\bar{k}_i \cdot \bar{r}_j} \chi_i(\sigma_j)\, \nu_i(\tau_j)$$

which are all mutually orthogonal.

$\chi_i(\sigma_j)$ and $\nu_i(\tau_j)$ are the spin and isotopic spin variables assigning the i'th spin and isotopic spin state to the j'th nucleon in the spatial state k_i and v, the volume, is defined in equation 7.1 .

If we consider the A nucleons as consisting of an equal number of spin up-and-down protons and neutrons, the total spin of the system is zero. Since the Λ-particle is not governed by the Pauli principle, we can write the total wave function for the A nucleons plus Λ-particle as

$$\Phi = \psi \cdot \psi_\Lambda \tag{7.4}$$

where we put

$$\psi_\Lambda = \frac{1}{\sqrt{V}} e^{i\bar{k}_\Lambda \cdot \bar{r}_\Lambda} \chi_\Lambda(\sigma_\Lambda) \tag{7.5}$$

where $\bar{k}_\Lambda$ can take on the value of any of the $\bar{k}_i$'s .

C. <u>Calculation of the Two- and Three-body Potential Expectation Values</u>

We first calculate the expectation value of the two-body potential. We use the potential V_{DF} given by equation 1.9 and consider only the part which is spin independent. The result is

$$\text{V(2-body)} = \sum_{i=1}^{A} \int \Phi^* V(i\Lambda) \Phi \, d\tau^{A+1}$$

$$= A \int \Phi^* V(1\Lambda) \Phi \, d\tau^{A+1} \tag{7.6}$$

where

$$V(1\Lambda) = -\mu \frac{g^2}{4\pi}\frac{\mu^2}{4m^2}\frac{g_\Lambda^2}{4\pi}\frac{\mu^2}{4m_\Lambda^2}\frac{6}{\pi}\frac{e^{-x}}{x^3}\left[\frac{4+4x+x^2}{x}K_1(x) + (2+2x+x^2)K_0(x)\right] \quad 7.7$$

and

$$x = \mu|\bar{r}_1 - \bar{r}_\Lambda|$$

Expanding the Slater wave function Φ in 7.6, we obtain

$$V(\text{2-body}) = A\frac{(A-1)!}{A!}\sum_{i=1}^{A}\int \phi_i^*(1)\psi_\Lambda^* V(1\Lambda)\phi_i(1)\psi_\Lambda \, d\tau_1 d\tau_\Lambda \quad 7.8$$

The factor A! comes from the normalization and (A-1)! from the fact that every term in the sum contains (A-1)! permutations of the remaining A-1 factors which integrate to unity.

Therefore since

$$\sum_{i=1}^{A} \phi_i^*(1)\psi_\Lambda^*\psi_\Lambda\phi_i(1) = \frac{A}{V^2}$$

$$\langle V(\text{2-body})\rangle = \frac{A}{V^2}\int V(1\Lambda)\, d\tau_1 d\tau_\Lambda$$

$$= \frac{A}{V}\int V(1\Lambda)\, d\tau_{1\Lambda} \quad 7.9$$

The surviving terms of the three-body potentials given by equations 2.11 and 2.12 reduce to the form

$$(\bar{\tau}_1 \cdot \bar{\tau}_2)(\bar{\sigma}_1 \cdot \bar{\sigma}_2)\, V(1,2,\Lambda)$$

where

$$V_{NB}(1,2,\Lambda) = \mu \frac{g^2}{4\pi}\frac{\mu^2}{4m^2}\frac{g_\Lambda^2}{4\pi}\frac{\mu^2}{4m_\Lambda^2} \cdot \frac{2}{9\pi xy}\left[e^{-x}(K_1(y) - yK_0(y)) + e^{-y}(K_1(x) - xK_0(x))\right] \qquad 7.10$$

$$V_B(1,2,\Lambda) = -\mu \frac{g^2}{4\pi}\frac{\mu^2}{4m^2} \cdot \frac{g_\Lambda^2}{4\pi}\frac{\mu^2}{4m_\Lambda^2} \cdot \frac{\mu}{\Delta}\frac{2}{9}\frac{e^{-x-y}}{xy} \qquad 7.11$$

The expectation value of the three-body potentials is given by

$$\langle V(3\text{-body})\rangle = \sum_{pairs} \int \Phi^* V(ij,\Lambda)(\bar{\tau}_i \cdot \bar{\tau}_j)(\bar{\sigma}_i \cdot \bar{\sigma}_j)\, \Phi \, d\tau^{A+1}$$

$$= \binom{A}{2} \frac{(A-2)!}{A!} \sum_{i,j=1}^{A} \int \phi_i^*(1)\, \phi_j^*(2)\, \psi_\Lambda^* V(12,\Lambda)(\bar{\sigma}_1 \cdot \bar{\sigma}_2)(\bar{\tau}_1 \cdot \bar{\tau}_2)\, \times$$

$$\psi_\Lambda \left[\phi_i(1)\phi_j(2) - \phi_j(1)\phi_i(2)\right] d\tau_1 d\tau_2 d\tau_\Lambda \qquad 7.12$$

where $\binom{A}{2}$ comes from the number of pairs and (A-2)! from the number of permutations of the A-2 factors after a pair is chosen. Since the potential does not contain the spin of the Λ particle, we can put

$$\psi_\Lambda^* \psi_\Lambda = \frac{1}{v}$$

$$\langle V(\text{3-body})\rangle = \frac{1}{2V}\sum_{i,j=1}^{A}\int \phi_i^*(1)\,\phi_j^*(2)\,V(12,\Lambda)(\bar{\sigma}_1\cdot\bar{\sigma}_2)(\bar{\tau}_1\cdot\bar{\tau}_2)\times$$

$$\left[\phi_i(1)\,\phi_j(2) - \phi_j(2)\,\phi_i(1)\right] d\tau_1\, d\tau_2\, d\tau_\Lambda \qquad 7.13$$

The spin-isotopic spin parts are done as follows: since we have assumed A nucleons with $A/2$ protons and $A/2$ neutrons, and A to be large, $5/8$ of the pairs are antisymmetric in their space co-ordinates and $3/8$ are symmetric (see Blatt and Weisskopf p. 147). It therefore follows from the Pauli principle that $5/8$ of the pairs are symmetric and $3/8$ are antisymmetric in the spin-isotopic spin variables.

Table 7.1 therefore gives the spin-isotopic spin expectation values of $(\bar{\sigma}_1\cdot\bar{\sigma}_2)(\bar{\tau}_1\cdot\bar{\tau}_2)$ for the different states.

Space Antisymmetric	Spin triplet I-Spin triplet 1	Spin singlet I-Spin singlet 9
Space Symmetric	Spin triplet I-Spin singlet -3	Spin singlet I-Spin triplet -3

Table 7.1

We then write

$$\langle V(\text{3-body})\rangle = \frac{1}{2V}\sum_{i,j=1}^{A}\int V(12,\Lambda)\,\phi_i^*(1)\,\phi_j^*(2)(\bar{\sigma}_1\cdot\bar{\sigma}_2)(\bar{\tau}_1\cdot\bar{\tau}_2)\phi_i(1)\phi_j(2)\,d\tau_1 d\tau_2\, d\tau_\Lambda$$

$$-\frac{1}{2V}\sum_{i,j=1}^{A}\int V(12,\Lambda)\,\phi_i^*(1)\,\phi_j^*(2)(\bar{\sigma}_1\cdot\bar{\sigma}_2)(\bar{\tau}_1\cdot\bar{\tau}_2)\phi_j(1)\phi_i(2)\,d\tau_1 d\tau_2\, d\tau_\Lambda \qquad 7.14$$

The spin-isotopic spin integrations of the first term of 7.14 give

$$\frac{5}{8} \times \frac{1}{10}(9 \times 1 + 9) + \frac{3}{8} \times \frac{1}{6} (3 \times -3 + 3 \times -3)$$

$$= 0$$

The $^5/_8$ term is multiplied by $9 \times 1 + 9$ since the value 1 can occur in 9 ways (3 spin triplets and 3 I-spin triplets).

The second term on the right-hand side of 7.14 gives

$$- \frac{5}{8} \times \frac{1}{10} (9 \times 1 + 9) - \frac{3}{8} \times \frac{1}{6} (3 \times -3 + 3 \times -3)$$

$$= -\frac{9}{4}$$

where the minus sign before the $^3/_8$ term results from the fact that these terms are antisymmetric in spin-isotopic spin exchange. Equation 7.14 therefore becomes

$$\langle V(\text{3-body}) \rangle = \frac{-9}{8V^3} \sum_{i,j=1}^{A} \int e^{i(\bar{k}_i - \bar{k}_j)\cdot(\bar{r}_i - \bar{r}_2)} V(12,\Lambda)\, d\tau_1 d\tau_2 d\tau_\Lambda$$

$$= -\frac{9}{8} \frac{A^2}{V^3} \int D^2(k_m r_{12})\, V(12,\Lambda)\, d\tau_1 d\tau_2\, d\tau_\Lambda \qquad 7.15$$

$$\text{where} \quad D^2(k_m r_{12}) = \left[\frac{3 j_1 (k_m r_{12})}{k_m r_{12}} \right]^2 \qquad 7.16$$

and $k_m = 1.52\ \mu/\hbar$ or the largest momentum of the highest filled level for the free particle states in the nuclear well. Here we use the same notation as Drell and Huang.

From equations 7.9 and 7.15, the ratio of the expectation values of the three-body to two-body potentials becomes

$$R = \frac{\langle V(\text{3-body})\rangle}{\langle V(\text{2-body})\rangle} = -\frac{9}{8}\,\frac{A}{v}\,\frac{\int D^2(k_m r_{12})\,V(12,\Lambda)\,d^3\bar{r}_{1\Lambda}\,d^3\bar{r}_{2\Lambda}}{\int V(1\Lambda)\,d^3\bar{r}_{1\Lambda}} \qquad 7.17$$

If we first consider the three-body potential with no bare lines, we get from equation 7.7 and 7.10

$$R_{NB} = \frac{A}{12v}\,\frac{\int_{x,y\geq\epsilon} D^2(\alpha\rho)\,\frac{e^{-x}}{xy}\,\left(k_1(y) - y\,k_0(y)\right)d^3x\,d^3y}{\int_{x\geq\epsilon} \frac{e^{-x}}{x^3}\left[(4+4x+x^2)\,\frac{k_1(x)}{x} + (2+2x+x^2)\,k_0(x)\right]d^3x} \qquad 7.18$$

The ratio using the three-body potential given by 7.11 gives

$$R_B = \frac{-\frac{\pi}{24}\,\frac{\mu}{\Delta}\left(\frac{A}{V}\right)\int_{x,y\geq\epsilon} D^2(\alpha\rho)\,\frac{e^{-x\cdot y}}{xy}\,d^3x\,d^3y}{\int_{x\geq\epsilon} \frac{e^{-x}}{x^3}\left[(4+4x+x^2)\,\frac{k_1(x)}{x} + (2+2x+x^2)\,k_0(x)\right]d^3x} \qquad 7.19$$

where we have changed to meson units in 7.18 and 7.19 and we take $\epsilon = .386$. From equations 4.25 and 4.26 in Section IV, we expect that $|R_B|$ is larger than R_{NB}. We therefore evaluate R_B and first choose $\eta = 1$. The numerator of 7.19 becomes, apart from the factor $\frac{-\pi}{24}\frac{\mu}{\Delta}\left(\frac{A}{V}\right)$

$$\frac{9}{\alpha^2}\int_{x,y\geq\epsilon} \mathcal{J}_1^2\frac{(\alpha\rho)}{\rho^2}\,\frac{e^{-x-y}}{xy}\,d^3x\,d^3y$$

$$= \quad \frac{72\pi^2}{\alpha^2}\int_0^\infty \mathcal{J}_1^2\frac{(\alpha\rho)}{\rho}\,d\rho\int_\epsilon^\infty e^{-x}dx\int_{|\rho-x|\geq\epsilon}^{\rho+x\geq\epsilon} e^{-y}dy$$

$$= \quad -\frac{72\pi^2}{\alpha^2}\int_0^\infty \mathcal{J}_1^2(\alpha\rho)\,\frac{e^{-\rho}d\rho}{\rho}\int_\epsilon^\infty e^{-2x}dx$$

$$+\frac{72\pi^2}{\alpha^2}\int_0^\infty \mathcal{J}_1^2\frac{(\alpha\rho)}{\rho}e^{\rho}d\rho\int_{\rho+\epsilon}^\infty e^{-2x}dx$$

$$+\frac{72\pi^2}{\alpha^2}\int_{2\epsilon}^\infty \mathcal{J}_1^2(\alpha\rho)\frac{e^{-\rho}}{\rho}d\rho\int_\epsilon^{\rho-\epsilon}dx$$

$$= \quad \frac{72\pi^2}{\alpha^2}\int_{2\epsilon}^\infty \mathcal{J}_1^2(\alpha\rho)\,\frac{e^{-\rho}}{\rho}(\rho-2\epsilon)\,d\rho$$

This integral and the denominator of 7.19 were evaluated numerically.

Using the relation

$$v = \frac{4}{3}\pi\eta^3 A$$

the result gave

$$R_B = -.0047$$

R_B was also calculated for η = .8 and 1.2 . The results are shown in Table 7.2

η	R_B
.8	-.0051
1	-.0047
1.2	-.0043

Table 7.2

Since R_{NB}, which is given by 7.18, is of opposite sign, it will tend to reduce the value of the ratio if included. From the calculations in Section IV, however, we expect that it would be small. Table 7.2 then shows that three-body forces are not likely to contribute more than about .5% to the hypernuclear force in heavy hypernuclei if the correlations average out to zero. We can obtain a qualitative upper estimate of the size of three-body forces in heavy hypernuclei from the above table if correlations are present. The radius (in meson units) of a nucleus can be represented by the formula

$$R = \eta A^{1/3} \qquad 7.20$$

where the parameter η is a measure of the nuclear density. If the values of η , corresponding to a heavy hypernucleus and the hypertriton, are estimated, and the corresponding values of R_B taken from Table 7.2, the ratio of the two values of R_B gives a measure of the relative importance of three-body forces for the two cases. This ratio may then be multiplied by 4.35% to yield a qualitative upper estimate of the relative importance of three-body forces in heavy hypernuclei in the presence of correlations, since it was shown in Section V that the relative importance of three-body forces in the hypertriton could be as large as 4.35% when correlations were included.

An average taken over the data for heavy nuclei on page 15 of Blatt and Weisskopf gives

$$\eta = \frac{1.342}{1.42} = .94$$

The value of η corresponding to the hypertriton is more difficult to estimate since the meaning of η is lost in this case. This can be illustrated as follows: for heavy hypernuclei the Λ-particle and the A nucleons are all contained in the volume $v = \frac{4}{3}\pi\eta^3 A$ but in the hypertriton the question arises as to what volume should be considered for the evaluation of η, the deuteron volume or the total volume of the hypertriton.

Gerjuoy and Schwinger (19) have performed a variational calculation on the deuteron considering S and D state wave functions. The S part of their wave function was of the form $e^{-\frac{1}{2}\mu\rho^2}$. If we put $\mu = \frac{1}{R^2}$ where R is the effective radius of the deuteron, the results of their calculation give

$$R = \frac{2.8 \times \sqrt{2}}{1.5 \times 1.42}$$

$$= 1.86 \text{ meson units}$$

Taking A equal to 2 in this case corresponds to

$$\eta = \frac{1.86}{2^{1/3}}$$

$$= 1.475$$

Interpolating and extrapolating Table 7.2 graphically we obtain

$$R_B(\eta = .94) = -.0048$$

$$R_B(\eta = 1.475) = -.0038$$

We therefore conclude that if strong correlations were present in a heavy hypernucleus, the relative importance of three-body forces could be as important as 5.5%.

Section VIII

CONCLUSIONS

It must be admitted that there are many weak points in the derivation of the Λ-nucleon three-body potential. To begin with, the interaction Hamiltonian given by equation 1.7 may not be correct. In fact, it would be quite different if the spins of the Λ and Σ particles were not $\frac{1}{2}$. In addition, if the parities of Λ and Σ were not the same, this would also change the form of the interaction Hamiltonian.

Another assumption was the fact that K meson exchange terms were small. For the two-body potential, the K meson may play a very important role, but one might expect that they would not be so important in three-body forces. One would in fact expect that if the entire Λ-nucleon potential were due to the exchange of the K meson, then three-body forces would be about as important in hypernuclei as they are in ordinary nuclei.

Another question is the validity of the use of perturbation theory for strong interactions, which is still unresolved. Furthermore, approximations were made in the perturbation theory, the most important of which was the assumption that the baryon wave functions were delta functions and the baryon recoil was neglected, which would introduce an appreciable error in the exchange of mesons of large momentum. On the other hand, disregarding the mass difference $\Delta = M_\Sigma - M_\Lambda$, in the energy denominators, introduced an error in the exchange of mesons of very small momentum.

Finally, after deriving the three-body potentials 2.11, 2.12 and 2.13, it was assumed that the pair term 2.13 was damped.

If we examine all these items we see that the most critical factor was probably the assumption of the form of the interaction Hamiltonian, because once it is chosen the qualitative form of the three-body potential is fixed. Therefore, if the interaction Hamiltonian given by equation 1.7 is reasonable, we conclude that the three-body potentials that were subsequently derived have the correct form, i.e. they depend very strongly on correlations and are less important if no correlations exist in the hypernucleus.

The next item subject to criticism is the method of determining the wave function for the hypertriton and core ϵ associated with the Λ-nucleon three-body potential. This was done in Section III by performing a calculation on the hypertriton, considering only two-body forces. To begin with, the choice of the Λ-nucleon two-body potential V_{DF}, given by equation 1.9, may be criticized. Also, the trial wave function was not suitable for use with the phenomenological nucleon-nucleon potential, since the deuteron part did not vanish at the hard core associated with it. These two factors are discussed below.

(1) The Λ-Nucleon Two-body Potential

One of the primary results that was desired from this calculation was to determine what core size ϵ should be used for the Λ-nucleon three-body potential. To achieve this, one must use a two-body Λ-nucleon potential in the hypertriton, which is derived in a similar manner to that of the three-body potential. It is essential also that the two-body potential possess the desirable features, i.e. it should be more attractive for the Λ-nucleon singlet state in order to be consistent with the recent evidence (11). The potential V_{LR} given by 1.10 is more attractive for triplet states. These considerations lead to the choice of V_{DF}, given by equation 1.9.

(2) The Error Involved In Calculating The Deuteron Part Of The Energy

Fig. 3.3 shows the qualitative effect of incurring errors by the use of the trial wave function 3.1. In the first place, the computed kinetic energy is too small because the trial wave function is too smooth, and the absolute value of the potential energy is too large since the wave function is large at the core radius. The phenomenological potential given by equation 3.5 was taken from the recent work of Gammel and Thaler (16), and this potential was chosen since it was of the same form as the Λ-nucleon two-body potential V_{DF} inasmuch as they both have a core. An earlier paper by Gammel, Christian and Thaler (20) shows other phenomenological potentials. We see that the potential given in Table 1, identification number 4 of reference 20, corresponds to the potential used in our calculation. If we consider the potential corresponding to identification number 15, the central part is given by

$$-V_c \frac{e^{-\gamma\rho}}{\gamma\rho}$$

where V_c = 28.28 Mev and γ = .972 inverse meson units, and the hard core radius is zero. We do not consider the tensor part. This potential may be more appropriate to our wave function (equation 3.1).

Calculating the expectation value of this potential with our wave function 3.1 with α , β = 1.8 and 1.4, we obtain

$$\langle V_D(\text{zero core})\rangle = -9.75 \text{ Mev} \qquad 8.1$$

Now if we return to equation 3.12, we see that there

$$\langle V_D(\text{core} = 0.283)\rangle = -14.2 \text{ Mev} \qquad 8.2$$

If we now assume that with zero core in the deuteron potential the trial wave function is fairly good in the region of the potential, then we see that $\langle V_D \rangle$ and $\langle T_{tot} \rangle$ are given by -9.75 and 58 Mev, to a fair approximation. However, if we use the potential with a hard core (0.283 meson units) we see that, using the same trial function $\langle V_D \rangle$ and $\langle T_{tot} \rangle$ given by -14.2 and 58 Mev are both in error. The error involved, therefore, by using the trial wave function is approximately given by the difference, that is (58 - 9.75) - (58 - 14.2) = 4.45 Mev. If we now examine Fig. 3.2, we see that B_Λ is therefore too small by about 4.45 Mev. This would not affect the parameter α in the wave function very much since the minimum is quite shallow. ϵ , the nucleon core radius, will have to be chosen slightly smaller than the value 0.386. To estimate this, we note that the minima for ϵ =0.388 and 0.332 occur at B_Λ = 1 and -58 Mev. If we interpolate we find that the core ϵ may be reduced to 0.380, which is a small change from 0.386.

It therefore can be seen that this correction would not appreciably effect the ratio

$$\frac{\langle V(3\text{-body})\rangle}{\langle V(2\text{-body})\rangle}$$

The methods of evaluating the expectation values of the three-body potentials in Section IV were also in error. An estimate was made of the errors involved and it was found to be about 10%. A further approximation was also made when calculating the ratio

$$\frac{\langle V(3\text{-body})\rangle}{\langle V(2\text{-body})\rangle}$$

for D state admixtures. This was due to the fact that $\langle V(2\text{-body})\rangle$ was calculated only using the S state wave function 3.1, whereas $\langle V(3\text{-body})\rangle$ was calculated using D state admixtures in the wave function. We therefore expect that the ratio is not too good for large D state admixtures. Fig. 3.1 therefore only shows qualitatively how three-body forces depend on D state admixtures.

In Section VI, the expectation values of the three-body potentials were evaluated showing the sensitivity with respect to the quantities α, β and ϵ. We quote again the result for the three-body expectation value.

$$E_{\Lambda NB}(\alpha,\beta,\epsilon) + E_{\Lambda B}(\alpha,\beta,\epsilon) = 2.00 + 3.61\,\delta\alpha + 2.49\,\delta\beta - 8.46\,\delta\epsilon \text{ Mev} \qquad 6.8$$

The variations expected in the parameters are small. For example, a better value of β may be nearer 1.52, and perhaps ϵ should be nearer 0.380. If we retain α constant, we see that the three-body expectation value would go from 2 to 2.35 Mev, which is a change of 17%. We expect the two-body expectation value to change slowly also, so that the ratio would not be effected greatly.

In Section VII, three-body forces were considered in the heavy hypernuclei. Obvious assumptions made there were that only one Λ-particle was present and that no effects due to the presence of the Σ-particle were included. As was stated in the introduction, the Σ- particle probably plays an important role in heavy hypernuclei, because then the binding energies involved become comparable to $\Delta = M_\Sigma - M_\Lambda$. However, since a complete calculation of that nature was not within the scope of this work, a qualitative estimate was made considering only one Λ-particle in a large nucleus.

The main conclusions may then be summarized as follows.

(1) Since the Λ-nucleon potential can be obtained by a consideration of a double pion exchange, one would expect that the two and three-body Λ- nucleon potentials, which have the same order of coupling constant, would have comparable magnitudes. It is found, however, that the three-body potential is much smaller than the two-body potential due to its weak singularity.

(2) The largest terms in the Λ-nucleon three-body potential are non-central, whereas the central part is small.

(3) For the hypertriton, where correlations between spin and directions and between directions will be large, the relative importance of the

three to two-body Λ-nucleon forces may be as large as 4.35%, and repulsive. The admixture of D state tends to decrease the importance, and to make the three-body forces less repulsive.

(4) In heavy hypernuclei, two factors play an important part in determining the magnitude of three-body Λ-nucleon forces. The first factor is the increased nuclear density which enhances the effect of the three-body potential and the second factor is the reduced amount of correlations in the nucleus. The latter effect is the more important since the most singular parts of the Λ-nucleon three-body potential are non-central. It is expected that with no correlations, the relative importance of three to two-body Λ-nucleon forces is only about .5%. With correlations as strong as in the hypertriton, the relative importance may be as large as 5.5%. In either case the contribution is repulsive.

(5) If the three-body potential can only be represented by graphs with no bare nucleon lines, then the relative importance of three to two-body forces in hypernuclei is less than $\frac{1}{2}$% even with correlations present.

APPENDIX A

Spin Integrations Of The Λ-Nucleon Two-Body Potential

(1) Consider first $(\chi_s | \bar{\sigma}_1 \cdot \bar{\sigma}_\Lambda | \chi_s)$

where $$\chi_s = -\sqrt{\tfrac{2}{3}}\,\alpha(1)\alpha(2)\beta(\Lambda) + \sqrt{\tfrac{1}{6}}\left(\alpha(1)\beta(2) + \beta(1)\alpha(2)\right)\alpha(\Lambda) \qquad \text{A1}$$

From the relations

$$\begin{array}{lll} \sigma_x \alpha = \beta & \qquad \sigma_x \beta = \alpha & \\ \sigma_y \alpha = i\beta & \qquad \sigma_y \beta = -i\alpha & \qquad \text{A2} \\ \sigma_z \alpha = \alpha & \qquad \sigma_z \beta = -\beta & \end{array}$$

$$\begin{aligned} (\bar{\sigma}_1 \cdot \bar{\sigma}_\Lambda)\chi_s &= (\sigma_{1x}\sigma_{\Lambda x} + \sigma_{1y}\sigma_{\Lambda y} + \sigma_{1z}\sigma_{\Lambda z})\chi_s \\ &= -\sqrt{\tfrac{2}{3}}\,\alpha(2)\left(2\beta(1)\alpha(\Lambda) - \alpha(1)\beta(\Lambda)\right) + \sqrt{\tfrac{1}{6}}\,\beta(2)\alpha(1)\alpha(\Lambda) \\ &\quad + \sqrt{\tfrac{1}{6}}\,\alpha(2)\left(2\alpha(1)\beta(\Lambda) - \beta(1)\alpha(\Lambda)\right) \\ &= 2\sqrt{\tfrac{2}{3}}\,\alpha(1)\alpha(2)\beta(\Lambda) - \sqrt{\tfrac{1}{6}}\left(5\alpha(2)\beta(1) - \alpha(1)\beta(2)\right)\alpha(\Lambda) \end{aligned} \qquad \text{A3}$$

Therefore $$(\chi_s | \bar{\sigma}_1 \cdot \bar{\sigma}_\Lambda | \chi_s) = -\frac{4}{3} - \frac{2}{3}$$

$$= -2 \qquad \text{A4}$$

(2) We next evaluate $(\chi_s \mid S_{1\Lambda} \mid \chi_s)$

$$\text{where} \quad S_{1\Lambda} = \frac{3(\bar{\sigma}_1 \cdot \bar{r}_{1\Lambda})(\bar{\sigma}_\Lambda \cdot \bar{r}_{1\Lambda})}{\bar{r}_{1\Lambda}^2} - \bar{\sigma}_1 \cdot \bar{\sigma}_\Lambda \qquad \text{A5}$$

Taking $\bar{r}_{1\Lambda}$ as our z axis, we have

$$S_{1\Lambda}\chi_s = (2\sigma_{1z}\sigma_{\Lambda z} - \sigma_{1x}\sigma_{\Lambda x} - \sigma_{1y}\sigma_{\Lambda y})\chi_s$$

$$= -\sqrt{\tfrac{2}{3}}\,\alpha(2)\left(-2\alpha(1)\beta(\Lambda) - 2\beta(1)\alpha(\Lambda)\right) + 2\sqrt{\tfrac{1}{6}}\,\alpha(1)\alpha(\Lambda)\beta(2)$$

$$+\sqrt{\tfrac{1}{6}}\,\alpha(2)\left(-2\beta(1)\alpha(\Lambda) - 2\alpha(1)\beta(\Lambda)\right)$$

$$= \sqrt{\tfrac{2}{3}}\,\alpha(1)\alpha(2)\beta(\Lambda) + \sqrt{\tfrac{2}{3}}\left(\alpha(1)\beta(2) + \beta(1)\alpha(2)\right)\alpha(\Lambda) \qquad \text{A6}$$

$$\text{Therefore} \quad (\chi_s \mid S_{1\Lambda} \mid \chi_s) = -\frac{2}{3} + \frac{2}{3} = 0 \qquad \text{A7}$$

APPENDIX B

Spin Integrations Of The Λ-Nucleon Three-Body Potential

The Λ-nucleon three-body potential has the form

$$V(12,\Lambda) = (\bar{\tau}_1\cdot\bar{\tau}_2)\left[S_{12,xy}V_{xy} + S_{12,x}V_x + S_{12,y}V_y + \bar{\sigma}_1\cdot\bar{\sigma}_2 V_c\right] \qquad \text{B1}$$

where
$$S_{12,xy} = \frac{9(\bar{x}\cdot\bar{y})(\bar{\sigma}_1\cdot\bar{x})(\bar{\sigma}_2\cdot\bar{y})}{x^2y^2} - \bar{\sigma}_1\cdot\bar{\sigma}_2$$

$$S_{12,x} = \frac{3(\bar{\sigma}_1\cdot\bar{x})(\bar{\sigma}_2\cdot\bar{x})}{x^2} - \bar{\sigma}_1\cdot\bar{\sigma}_2 \qquad \text{B2}$$

$$S_{12,y} = \frac{3(\bar{\sigma}_1\cdot\bar{y})(\bar{\sigma}_2\cdot\bar{y})}{y^2} - \bar{\sigma}_1\cdot\bar{\sigma}_2$$

and the vectors $\bar{x}$ and $\bar{y}$ are directed from the Λ-particle to nucleon one and two. The spin of the Λ-particle does not appear.

We now compute the spin integrations with the spin part of the wave function χ_s where χ_s is given by equation A1. This results in four integrations which are

$$(\chi_s|\bar{\sigma}_1\cdot\bar{\sigma}_2|\chi_s),\ (\chi_s|S_{12,x}|\chi_s),\ (\chi_s|S_{12,y}|\chi_s) \text{ and } (\chi_s|S_{12,xy}|\chi_s)$$

(1) $(\chi_s|\bar{\sigma}_1\cdot\bar{\sigma}_2|\chi_s)$

Since χ_s is a spin wave function such that particles one and two are in a mutual triplet spin state, we have

$$(\chi_s|\bar{\sigma}_1\cdot\bar{\sigma}_2|\chi_s) = 1 \qquad \text{B3}$$

(2) We now consider $(\chi_s | S_{12,x} | \chi_s)$

Choosing our $\bar{x}$ axis as the axis of quantization, we obtain

$$S_{12,x}\chi_s = (2\sigma_{1z}\sigma_{2z} - \sigma_{1x}\sigma_{2x} - \sigma_{1y}\sigma_{2y})\chi_s$$

$$= -2\sqrt{\tfrac{2}{3}}\,\alpha(1)\alpha(2)\beta(\Lambda) - 2\sqrt{\tfrac{2}{3}}\,(\alpha(1)\beta(2)+\beta(1)\alpha(2))\alpha(\Lambda) \qquad \text{B4}$$

Therefore $(\chi_s | S_{12,x} | \chi_s) = \dfrac{4}{3} - \dfrac{4}{3}$

$$= 0 \qquad \text{B5}$$

(3) Similarly $(\chi_s | S_{12,y} | \chi_s) = 0$ B6

(4) $(\chi_s | S_{12,xy} | \chi_s)$

For $S_{12,xy}$, we choose again our $\bar{x}$ axis as the axis of quantization and let the polar angles of $\bar{y}$ referred to $\bar{x}$ be θ and ϕ.

$$S_{12,xy}\chi_s = \left(9\cos\theta\,\sigma_{1z}\left(\sigma_{2x}\sin\theta\cos\phi + \sigma_{2y}\sin\theta\sin\phi + \sigma_{2z}\cos\theta\right) - \bar{\sigma}_1\cdot\bar{\sigma}_2\right)\chi_s$$

$$= -\sqrt{\tfrac{2}{3}}\,\beta(\Lambda)\left[9\cos\theta\,\alpha(1)\left(\beta(2)\sin\theta\, e^{i\varphi} + \alpha(2)\cos\theta\right) - \alpha(1)\alpha(2)\right]$$

$$+\sqrt{\tfrac{1}{6}}\,\alpha(\Lambda)\Big[9\cos\theta\,\alpha(1)\left(\alpha(2)\sin\theta\, e^{-i\varphi} - \beta(2)\cos\theta\right)$$

$$-9\cos\theta\,\beta(1)\left(\beta(2)\sin\theta\, e^{i\varphi} + \alpha(2)\cos\theta\right) - \left(\alpha(1)\beta(2) + \alpha(2)\beta(1)\right)\Big]$$

$$= -\sqrt{\tfrac{2}{3}}\,\beta(\Lambda)\left[\alpha(1)\alpha(2)(9\cos^2\theta - 1) + 9\alpha(1)\beta(2)\sin\theta\cos\theta\, e^{i\varphi}\right]$$

$$+\sqrt{\tfrac{1}{6}}\,\alpha(\Lambda)\Big[9\alpha(1)\alpha(2)\sin\theta\cos\theta\, e^{-i\varphi} - (\alpha(1)\beta(2) + \beta(1)\alpha(2))(9\cos^2\theta + 1)$$

$$-9\beta(1)\beta(2)\sin\theta\cos\theta\, e^{i\varphi}\Big] \qquad \text{B7}$$

Therefore

$$(\chi_s | S_{12,xy} | \chi_s) = \frac{2}{3} (9 \cos^2\theta - 1) - \frac{1}{3} (9 \cos^2\theta + 1)$$

$$= 3 \cos^2\theta - 1 \qquad \text{B8}$$

APPENDIX C

Evaluation Of The S and D State Normalization Constants

(1) The S state wave function ψ_s is given by

$$\psi_s = \frac{\chi_s}{\sqrt{N_s}} e^{-\frac{\alpha}{2}(x+y) - \frac{\beta}{2}\rho} \qquad \text{C1}$$

We therefore have by equation 3.7

$$N_S = \int_0^\infty e^{-\alpha x} x\,dx \int_0^\infty e^{-\alpha y} y\,dy \int_{|x-y|}^{x+y} e^{-\beta\rho}\rho\,d\rho$$

where the factor $8\pi^2$ has been omitted. (See page 28)

$$N_S = \int_0^\infty x e^{-\alpha x} dx \int_0^\infty \rho e^{-\beta\rho} d\rho \int_{|\rho-x|}^{\rho+x} y e^{-\alpha y} dy$$

$$= -\frac{1}{\alpha}\int_0^\infty x e^{-2\alpha x} dx \int_0^\infty \rho e^{-(\alpha+\beta)\rho} d\rho \left(x+\rho+\frac{1}{\alpha}\right)$$

$$+\frac{1}{\alpha}\int_0^\infty \rho e^{-(\alpha+\beta)\rho} d\rho \int_0^\rho x\,dx\left(\rho - x + \frac{1}{\alpha}\right) + \frac{1}{\alpha}\int_0^\infty e^{-(\beta-\alpha)\rho}\rho\,d\rho \int_\rho^\infty x e^{-2\alpha x} dx\left(x-\rho+\frac{1}{\alpha}\right)$$

$$= \frac{1}{\alpha}\int_0^\infty e^{-(\alpha+\beta)\rho}\rho\,d\rho\left(\frac{\rho^3}{6} + \frac{\rho^2}{2\alpha} + \frac{\rho}{2\alpha^2}\right)$$

$$= \frac{8\alpha^2 + 5\alpha\beta + \beta^2}{\alpha^3(\alpha+\beta)^5} \qquad \text{C2}$$

(2) The D state wave function is

$$\psi_D = (x^2 S_{1\Lambda,x} + y^2 S_{2\Lambda,y}) \chi_s e^{-\frac{\alpha}{2}(x+y) - \frac{\beta}{2}\rho} \qquad C3$$

where χ_s is given by equation A1.

Our normalization constant N_D is therefore given by (omitting $8\pi^2$ again)

$$N_D = \int_0^\infty \int_0^\infty \int_{|x-y|}^{x+y} \psi_D^* \psi_D \, x\,dx\, y\,dy\, \rho\, d\rho \qquad C4$$

We perform the spin integrations first.

We have $\chi_D = (x^2 S_{1\Lambda,x} + y^2 S_{2\Lambda,y}) \chi_s$

$$= x^2 \sqrt{\tfrac{2}{3}} \left[\alpha(1)\alpha(2)\beta(\Lambda) + (\alpha(1)\beta(2) + \beta(1)\alpha(2))\alpha(\Lambda) \right]$$

$$+ \frac{y^2}{\sqrt{6}} \Big\{ \Big[\alpha(1)\alpha(2)(3\cos^2\theta - 1) + 3(\alpha(1)\beta(2) + \beta(1)\alpha(2)) \sin\theta\cos\theta\, e^{i\varphi}$$

$$+ 3\beta(1)\beta(2)\sin^2\theta\, e^{2i\varphi} \Big] \beta(\Lambda)$$

$$+ \Big[-3\alpha(1)\alpha(2)\sin\theta\cos\theta\, e^{-i\varphi} + (\alpha(1)\beta(2) + \beta(1)\alpha(2))(3\cos^2\theta - 1)$$

$$+ 3\beta(1)\beta(2)\sin\theta\cos\theta\, e^{i\varphi} \Big] \alpha(\Lambda) \qquad C5$$

and $(\chi_D | \chi_D) = 2(x^4 + x^2 y^2 (3\cos^2\theta - 1) + y^4) \qquad C6$

We also have

$$\cos\theta = \frac{x^2 + y^2 - \rho^2}{2xy} \qquad \text{C7}$$

From these relations, equation C4 becomes

$$N_D = 2\int_0^\infty\int_0^\infty\int_{|x-y|}^{x+y} e^{-\alpha(x+y)-\beta\rho}\left[y^4 + x^2y^2\left[3\left(\frac{x^2+y^2-\rho^2}{2xy}\right)^2 - 1\right] + x^4\right] x\,dx\,y\,dy\,\rho\,d\rho$$

$$= \int_0^\infty\int_0^\infty\int_{|x-y|}^{x+y} e^{-\alpha(x+y)-\beta\rho}\left(7x^4 + x^2y^2 - 6x^2\rho^2 + \frac{3}{2}\rho^4\right) x\,dx\,y\,dy\,\rho\,d\rho \qquad \text{C8}$$

This result when explicitly evaluated is rather long. We therefore give the result for $\alpha = 1.8$ and $\beta = 1.4$. It is

$$N_D = 0.64766 \qquad \text{C9}$$

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