# FREE CONVECTION FLOWS ABOUT INCLINED CYLINDERS

by S.B. SAVAGE

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McGILL UNIVERSITY

DEPARTMENT OF CIVIL ENGINEERING AND APPLIED MECHANICS

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# FREE CONVECTION FLOWS ABOUT INCLINED CYLINDERS

by S.B. Savage*

McGill University, Montreal, Canada

*Assistant Professor, Department of Civil Engineering and Applied Mechanics. Member AIAA.

## NOMENCLATURE

| | | |
|---|---|---|
| C,n | = | constants used to define cylinder cross-section |
| e(x) | = | boundary layer thickness function |
| $f(x)$, $\ell(x)$ | = | velocity growth functions |
| $F(\eta)$ | = | stream function profile |
| g | = | gravitational acceleration |
| $G(\eta)$ | = | velocity profile parallel to cylinder generators |
| h(x) | = | temperature growth function |
| $H(\eta)$ | = | temperature profile |
| L | = | reference length |
| Pr | = | Prandtl number |
| $r(x) = \frac{R}{L}$ | , | dimensionless body radius |
| $R(X)$ | = | body radius |
| $t = \frac{T - T_\infty}{T_w - T_\infty}$ | , | dimensionless temperature |
| T | = | dimensional temperature |
| $T_\infty$ | = | temperature far from cylinder |
| $T_w$ | = | surface temperature |
| $\Delta T_\infty = T_w - T_\infty$ | | |
| $u = \frac{U}{U^*}$ | = | dimensionless velocity in x-direction |
| U | = | dimensional velocity in X-direction |
| $U^* = (\beta \Delta T_\infty L g \cos \phi)^{1/2}$ | , | reference velocity |
| $v = \sqrt{\frac{U^* L}{\nu}} \frac{V}{U^*}$ | , | dimensionless velocity in y-direction |
| V | = | dimensional velocity in Y-direction |
| $w = \frac{W}{U^*}$ | , | dimensionless velocity in z-direction |
| W | = | dimensional velocity in Z-direction |

$x = \frac{X}{L}$ , dimensionless spatial coordinate tangential to surface and normal to cylinder generators

$X, Y, Z$ = dimensional spatial coordinates

$y = \frac{Y}{L}\sqrt{\frac{U^{*}L}{\nu}}$ , dimensionless spatial coordinate normal to surface

$z = \frac{Z}{L}$ , dimensionless spatial coordinate parallel to cylinder generators

$\beta$ = volumetric expansion coefficient

$\eta = \frac{Y}{e(x)}$ , boundary layer similarity coordinate

$\nu$ = kinematic viscosity

$\phi$ = angle between cylinder generators and horizontal

$\psi$ = dimensionless stream function

Superscript

Primes denote differentiation

The purpose of this note is to point out the existence of certain classes of three-dimensional free convection flows in which the equations are separable. The particular case of free convection about infinite cylinders which are inclined to the horizontal and have isothermal surfaces will be considered. These flows are somewhat similar to the problem of forced flow about yawed infinite cylinders studied by Prandtl[1], Jones[2], Sears[3] and Cooke[4].

Consider the free convection about an inclined, isothermal, infinite cylinder as shown in Fig. 1. Since none of the flow quantities vary along the direction parallel to the generators of the cylinder the governing boundary layer equations may be expressed in non-dimensional form as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \sqrt{1 - (r')^2}\, t + \frac{\partial^2 u}{\partial y^2} \tag{2}$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} = \tan \phi \; t + \frac{\partial^2 w}{\partial y^2} \tag{3}$$

$$u\frac{\partial t}{\partial x} + v\frac{\partial t}{\partial y} = \frac{1}{Pr}\frac{\partial^2 t}{\partial y^2} \tag{4}$$

where the non-dimensionalization is given in the Nomenclature and the Boussinesq approximation has been adopted. Equations (1) to (4) are subject to the boundary conditions

$$\left.\begin{aligned} &x > 0,\quad y = 0 \qquad && u = v = w = 0,\quad t = 1 \\ &x > 0,\quad y = \infty \qquad && u = v = w = 0 \end{aligned}\right\} \tag{5}$$

As in the analogous forced convection case the flow in the x-y plane can be solved independent of the flow in the x-z plane. The cylinder cross-sections required for similarity are the same as those found previously for free convection about horizontal cylinders by Braun, et al[5]. The reference velocity has been chosen for sake of convenience as:

$$U^* = (\beta \Delta T_\infty L g \cos \phi)^{1/2} \tag{6}$$

in order that Eq. 2 take on the same form as in the two-dimensional flow (horizontal cylinder) case.

A stream function is defined such that

$$u = \frac{\partial \Psi}{\partial y}, \qquad v = -\frac{\partial \Psi}{\partial x} \tag{7}$$

We attempt to find similarity solutions of the form:

$$\Psi = f(x)\, F(\eta), \quad w = \ell(x)\, G(\eta), \quad t = h(x)\, H(\eta) \tag{8}$$

$$\text{where} \quad \eta = \frac{y}{e(x)} \tag{9}$$

If we consider the case of an isothermal wall such that $h' = 0$, $h = 1$, then a similarity analysis leads to the requirements that

$$\left.\begin{aligned} f &= C^{1/8} (x)^{\frac{3+n}{4}} \\ \ell &= C^{-1/4} (x)^{\frac{1-n}{2}} \\ e &= C^{-1/8} (x)^{\frac{1-n}{4}} \\ r' &= \sqrt{1 - Cx^{2n}} \end{aligned}\right\} \tag{10}$$

where C and n are constants which determine the cross-sectional shape of the cylinder.

From Eqn. (2), (3) and (4) we thus obtain the following equations for the velocity profiles in the x and z-directions given by $F'(\eta)$ and $G(\eta)$ and the temperature profile given by $H(\eta)$:

$$F''' + \left(\frac{3+n}{4}\right)FF'' - \left(\frac{n+1}{2}\right)(F')^2 = -H \tag{11}$$

$$G'' + \left(\frac{3+n}{4}\right)FG' - \left(\frac{1-n}{2}\right)F'G = -\tan \phi\, H \tag{12}$$

$$H'' + \left(\frac{3+n}{4}\right)Pr\, FH' = 0 \tag{13}$$

with the boundary conditions

$$\left.\begin{aligned} &F(0) = F'(0) = G(0) = F'(\infty) = G(\infty) = H(\infty) = 0 \\ &H(0) = 1 \end{aligned}\right. \tag{14}$$

Braun, et al.[5] and Ostrach[6] have discussed the various cylinder cross-sections (wedges, parabolic-nose bodies, pointed-nose bodies, etc.) required for the case of two-dimensional similarity flows described by Eqns. (11) and (12). We note that the temperature profiles and the velocity profiles in planes perpendicular to the generators of the cylinder (x - y planes) are the same for the inclined cylinder as for the horizontal cylinder after making an obvious transformation which accounts for the appropriate body force component in the x - y plane (cf. Eqn.(6)). Having obtained the temperature profile $H(\eta)$ and the velocity function $F(\eta)$ in the x - y plane from Eqns. (11) and (12) the velocity profile $G(\eta)$ in the z-direction is easily found from Eqn. (12).

As an example, consider the case of free convection about an infinite isothermal inclined cylinder having a parabolic nose ($n = 1$) immersed in air ($Pr = .72$). Note that the flow developed here would correspond to the flow developed in the stagnation region of an inclined circular cylinder. It is convenient to set $C = 1$, such that the characteristic length L becomes equal to the dimensional radius of curvature at the origin. Thus from Eqn. (10) we have

$$f = x\ , \quad \ell = 1\ , \quad e = 1\ , \quad r = 1/2[x\sqrt{1 - x^2} + \sin^{-1}x] \qquad (15)$$

Equations (11) - (13) subject to the boundary conditions (14) were integrated numerically on an IBM 360-50 digital computer using a Runge-Kutta technique. The temperature and velocity profiles, $H(\eta)$ and $F'(\eta)$, for $Pr = .72$ are shown in Fig. 2. The velocity profiles in the z -direction, $G(\eta)$, for $Pr = .72$ and for several angles of inclination $\phi$ are shown in Fig. 3.

Finally, in passing we note that the Goertler-type series adopted by Saville and Churchill (7) to treat free convection flows about arbitrary shaped horizontal cylinders can be extended to treat the three-dimensional free convection flows about arbitrary shaped inclined cylinders. Thus the solutions just given for the inclined parabolic-nosed cylinder would form the zero order solution for the flow about the inclined circular cylinder.

REFERENCES

1. Prandtl, L., "On Boundary Layers in Three-Dimensional Flow", Rept. and Transl.64, 1946, British Ministry of Aircraft Production.

2. Jones, R.T., "Effects of Sweepback on Boundary Layer and Separation", Rept. 884, 1949, NACA.

3. Sears, W.R., "The Boundary Layer of Yawed Cylinders", Journal of the Aeronautical Sciences, Vol. 19, No. 8, August 1952, pp. 553-565.

4. Cooke, J.C., "The Boundary Layer of a Class of Infinite Yawed Cylinders", Proceedings of the Cambridge Philosophical Society, Vol. 46, 1950, pp. 645-648.

5. Braun, W.H., Ostrach, S. and Heighway, J.E., "Free-Convection Similarity Flows about Two-Dimensional and Axisymmetric Bodies with Closed Lower Ends", International Journal of Heat and Mass Transfer, Vol. 2, March 1961, pp. 121-135.

6. Ostrach, S., "Laminar Flows with Body Forces", Theory of Laminar Flows, edited by F.K. Moore, Princeton University Press, Princeton, N.J., 1964, pp. 528-718.

7. Saville, D.A. and Churchill, S.W., "Laminar Free Convection in Boundary Layers Near Horizontal Cylinders and Vertical Axisymmetric Bodies", Journal of Fluid Mechanics, Vol. 29, Part 2, 1967, pp 391-399.

Savage, S.B.

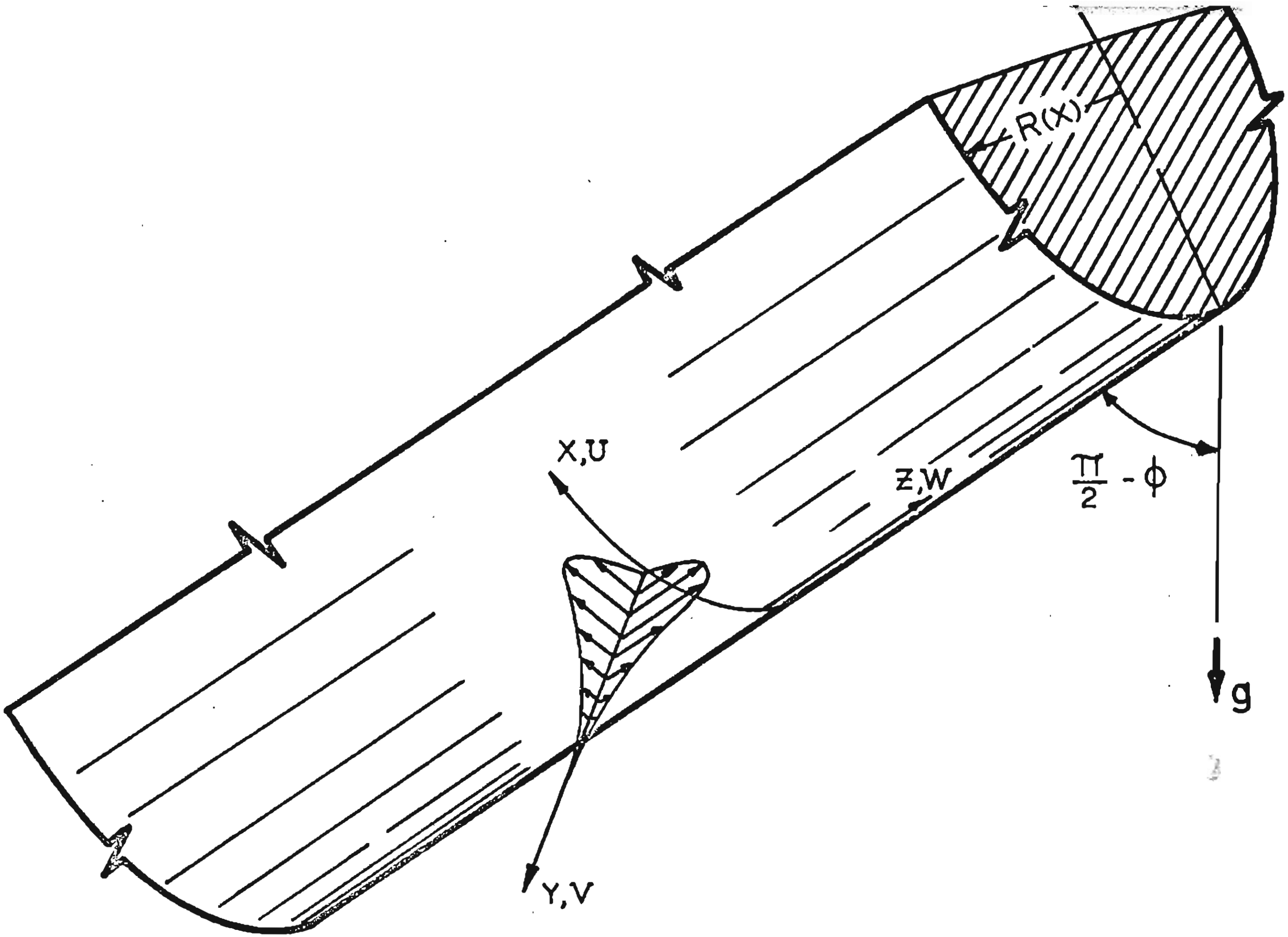


Fig. 1 Inclined cylinder geometry and coordinate system.

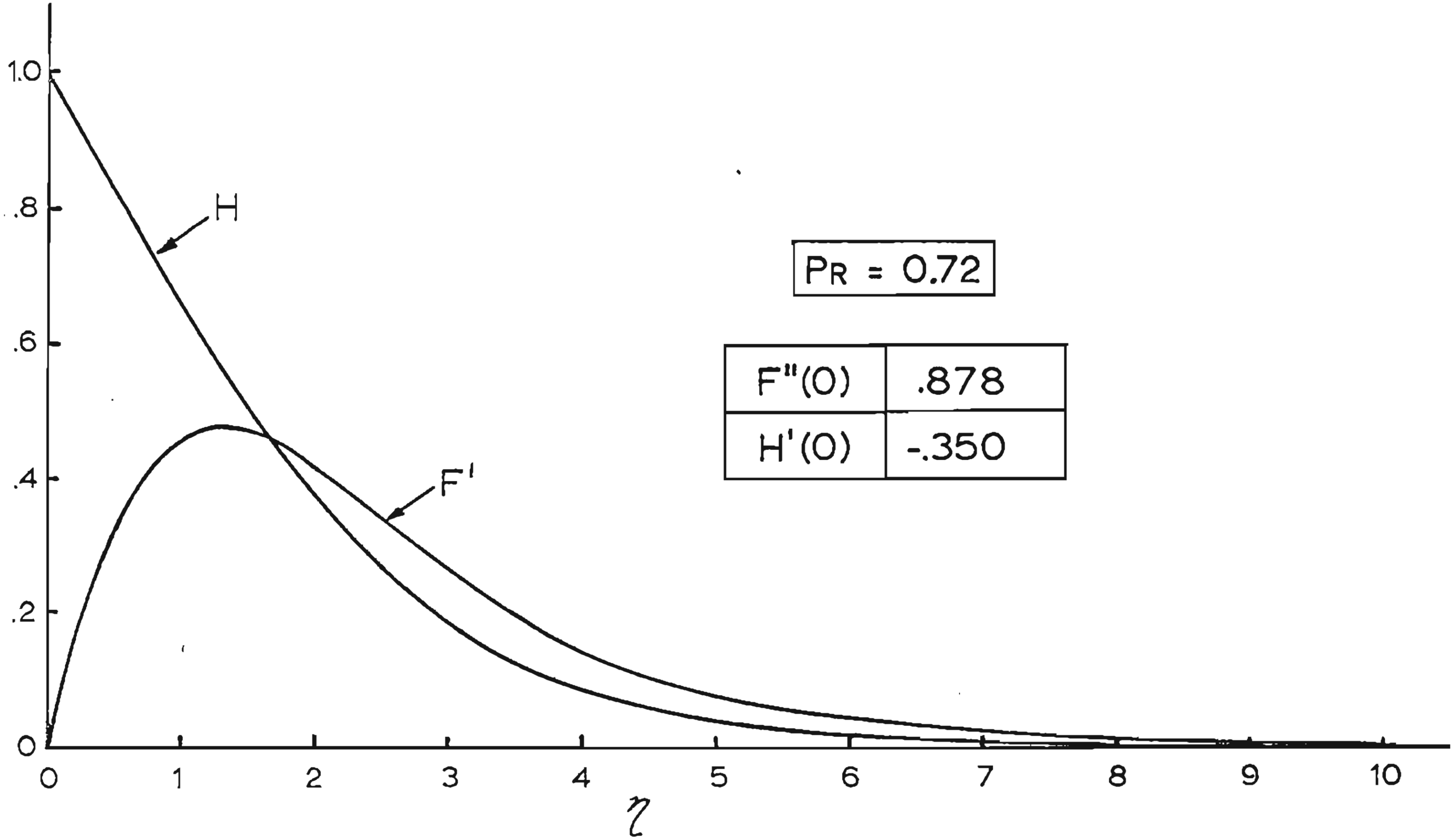


Fig. 2 Velocity and temperature profiles in planes normal to cylinder axis.

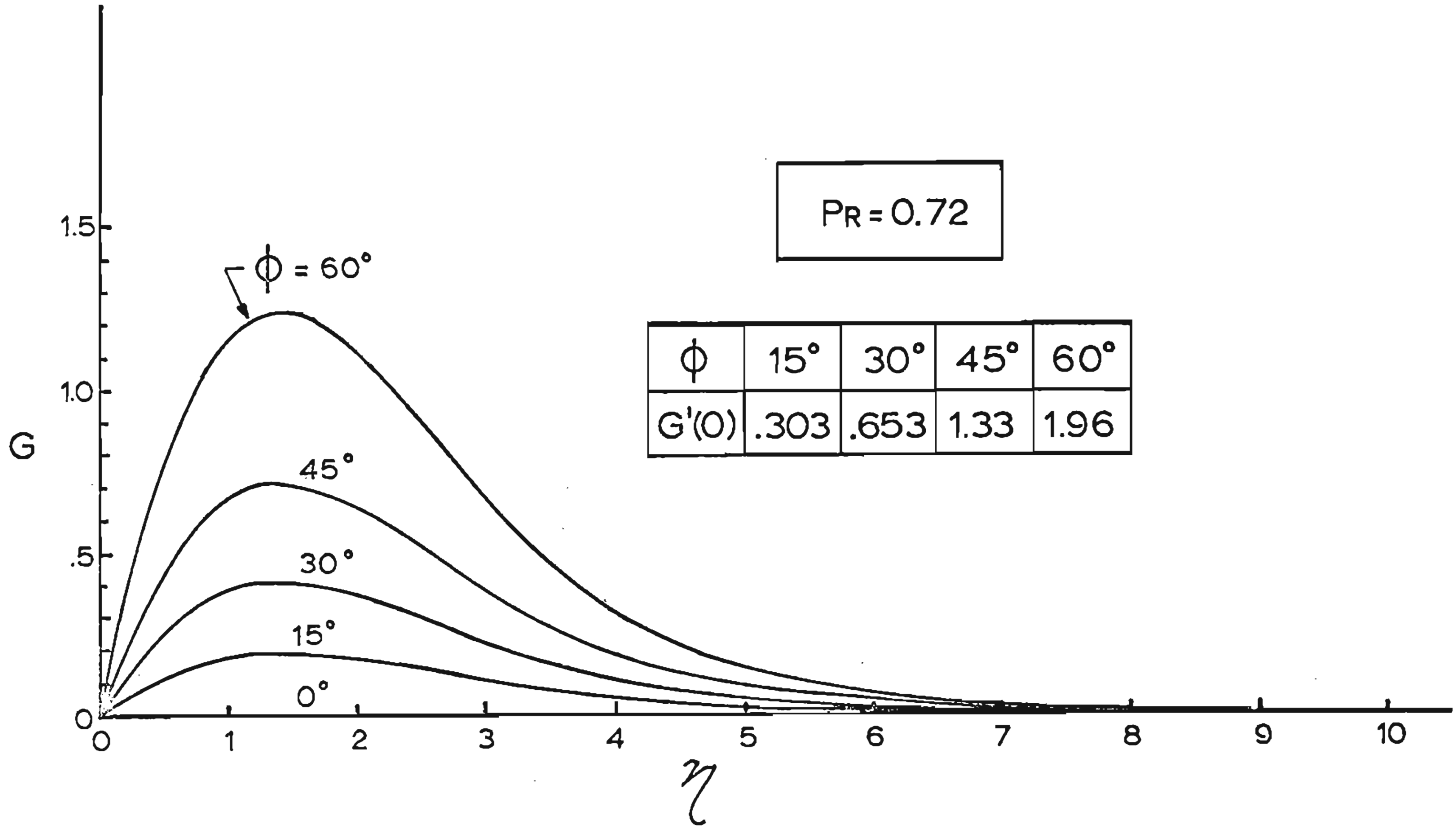


Fig. 3 Velocity profiles parallel to cylinder axis for several angles of inclination.