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R. S. WEAVER and R. E. BELL

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METHOD OF EVALUATING DELAYED COINCIDENCE EXPERIMENTS

R. S. WEAVER and R. E. BELL

Radiation Laboratory, McGill University, Montreal

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Frequently in delayed coincidence work a resolution curve is measured with good accuracy, but no exactly comparable prompt resolution curve is available. The centroid-shift method of evaluating the lifetime τ cannot be applied because the zero of time (centroid of the prompt curve) is not known. If τ is close to the instrumental limit, it cannot be measured from the logarithmic slope of the measured resolution curve. We then require an objective measure of the asymmetry of the measured curve, calculable without reference to any other curve. Such a quantity is the third moment of the measured curve about its own centroid. We show that the normalized third moment is equal to $(2\tau^3 + \varepsilon)$, where ε is the normalized third moment of the prompt curve, usually small and frequently negligible. Thus τ can be evaluated with at most only rough knowledge of the prompt curve. The McGill IBM 650 computer has been programmed to perform the arithmetical work, which is otherwise laborious. An application to an actual example is shown. Estimates are given of the accuracy to be expected from the method.

1. Introduction

In measuring short lifetimes by the delayed coincidence method, the usual procedure is to measure the time resolution curve for the radiation in question (the "delayed" curve), and a curve of the time resolution of the apparatus for a source of radiation of very short lifetime (the "prompt" curve). Ideally the prompt curve should be measured with a source identical in all respects to the delayed source, except for the lifetime. Such a source is almost never available. The position of the centroid of the prompt curve defines the zero of time for the experiment. Unfortunately this centroid position is sensitive to the detailed properties of the prompt source, and is also subject to instrumental drifts. These effects may cause large errors in evaluating the lifetime by the centroid-shift method of Bay[1]) or the area analysis method of Newton[2]), if the lifetime being measured is close to the instrumental limit. Birk, Goldring and Wolfson[3]) have recently introduced a method of analysis making detailed use of the moments of the prompt and delayed resolution curves. Their method thus depends on accurate knowledge of the prompt curve, and in their experiments they adopt a fast cycling procedure to minimize drifts.

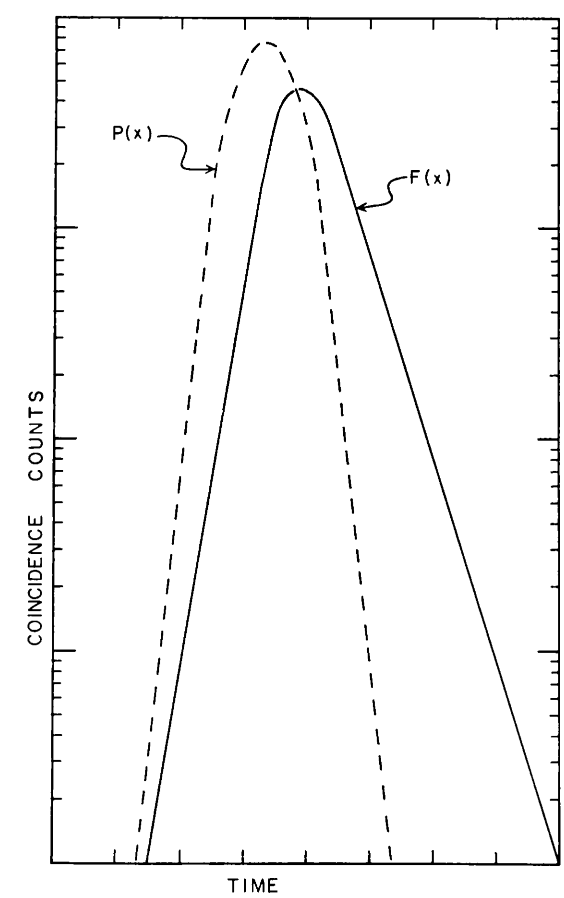

Fig. 1. Normalized prompt and delayed coincidence curves. The prompt curve is drawn with a broken line because we assume that it has not been observed accurately.

[1]) Z. Bay, Phys. Rev. **77** (1950) 419.
[2]) T. D. Newton, Phys. Rev. **78** (1950) 490.
[3]) M. Birk, G. Goldring and Y. Wolfson, Phys. Rev. **116** (1959) 730.

2. Derivation of Moment Relations

We now show that it is possible to evaluate the required lifetime from the shape of the delayed curve with only rough knowledge of the shape of the prompt curve. The relative positions of the centroids of the two curves do not enter the analysis.

In fig. 1, $P(x)$ is the unobserved true prompt curve and $F(x)$ is the measured delayed curve, both normalized to unit area and plotted on a logarithmic scale as a function of x, the delay time. The true zero of time coincides with the centroid of the unobserved prompt curve $P(x)$, and on this scale the centroid of $F(x)$ lies at $+\tau$, where τ is the mean life to be evaluated. Bay[1]) has shown that the following relation holds among the moments of the two curves:

$$M_r(F) = \sum_{k=0}^{r} \frac{r!}{k!(r-k)!} M_{r-k}(P)\, M_k(\omega) \qquad (1)$$

where $M_r(F)$ is the rth moment of $F(x)$,

$$M_r(F) = \int_{-\infty}^{\infty} x^r F(x)\,\mathrm{d}x \qquad (2)$$

and $\omega(x) = [\tau^{-1} \exp(-x/\tau)]$ is the radioactive decay law for the radiation of mean life τ. We have at once by integration $M_n(\omega) = n!\tau^n$; in particular

$$\begin{aligned} M_1(\omega) &= \tau \\ M_2(\omega) &= 2\tau^2 \\ M_3(\omega) &= 6\tau^3, \end{aligned} \qquad (3)$$

and by definition of the zero of time,

$$M_1(P) = 0. \qquad (4)$$

We also define $M_2(P) = \sigma^2$ (this does not imply that $P(x)$ is a Gaussian), and $M_3(P) = \varepsilon$. Now writing out Bay's relation (1) for $r = 1, 2$ and 3, we have

$$M_1(F) = \tau \qquad (5)$$

$$M_2(F) = \sigma^2 + 2\tau^2 \qquad (6)$$

$$M_3(F) = 3\sigma^2\tau + 6\tau^3 + \varepsilon. \qquad (7)$$

These are the first three moments of the delayed curve about the zero of time, i.e. about the (unknown) centroid of the prompt curve.

Now let us take moments about the (calculable) centroid of the delayed curve. We denote such moments by the letter N rather than the previous M. Since the centroid of $F(x)$ lies at $+\tau$, we now have

$$N_r(F) = \int_{-\infty}^{\infty} (x-\tau)^r F(x)\,\mathrm{d}x. \qquad (8)$$

Expanding (8) for the cases $r = 1, 2, 3$, and substituting known values, we get

$$N_1(F) = 0 \quad \text{(definition of centroid of } F\text{)} \qquad (9)$$

$$N_2(F) = \sigma^2 + \tau^2 \qquad (10)$$

$$N_3(F) = 2\tau^3 + \varepsilon. \qquad (11)$$

Thus the procedure is to calculate the centroid of the observed delayed curve $F(x)$, and adopt it as the origin of x, ensuring that (9) is obeyed. Then upon computing $N_3(F)$ from the experimental data, we have

$$\tau = \left\{\frac{N_3(F)-\varepsilon}{2}\right\}^{\frac{1}{3}}. \qquad (12)$$

Here ε, the third moment of the prompt curve $P(x)$ about its own centroid, is usually small and frequently negligible, so that for many cases it is sufficient to write

$$\tau = \left\{\frac{N_3(F)}{2}\right\}^{\frac{1}{3}}. \qquad (13)$$

In any case, rough knowledge of $P(x)$ will give a value of ε ($\equiv M_3(P)$) sufficiently accurate to make a correction. Even if ε is as large as $\frac{1}{3}$ of $N_3(F)$, the error in τ caused by ignoring it altogether is only about 10 %.

3. Sample Computations

The experimental data usually consist of total counts in a number of time channels, recorded in the multichannel analyzer that forms the output of a time-to-amplitude converter. The arithmetical operations are simple, but laborious if performed by hand. The McGill IBM 650 computer has been programmed to do the calculations. The output data consists of the total area of the curve, and the normalized values of the first three moments.

As an example, we use the data from the measurement of the half life of the 1507 keV excited state of Gd^{156}, published by Bell and Jorgensen[4]) (fig. 2). In this case the half-life was

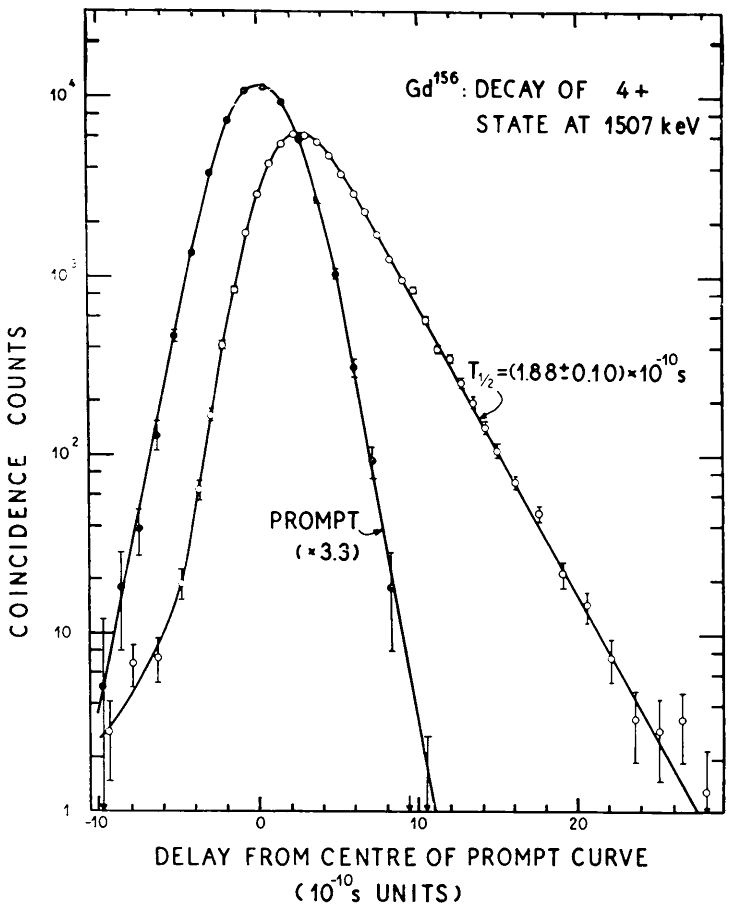

Fig. 2. A delayed resolution curve measured by Bell and Jørgensen[4]), with an approximate prompt curve for comparison, used in sample calculations by the third moment method.

read from the slope of the exponential decay of the delayed curve, and was given as $(1.88 \pm 0.10) \times 10^{-10}$ s. The assigned error of about 5% was mainly due to causes other than statistical. The prompt curve in the diagram is an example of the case discussed above, representing the approximate shape, but not the exact position, of the true prompt curve. The half-life of this state is short enough to permit a useful application of the moment procedure, and long enough to provide independent means of determination of the half life. A computer calculation of the third moment of the delayed curve yielded a result equivalent to a half-life of 1.88×10^{-10} s, the exact agreement with the published value being fortuitous. The calculated third moment of the prompt curve was 0.12% of that of the delayed curve, so that the error caused by neglecting it in calculating τ was only 0.04%. This implies that the rough practical limit mentioned above ($\varepsilon = \frac{1}{3} N_3(F)$) would have been reached if the computed third moment of the delayed curve had been about 300 times smaller, i.e. if the half life had been about 3×10^{-11} s.

As the computer lends itself to fast calculations of such quantities as moments, various rearrangements of the actual experimental data were tried. For example, the leading edge of the delayed coincidence curve has a slight tail at low values of time, due to instrumental resolution difficulties; at low values of counts on both leading and trailing edges the curve has poor statistics. The curve can be smoothed out, and new values of counts in a channel read from this smoothed curve. Using these "smoothed" values, a value of τ was obtained differing by less than 0.1% from the value obtained with the original figures. In another test, successively more and more values at the extremes of the curve were eliminated, and the value of τ obtained was found to be quite insensitive. For example, the curve for Gd^{156} (fig. 2) has values extending over more than 4 decades. If all values more than 3 decades below the peak are neglected, the value of τ obtained is only 4% lower than the "correct" value. Thus even a rough extrapolation of the 3-decade curve to correct for this neglect will give an error much less than 4%. The second moment of the prompt curve, as computed from the delayed curve, was 5.76 units; the second moment, as found directly from the approximate prompt curve itself, was 5.43 units.

4. Errors

The statistical errors that one should assign to these values of σ and τ could have been computed by having the machine carry the standard deviations of the individual points through the calculation. However, in order to give the error in closed form, we now assume that the prompt curve is a Gaussian of standard deviation σ. (The result is not sensitive to this assumption.) In that case it is easily shown (see Appendix) that

[4]) R. E. Bell and M. H. Jørgensen, Nucl. Phys. **12** (1959) 413.

the fractional standard deviation in the computed value of the lifetime is

$$\frac{\Delta\tau}{\tau} = \frac{1}{\sqrt{A}}\left(\frac{5}{12}\right)^{\frac{1}{2}}\left[\left(\frac{\sigma}{\tau}\right)^6 + 3\left(\frac{\sigma}{\tau}\right)^4 + 9\left(\frac{\sigma}{\tau}\right)^2 + \frac{53}{3}\right]^{\frac{1}{2}}$$

where A is the total of all the counts in the delayed curve. Applied to our present example, this estimate gives a standard deviation in τ of 1.3%. In an experiment like this, a systematic error of about 3% is assigned to cover instrument calibration and linearity; the statistical error in this case is therefore negligible. In our example, if the half life became smaller, the fractional standard deviation would rise, reaching 20% for a half life of $\approx 3 \times 10^{-11}$ sec.

5. Concluding Remarks

Another use of this third-moment technique would be in the "self-comparison" method of Bell, Graham and Petch[5]). The two delayed curves obtained would have a difference between their third moments of $4\tau^3$, while the third moment of the prompt curve would contribute nothing, to first order; hence the lower limit of measurement of the half life could be somewhat decreased.

[5]) R. E. Bell, R. L. Graham, and H. E. Petch, Can. J. Phys. **30** (1952) 35.

Throughout this paper we have assumed that the delayed curve corresponds to a single exponential decay, in particular that chance coincidences have been subtracted and that no prompt events are occurring in the delayed curve. The neglected presence of either kind of event will cause errors, as it does in the centroid-shift case. A moment analysis can be made to take account of either type of event, as shown by Birk *et al.*[3]) for the centroid-shift case, but the work is laborious and dependent on the exact circumstances of the measurement.

One of us (R.S.W.) expresses his thanks to the National Research Council of Canada for a Studentship held during the course of this work.

Appendix

CALCULATION OF FRACTIONAL STANDARD DEVIATION

To calculate the standard deviation in our computed τ, we assume that the number of counts, $F(x)\,\mathrm{d}x$, in a time channel of width $\mathrm{d}x$ has a standard deviation $[F(x)\,\mathrm{d}x]^{\frac{1}{2}}$. This error contributes an error in the third moment of amount $(x-\tau)^3\,[F(x)\,\mathrm{d}x]^{\frac{1}{2}}$. Combining the various channel contributions quadratically and replacing the sum over channels by an integral, we get

$$\Delta(N_3) = \left[\int_{-\infty}^{\infty}(x-\tau)^6\,F(x)\,\mathrm{d}x\right]^{\frac{1}{2}}\left[\int_{-\infty}^{\infty}F(x)\,\mathrm{d}x\right]^{-1},$$

the second factor, for normalization, being just A, the total number of counts in the curve. Expanding this expression and making use of eq. (1) and eq. (3), we get

$$\Delta(N_3) = \frac{1}{\sqrt{A}}\left[M_6(P) + 15\tau^2 M_4(P) + 40\tau^3 M_3(P) + 135\tau^4 M_2(P) + 264\tau^5 M_1(P) + 265\tau^6\right]^{\frac{1}{2}}.$$

If P has been observed, the moments given here may be computed. In order to give a definite form for the error, however, we now assume that P is a Gaussian with standard deviation σ; i.e. $M_2(P) = \sigma^2$, $M_4(P) = 3\sigma^4$, $M_6(P) = 15\sigma^6$, and all odd moments are zero. We then get

$$\Delta(N_3) = \frac{1}{\sqrt{A}}\left[15\sigma^6 + 45\tau^2\sigma^4 + 135\tau^4\sigma^2 + 265\tau^6\right]^{\frac{1}{2}}.$$

This is the total error in $2\tau^3$: hence the fractional statistical error can be expressed as

$$\frac{\Delta\tau}{\tau} = \frac{1}{\sqrt{A}}\left(\frac{5}{12}\right)^{\frac{1}{2}}\left[\left(\frac{\sigma}{\tau}\right)^6 + 3\left(\frac{\sigma}{\tau}\right)^4 + 9\left(\frac{\sigma}{\tau}\right)^2 + \frac{53}{3}\right]^{\frac{1}{2}}.$$

In fig. 3 is plotted a curve of $(\Delta\tau/\tau)\sqrt{A}$ vs the ratio of σ/τ, and also the ratio of $W_{\frac{1}{2}}/T_{\frac{1}{2}}$ where $W_{\frac{1}{2}}$ is the full width at half maximum of the promptcurve and $T_{\frac{1}{2}}$ is the half life being measured. If $A = 10\,000$ counts, then the fractional error is read directly as percent.

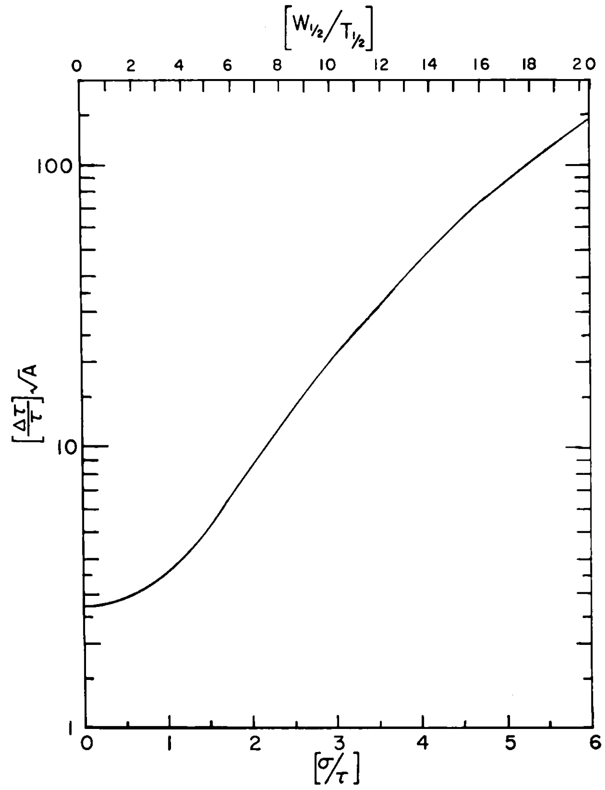

Fig. 3. Fractional standard deviation in τ plotted as a function of σ/τ (lower abscissa) or $W_{\frac{1}{2}}/T_{\frac{1}{2}}$ (upper abscissa). A is the total number of counts in the observed resolution curve.

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