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RESEARCH ANNOUNCEMENT: THE STRUCTURE OF GROUPS WITH A QUASICONVEX HIERARCHY

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Abstract. Let G be a word-hyperbolic group with a quasiconvex hierarchy. We show that G has a finite index subgroup G' that embeds as a quasiconvex subgroup of a right-angled Artin group. It follows that every quasiconvex subgroup of G is a virtual retract, and is hence separable. The results are applied to certain 3-manifold and one-relator groups.

1. Introduction and main results

This announcement concerns a growing body of work much of which is joint with Frédéric Haglund, Chris Hruska, Tim Hsu, and Michah Sageev.

Many groups that arise naturally in topology and combinatorial group theory (e.g. a one-relator group, or 3-manifold group, or HNN extension of a free group along a cyclic subgroup) are associated with small low-dimensional objects. The overall picture presented here suggests that it can be very fruitful to sacrifice the small initial "presentation" in favor of a higher-dimensional but much more organized structure, since this can reveal many hidden properties of the group.

1.1. Main theorem.

Definition 1.1 (Quasiconvex hierarchy)**.** A trivial group has a *length* 0 *quasiconvex hierarchy.* For $h \geq 1$, a group G has a *length* h *quasiconvex hierarchy* if $G \cong A *_C B$ or $G \cong A *_{C^t = C'}$ where A, B have quasiconvex hierarchies of length $\leq (h-1)$, and C is a finitely generated group such that the map $C \to G$ is a quasi-isometry with respect to word metrics.

The main result is the following [47]:

Theorem 1.2. *If G is word-hyperbolic and has a quasiconvex hierarchy then G is the fundamental group of a compact nonpositively curved cube complex X that is virtually special.*

A similar result holds for many groups that are hyperbolic relative to virtually abelian subgroups, and I expect that it holds in general for such groups. However, hyperbolicity cannot be relaxed too much here: For instance, the Baumslag-Solitar

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group $\langle a, t \mid (a^2)^t = a^3 \rangle$ is an example of a one-relator group with a (nonquasiconvex) hierarchy, and there are torsion-free irreducible lattices in $\mathrm{Aut}(T \times T)$ that have quasiconvex hierarchies [7, 51], but none of these groups is virtually special.

1.2. **Application to one-relator groups.** A *one-relator group* is a group having a presentation $\langle a, b, \cdots \mid W^n \rangle$ with a single defining relation. Assuming that W is reduced and cyclically reduced and is not a proper power, the one-relator group has torsion if and only if $n \geq 2$. In this case, all torsion is conjugate into $\langle W \rangle \cong \mathbb{Z}_n$ and the group is virtually torsion-free. We refer to [32] for more information on one-relator groups. A significant feature of one-relator groups with torsion is that they are word-hyperbolic, since the Newman Spelling Theorem provides very strong small-cancellation behavior. It became clear in the 60's that one-relator groups with torsion are better behaved than general one-relator groups, and to test this Gilbert Baumslag made the following:

Conjecture 1.3 ([4])**.** *Every one-relator group with torsion is residually finite.*

The main tool for studying one-relator groups is the Magnus hierarchy. Roughly speaking, every one-relator group G is an HNN extension $H*_{M^t=M'}$ of a simpler one-relator group H where M and M' are free subgroups generated by subsets of the generators of the presentation of G. The hierarchy terminates at a virtually free group of the form $\mathbb{Z}_n * F$. For one-relator groups with torsion, we show that the subgroups M, M' are quasiconvex at each level of the hierarchy in [47]. This result depends upon a variant of the Newman spelling theorem [24, 31]. When G is a one-relator group with torsion, and G' is a torsion-free finite index subgroup, the induced hierarchy for G' is a quasiconvex hierarchy that terminates at trivial groups (instead of finite groups) and is thus covered by Theorem 1.2.

Theorem 1.4. *Every one-relator group with torsion is virtually special.*

As discussed in Section 3, a virtually special word-hyperbolic group has very strong properties, and in particular it is residually finite, so Conjecture 1.3 follows from Theorem 1.4.

1.3. **Application to 3-manifolds.** Prior to Thurston's work, the main tool used to study 3-manifolds was a *hierarchy* which is a sequence of splittings along incompressible surfaces until only 3-balls remain. (An *incompressible surface* is a 2-sided π_1-injective surface along which the fundamental group splits as either an HNN extension or an amalgamated free product.)

It is well-known that every irreducible 3-manifold with an incompressible surface has a hierarchy and every irreducible 3-manifold with boundary has an incompressible surface. It is a deeper result that for a finite volume 3-manifold with cusps, there is always an incompressible geometrically finite surface [10]. In general, an incompressible surface in a hyperbolic 3-manifold is either geometrically finite or virtually corresponds to a fiber (see [6]). A fundamental result of Thurston's about subgroups of fundamental groups of infinite volume hyperbolic manifolds ensures that if the initial incompressible surface is geometrically finite, then the further incompressible surfaces in (any) hierarchy are geometrically finite (see the survey in [8]). Finally, we note that the geometrical finiteness of an incompressible surface where the 3-manifold splits, corresponds precisely to the quasi-isometric embedding of the corresponding subgroup along which the fundamental group splits. Thus, if

M has an incompressible surface then $\pi_1 M$ has a quasiconvex hierarchy and we have:

Theorem 1.5. *If M is a hyperbolic 3-manifold with an incompressible geometrically finite surface then $\pi_1 M$ is virtually special.*

We expect that all hyperbolic fibered 3-manifolds have finite covers with incompressible geometrically finite surfaces, so that all Haken hyperbolic 3-manifolds are virtually special.

Corollary 1.6. *If M is a hyperbolic 3-manifold with an incompressible geometrically finite surface then $\pi_1 M$ is subgroup separable.*

In the 80's Thurston suggested that perhaps every hyperbolic 3-manifold is virtually fibered. The key to proving the virtual fibering problem is the following beautiful result which weaves together several important ideas from 3-manifold topology [2]:

Proposition 1.7 (Agol's fibering criterion)**.** *Let M be a compact 3-manifold, and suppose that $\pi_1 M$ is residually finite $\mathbb{Q}$-solvable. (This holds when $\pi_1 M$ is residually torsion-free nilpotent). Then M has a finite cover that fibers.*

For a Haken hyperbolic 3-manifold M, either it virtually fibers, or the first incompressible surface is geometrically finite. In this case the virtual specialness implies that M has a finite cover with $\pi_1 \widehat{M}$ contained in a graph group which is residually torsion-free nilpotent, so we have:

Corollary 1.8. *Every hyperbolic Haken 3-manifold is virtually fibered.*

1.4. **Application to limit groups.** *Fully residually free groups* or *limit groups* have been a recent focal point of geometric group theory. These are groups G with the property that for every finite set $g_1, \ldots, g_k$ of nontrivial elements, there is a free quotient $G \to \bar{G}$ such that $\bar{g}_1, \ldots, \bar{g}_k$ are nontrivial. Among the many wonderful properties proved for these groups is that they have a rather simple *cyclic hierarchy* terminating at free groups.

(1) $A *_Z B$ where Z is cyclic and malnormal in A, and A, B have such hierarchies.
(2) $A*_{Z^t=Z'}$ where Z is cyclic and malnormal in A and Z, Z' do not have nontrivially intersecting conjugates.
(3) $A *_Z B$ where Z is cyclic and malnormal in A and $B \cong Z \times \mathbb{Z}^n$ for some n.

This hierarchy was obtained in [30], and is also implicit in Sela's retractive tower description of limit groups [45]. This hierarchy allows one to prove that limit groups are hyperbolic relative to free abelian subgroups [11, 3]. Using this cyclic hierarchy and the relative hyperbolicity we are able to see that:

Corollary 1.9. *Every limit group is virtually special.*

Combined with subgroup separability results for virtually special groups that are hyperbolic relative to abelian subgroups we are able to recover Wilton's result that limit groups are subgroup separable [46].

2. Cubulating groups

2.1. **Nonpositively curved cube complexes.** Gromov introduced nonpositively curved cube complexes as a source of simple examples, but they have turned out to play an unexpectedly wide role. An n-*cube* is a copy of $[-1,1]^n$ and a 0-*cube* is a single point. We regard the boundary of an n-cube as consisting of the union of lower dimensional cubes. A *cube complex* is a cell complex formed from cubes, such that the attaching map of each cube is combinatorial in the sense that it sends cubes homeomorphically to cubes by a map modelled on a combinatorial isometry of n-cubes. The *link* of a 0-cube v is the complex whose 0-simplices correspond to ends of 1-cubes adjacent to v, and these 0-simplices are joined up by n-simplices for each corner of an $(n+1)$-cube adjacent to v.

A *flag complex* is a simplicial complex with the property that any finite pairwise-adjacent collection of vertices spans a simplex. A cube complex C is *nonpositively curved* if $\text{link}(v)$ is a flag complex for each 0-cube $v \in C^0$. A map $\phi : Y \to X$ between nonpositively curved cube complex is a *local-isometry* if for each $y \in Y^0$ the induced map $\text{link}(y) \to \text{link}(\phi(y))$ is an adjacency preserving embedding.

2.2. **CAT(0) cube complexes and hyperplanes.** Simply-connected nonpositively curved cube complexes are called *CAT(0) cube complexes* because they admit a CAT(0) metric where each n-cube is isometric to $[-1,1]^n \subset \mathbb{R}^n$ however we rarely use this metric. Instead, the crucial characteristic properties of CAT(0) cube complexes are the separative qualities of their hyperplanes: A *midcube* is the codimension-1 subspace of the n-cube $[-1,1]^n$ obtained by restricting exactly one coordinate to 0. A *hyperplane* is a connected nonempty subspace of C whose intersection with each cube is either empty or consists of one of its midcubes. The 1-cells intersected by a hyperplane are *dual* to it.

Remark 2.1. Hyperplanes have several important properties [42]:

(1) If D is a hyperplane of C then $C - D$ has exactly two components.
(2) Each midcube of a cube of C lies in a unique hyperplane.
(3) A hyperplane is itself a CAT(0) cube complex.
(4) The union of all cubes that D passes through is a convex subcomplex of C (with respect to both the combinatorial and path metrics).

2.3. **Hulls.** The smallest subcomplex containing one of the two components of $\widetilde{X} - D$ is a *halfspace* of $\widetilde{X}$. Each such halfspace is a convex subcomplex of $\widetilde{X}$. The combinatorial convex *hull* of a subspace $S \subset \widetilde{X}$ is the intersection of all halfspaces containing S. We showed in [17, 43] that:

Lemma 2.2 (Cocompact Convex Hulls)**.** *Let G be a word-hyperbolic group acting properly and cocompactly on a CAT(0) cube complex $\widetilde{X}$. Let H be a quasiconvex subgroup. For each compact set C, the subcomplex $\widetilde{Y} = \text{Hull}(HC)$ is H-cocompact.*

A similar result holds in the relatively hyperbolic case, but $\widetilde{Y}$ is H-cosparse *in the sense that $H\backslash\widetilde{Y}$ is quasi-isometric to the wedge of finitely many euclidean flats and half-flats.*

Such cores have proven invaluable for the study of subgroups of free groups, and began to appear in higher dimensions with the work in [44].

The immediate purpose of Lemma 2.2 is to provide a compact local-isometry $Y \to X$ representing a quasiconvex subgroup H of G. For instance, Theorem 3.6

holds by combining Theorem 3.3 with Lemma 2.2. However, Lemma 2.2 plays a larger role in facilitating the small-cancellation theory we discuss in Section 5.

2.4. **Cubulating groups and spaces.** H is a *codimension-1* subgroup of the finitely generated group G if for some r, the complement of its r-thickening $N_r(H)$ in the cayley graph $\Gamma(G)$ has at least two deep components, where a component D is *deep* if $D \not\subset N_s(H)$ for any $s > 0$. Examples of codimension-1 subgroups include edge groups of nontrivial splittings, closed surface subgroups of 3-manifold groups, and any copy of $\mathbb{Z}^n$ inside $\mathbb{Z}^{n+1}$.

For a separating subspace $A \subset B$, its full preimage $\widetilde{A} \subset \widetilde{B}$ consists of various components each of which separates $\widetilde{B}$. There is a tree dual to this data whose vertices correspond to components of $\widetilde{A} - \widetilde{B}$, and whose edge correspond to components of $\widetilde{B}$. In analogy with this, for a collection of codimension-1 subgroups H_i with r_i-thickenings $N^i = N_{r_i}(H_i)$, the translates gN^i form a collection of "walls" in $\Gamma(G)$ and Sageev defined a CAT(0) cube complex C that is *dual* to this system of walls, as well as an action of G on C [42] (see also [15, 41]). We note that the hyperplanes of C correspond to the walls of G. This situation has been abstracted to groups acting on the wallspaces of Haglund and Paulin [9, 35].

Sageev proved that G acts cocompactly on C when the H_i are quasiconvex and G is word-hyperbolic. We have investigated finiteness properties focusing on properness and relative cocompactness when G is relatively hyperbolic in [22, 23]. A quick summary is that the action is proper when there are sufficiently many codimension-1 subgroups, and the action is relatively cocompact when the subgroups are quasi-isometrically embedded and G is relatively hyperbolic. Following the structure that arose in [49], when G is hyperbolic relative to virtually abelian subgroups, we found that the action is *cosparse* in the sense that the quotient $G\backslash C$ is quasi-isometric to the union of finitely many euclidean flats and half-flats, in a sense vaguely reminiscent of a cusped hyperbolic manifold (but with thicker cusps).

Some further work along these lines is: Cubulation of Coxeter groups [34], of certain small-cancellation groups [49], of Gromov's random groups at density $< \frac{1}{6}$ [37], and of rhombus groups related to Penrose tilings [29].

2.5. **Cubulating malnormal amalgams.** A subgroup $M \subset G$ is *(almost) malnormal* if $M \cap M^g$ is trivial (finite) unless $g \in M$. In [26] we prove the following:

Theorem 2.3 (Cubulating Malnormal Amalgams)**.** *Let $G = A *_C B$ split as an amalgamated product with the following properties. Then G acts properly and cocompactly on a CAT(0) cube complex.*

1. *C is quasiconvex in G.*
2. *C is almost malnormal in G.*
3. *A and B act properly and cocompactly on a CAT(0) cube complex.*
4. *Every quasiconvex codimension-1 subgroup of C extends to a quasiconvex codimension-1 subgroup of A and B.*

A special case of this was described in [25] where we cubulated graphs of free groups with cyclic edge groups provided they do not contain Baumslag-Solitar subgroups $\langle a, t \mid (a^m)^t = a^n\rangle$ with $m \neq \pm n$ both nonzero. There is a similar statement for groups that are hyperbolic relative to virtually abelian subgroups, and also similar statement that hold for general graphs of groups. Note that for a subgroup

$H \subset G$, we say that a quasiconvex codimension-1 subgroup $U \subset H$ *extends* to a quasiconvex codimension-1 subgroup $V \subset G$ provided that $U = V \cap H$.

3. Special cube complexes

3.1. **Graph groups.** Let Γ be a simplicial graph. The *right-angled Artin group* or *graph group* $G(\Gamma)$ associated to Γ is presented by:

$$\langle \; v : v \in \text{ vertices}(\Gamma) \quad | \quad [u,v] : (u,v) \in \text{ edges}(\Gamma) \; \rangle$$

A fundamental example of a nonpositively curved cube complex arises from a graph group. This is the cube complex $C(\Gamma)$ containing a torus T^n for each copy of the complete graph $K(n)$ appearing in Γ. Note that the torus T^n is isomorphic to the usual product $(S^1)^n$ obtained by identifying opposite faces of an n-cube. We note that $\pi_1 C(\Gamma) \cong G(\Gamma)$ since the 2-skeleton of $C(\Gamma)$ is the standard 2-complex of the presentation above.

Proposition 3.1 (Properties)**.** *Graph groups have the following properties:*

1. *They are residually torsion-free nilpotent* [14].
2. *They are linear* [28].
3. *They embed in right-angled Coxeter groups and hence in* $SL_n(\mathbb{Z})$ [27, 12].

3.2. **Special cube complexes.** In [21] we defined "special cube complexes" and examined some of their properties. We first defined them in terms of illegal hyperplane pathologies, and subsequently found a simple characterization in terms of local isometries to the cube complex of a graph group. The hyperplane pathology definition of special cube complexes arose from our desire to define canonical completion and retraction above dimension one. In this sense, special cube complexes are "generalized graphs". The theory successfully generalizes to arbitrary dimensions the notion of "clean $\mathcal{VH}$-complex" which was studied in [50] and [48]. While a CAT(0) cube complex is a faithful generalization of a tree, it turns out that nonpositively curved cube complexes are less transparent. Special cube complexes are an effective and flexible high-dimensional generalization of a graph, and faithfully capture many of the properties of a graph that lead to the tractable study of free groups. The class of groups that are fundamental groups of special cube complexes is surprisingly rich.

3.3. **Hyperplane definition of special cube complex.** Let C be a cube complex and let M denote the disjoint union of the collection of midcubes of cubes of C. Let D denote the quotient space of M induced by identifying faces of midcubes under the inclusion map. The connected components of D are the *immersed hyperplanes* of C.

We shall define a special cube complex as a cube complex which does not have certain pathologies related to its immersed hyperplanes.

An immersed hyperplane D *crosses itself* if it contains two different midcubes from the same cube of C.

An immersed hyperplane D is *2-sided* if the map $D \to C$ extends to a map $D \times I \to C$ which is a combinatorial map of cube complexes.

A 1-cube of C is *dual* to D if its midcube is a 0-cube of D. When D is 2-sided, it is possible to consistently orient its dual 1-cubes so that any two dual 1-cubes lying (opposite each other) in the same 2-cube are oriented in the same direction.

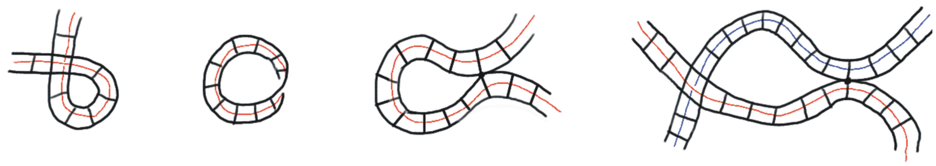

FIGURE 1. Immersed Hyperplane Pathologies

An immersed 2-sided hyperplane D *self-osculates* if for one of the two choices of induced orientations on its dual 1-cells, some 0-cube v of C is the initial 0-cube of two distinct dual 1-cells of D.

A pair of distinct immersed hyperplanes D, E *cross* if they contain midcubes lying in the same cube of C. We say D, E osculate, if they have dual 1-cubes which contain a common 0-cube, but do not lie in a common 2-cube. Finally, a pair of distinct immersed hyperplanes D, E *inter-osculate* if they both cross and *osculate*, meaning that they have dual 1-cubes which share a 0-cube but do not lie in a common 2-cube.

A cube complex is *special* if all the following hold:

(1) No immersed hyperplane crosses itself.
(2) Each immersed hyperplane is 2-sided.
(3) No immersed hyperplane self-osculates.
(4) No two immersed hyperplanes inter-osculate.

Example 3.2. (1) Any graph is special.
(2) Any CAT(0) cube complex is special.
(3) The cube complex associated to a right-angled Artin group is special.

Special cube complexes were fashioned to admit the following property:

Theorem 3.3 (Canonical Completion and Retraction). *Let $\phi : Y \to X$ be a local-isometry of nonpositively curved cube complexes where Y is compact and X is special. There exists a finite cover $\widehat{X} \to X$ so that $\phi : Y \to X$ lifts to an embedding $\widehat{\phi} : Y \hookrightarrow \widehat{X}$ and there is a retraction map $\widehat{X} \to \widehat{\phi}(Y)$.*

3.4. **Right-angled Artin group characterization:** We gave the following characterization of special cube complexes in [21]:

Proposition 3.4. *A cube complex is special if and only if it admits a combinatorial local isometry to the cube complex of a right angled Artin group.*

A quick explanation of Proposition 3.4 is that for a local isometry $B \to C$, the prohibited hyperplane pathologies on C induce the same prohibited pathologies in B. On the other hand, if C is special, then we define a graph Γ whose vertices are the immersed hyperplanes of C, and whose edges correspond to intersecting hyperplanes. Then there is a natural map $C \to C(\Gamma)$ which is a local isometry.

3.5. **Virtual specialness.** We refer the reader to [21] and especially [20] where a version of the following criterion is given that works in the presence of torsion.

Proposition 3.5 (Double Coset Criterion). *Let X be a nonpositively curved cube complex with finitely many immersed hyperplanes. Then X is virtually special if and only if for each pair of immersed hyperplanes A, B and choice of basepoint $x \in A \cap B$, the double coset $\pi_1 A \pi_1 B$ is separable in $\pi_1 X$.*

Using the double coset criterion we specialized Coxeter groups [20] and simple hyperbolic arithmetic groups [5, 18]. This generalizes the results of [1].

Theorem 3.6 (Separability)**.** *If G is word-hyperbolic and virtually special then every quasiconvex subgroup is separable.*

In [19] we prove the following result which is considerably deeper than the virtual specialness criterion of Proposition 3.5:

Theorem 3.7 (Special Malnormal Combination)**.** *Let Q be a compact nonpositively curved cube complex with an embedded 2-sided hyperplane H. Suppose that $\pi_1 Q$ is word-hyperbolic. Suppose that $\pi_1 H \subset \pi_1 X$ is malnormal. Let $N(H)$ denote the open cubical neighborhood of H. Suppose that each component of $Q - N(H)$ is virtually special. Then Q is virtually special.*

4. Working under the LERF assumption

If we assume that a word-hyperbolic group G has separable quasiconvex subgroups, and that G has a quasiconvex hierarchy, then it is substantially easier to prove that G is virtually special, and indeed, much of our introduced technology can be sidestepped. One first finds a collection of sufficiently many codimension-1 quasiconvex subgroups in G so that the action on the dual cube complex given by Sageev's construction is proper and cocompact. One then applies Proposition 3.5 to the resulting cube complex in order to pass to a finite cover. It is easy to imagine approaches to do this for many groups (such as generic hyperbolic Haken 3-manifold groups).

One very specific way to implement this is as follows: Using separability we can pass to a finite index subgroup so that each edge group arising in the splittings of the hierarchy is malnormal [22]. We can then utilize Theorem 2.3 to obtain the desired cubulation as we proceed up the hierarchy, and at each stage obtain virtually specialness by Proposition 3.5.

Theorem 2.3 and Theorem 3.7 taken together can prove the virtual specialness of a malnormal quasiconvex hierarchy, which we reckon is the generic situation. It appears to require more elaborate methods to prove the theorem for an arbitrary quasiconvex hierarchy.

5. Small-cancellation theory over cube complexes

The third ingredient in the proof of Theorem 1.2 is a "small-cancellation theory over cube complexes".

This work joins a variety of generalizations of small-cancellation theory including works of [38, 33, 39, 16]. One difference is that our theory doesn't require a hyperbolic or even relatively hyperbolic base, but is certainly facilitated by it. It is also a bit more combinatorial than geometric among the spectrum of theories, and I hope lends itself more easily to explicit production of examples. We note that related small-cancellation theories were developed recently in [13].

A *cubical relative presentation* $\langle X \mid Y_1, \dots, Y_r \rangle$ has "generators" consisting of a nonpositively curved cube complex X, together with "relators" consisting of a collection of local-isometries $Y_i \to X$. In the classical case, X is a bouquet of circles and each Y_i is a closed immersed path. Letting X^* denote the space obtained by attaching a cone $C(Y_i)$ along its base $Y_i \to X$, we define the group of the cubical relative presentation to be $\pi_1 X^*$.

A case of interest is the graphical small-cancellation theory [40] which reappeared more recently as a special case of a theory of Gromov's [36]. This is where X is 1-dimensional and each $Y_i \to X$ is a combinatorial immersion of graphs.

The "pieces" in the presentation are overlaps between relators and relators or hyperplanes (the latter pieces don't appear in dimension one). When the pieces in each relator Y_i are small relative to the systole of Y_i, then we obtain a small-cancellation theory equipped with a Greendlinger's lemma.

When each relator Y_i has a certain additional wallspace structure generalizing the $B(6)$ condition in [49], then the universal cover of X^* contains a natural system of walls. Under sufficiently stringent small-cancellation conditions we are able to verify that the walls are quasiconvex, and to obtain a proper action on a CAT(0) cube complex. Under further conditions, we are able to obtain splittings along walls, and a quasiconvex hierarchy for $\pi_1 X^*$.

It is hard to say what the main result is since the definitions are more important than the theorems. The following sample result gives some idea of the scope here. In ordinary small-cancellation theory, when $W_1, \ldots, W_r$ represent distinct conjugacy classes, the presentation $\langle a, b, \ldots \mid W_1^{n_1}, \ldots, W_r^{n_r} \rangle$ is "small-cancellation" for sufficiently large n_i. In analogy with this we have the following:

Theorem 5.1. *Let X be a nonpositively curved cube complex. Let $Y_i \to X$ be a compact local isometry for $1 \leq i \leq r$ such that each $\pi_1 Y_i$ is malnormal, and $\pi_1 Y_i, \pi_1 Y_j$ do not share any nontrivial conjugacy classes. Then $\langle X \mid \widehat{Y}_1, \ldots, \widehat{Y}_r \rangle$ is a "small-cancellation" cubical relative presentation for sufficiently large "girth" finite covers $\widehat{Y}_i \to Y_i$.*

Moreover, if X is compact and each of its immersed hyperplanes has separable fundamental group, then we can choose the $\widehat{Y}_i$ such that $\langle X \mid \widehat{Y}_1, \ldots, \widehat{Y}_r \rangle$ has a hierarchy (and so $\pi_1 X^*$ has a quasiconvex hierarchy).

The following theorem is already nontrivial in the case where G is free and each H_i is cyclic (although there is a simplified proof in that case). This plays a fundamental role in proving Theorem 1.2.

Theorem 5.2 (Special Quotient Theorem)**.** *Let G be a word-hyperbolic group that is virtually $\pi_1 X$ where X is compact and special. Let $H_1, \ldots, H_k$ be quasiconvex subgroups. There exist finite index subgroups $H_1', \ldots, H_k'$ such that $G/\langle\langle H_1', \ldots, H_k' \rangle\rangle$ is virtually special.*

6. Conclusion: A scheme for understanding groups

A "grand plan" for understanding many groups can be outlined by:

(1) "Find" codimension-1 subgroups in a group G.
(2) Use Sageev's construction to produce a CAT(0) cube complex C upon which G acts.
(3) Verify that G acts properly and relatively cocompactly on C by examining the extrinsic nature of the codimension-1 subgroups.
(4) Consequently G is the fundamental group of a nonpositively curved cube complex. $D = G \backslash C$ (note that D is an orbihedron if G has torsion).
(5) Find a finite covering space E of D, such that E is special.
(6) Conclude that G is linear - indeed it is contained in $SL_n(\mathbb{Z})$, and that the geometrically best behaved subgroups of G are separable

A general principle that one hopes to demonstrate by way of many classes of examples, and perhaps in the context of Gromov's theory of random groups is the following:

Contention: Most groups presented with relatively few relations compared to the number of generators, have a finite index subgroup that is the fundamental group of a special cube complex.

6.1. **The virtual Haken problem for cube complexes.** Resolution of the following problem would shed light on all aspects of this work:

Problem 6.1. Let X be a compact nonpositively curved cube complex such that $\pi_1 X$ is word-hyperbolic. Does X have a finite cover in which some immersed hyperplane embeds?

There are counterexamples when $\pi_1 X$ is not word-hyperbolic [47].

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