On plane curves with one place at infinity

by

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ABSTRACT

The main theorem we prove is the following:

Suppose $f = f(x,y)$ is a polynomial in two variables over a field k, and f "has one rational place at infinity". Then the polynomial $f - t$ over $k(t)$, t transcendental over k, has one purely inseparable place at infinity, i. e., has geometrically one place at infinity.

This result and its proof help to establish the following:

i) If f has one rational place at infinity, then so does $f - \lambda$, for all but finitely many λ in $\bar{k}$.

ii) Of the polynomials $f - \lambda$ satisfying i), all but finitely many have multiplicity sequences at infinity identical to that of $f - t$ over $\overline{k(t)}$.

These results generalize certain well-known theorems (among them the Epimorphism Theorem) of Abhyankar and Moh. Examples are given to show that i) and ii) cannot be strengthened. In the last section, partial results are given towards the problem of classifying all lines in the plane in positive characteristic.

RESUME

Le théorème principal qu'on prouve est le suivant:

Soit f = f(x,y) un polynôme en deux variables sur un corps k, f ayant une seule place rationelle à l'infini. Alors le polynôme f - t sur k(t), t transcendante par rapport à k, possède une seule place purement inséparable à l'infini, c'est-à-dire, f - t a <u>géométriquement</u> une seule place à l'infini.

Ce théorème, et sa démonstration, nous permettent de démontrer les résultats suivants:

i) Si f possède une seule place rationelle à l'infini, alors la même chose est vrai pour f - λ, pour tout λ de $\bar{k}$, à un nombre fini d'exceptions près.

ii) Parmis les polynômes f - λ satisfaisants i), tous, à un nombre fini près, ont des suites de multiplicités à l'infini identiques à celle de f - t sur $\overline{k(t)}$.

Ces résultats généralisent certains théorèmes bien connus d'Abhyankar et Moh (parmis lesquels le "Théorème d'Epimorphisme"). On prouve par des examples que i) et ii) ne peuvent pas être améliorés. Dans la section finale on donne certains résultats partiels sur la classification des droites dans le plan en caractéristique positive.

(On plane curves with one place at infinity.

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CONTENTS

INTRODUCTION

This thesis began in an attempt to answer the following question:

1) Suppose k is an algebraically closed field, and

$$\begin{array}{ccc} \mathbb{A}_k^2 & & (x,y) \\ \phi \downarrow & & \downarrow \\ \mathbb{A}_k^1 & & f(x,y) \end{array}$$

is an algebraic map of the affine plane onto the affine line. Suppose the fibre over some point is biregular to a line. Are then all fibres over closed points lines? (The question seems first to have been asked in print in [28], Q(1,2).)

The work of Abhyankar and Moh ([1], [2], [3]) within the last five years has provided, as a corollary of both stronger and more general results, an affirmative answer to 1) in the characteristic zero case, and in positive characteristic, in the presence of certain hypotheses of "tameness" on the polynomial f. (Even the generic fibre is a line, in these cases.) A question related to 1) is this:

2) Suppose the fibre of ϕ over some closed point has one place at infinity. Do then all fibres (over closed points) have one place at infinity? ([1], (11.18))

Abhyankar and Moh were able to answer 2) affirmatively in the above-mentioned cases. (Even the generic fibre has one rational place at infinity, in these cases.)

In considering these questions, we have been able to recapture many of the results of Abhyankar and Moh, by methods quite different from theirs. In particular, we have obtained affirmative answers to 1) and 2) (generic fibre included) in the cases cited, and have reproved the celebrated Epimorphism Theorem ([3], Theorem (1.2)). Moreover, in § 1 below we answer question 2) quite generally:

i) The generic fibre of ϕ has one purely inseparable place at infinity.

ii) The general fibre has one place at infinity.

iii) The multiplicity sequences at infinity (relative to any choice of variables x,y) of almost all fibres satisfying ii) are identical.

These results are all new. Several examples in § 2 below show various ways in which ii) and iii) are "best possible". In the last section, some progress is made towards answering 1).

§ 0 consists almost entirely of well-known results, which the writer first learned from Peter Russell. ([24]) Many of them can be found in [1], with somewhat different proofs. Where credits are not explicitly given in § 0, the proofs are basically those of [24], with some minor tampering on the writer's part.

Lemma 1.8 is due to Peter Russell; it has undergone some recasting and embellishment for its rôle in the proofs of 1.1 and 1.2.

Undoubtedly the "deepest" single result in the thesis is 1.15, owing to its reliance on the Connectedness Theorem of Zariski. The version of this theorem which we have used is Grothendieck's. It is possible that a proof of a special case of the theorem, shorter than the very long proofs which appear in [32] and [8], and tailored to the needs of our 1.15 or 1.16, can be adduced.

We should also mention the genesis of the "algebraic Kodaira lemma" (1.15) itself. Its ancestor is Lemma 6.1 of [10], in which essentially the following is proven:

> Suppose V is a complex analytic surface and $\Phi : V \to \Delta$ is a proper holomorphic map of V into a disc in the complex plane. Suppose the general fibre of Φ is connected. If $\mu\Gamma$ is a "singular fibre" of Φ with simply connected support, then $\mu = 1$.

As is so often the case in algebraic geometry, the modular analogue of the classical result - in this case 1.15 - requires a considerably more involved proof.

2.9 is a variation on a lemma in [24]. The irreducibility criteria for power series in § 2 were developed by the writer; we do not know to what extent they were already known. Of the examples of § 2 whose source is not indicated, all are new, and due to the writer.

3.11 and 3.14 arose in discussions with Peter Russell. He also pointed out to us the result of Samuel which appears in, and is vital to, the proof of 3.9. 3.4 and the idea of studying the derivation D_f and its ring of constants are due to the writer. Looking hard at 3.8 has yielded some results, most of which do not appear here.

Some remarks on notation and terminology are in order.

k always denotes a field.

Following Wright, Russell, Sathaye and other authors, we use the symbol $A^{[n]}$ to denote a polynomial ring in n variables over the ring A. The real advantage of this notation is that it enables one to consider such rings intrinsically, <u>without</u> specifying variables for them. If B is a given A - algebra, the statement $B \cong A^{[n]}$ means that B is, in an obvious way, isomorphic as an A - algebra to a polynomial ring over A in n variables.

The term "place" has basically the same meaning as in [6], i. e., valuation ring.

The "generic" point of an irreducible variety is the unique point whose closure is the whole variety. A property is said to hold for the "general" point of a variety if it holds for all algebraic (i. e., closed) points of a nonempty dense Zariski-open subset of the variety. We refer the reader to the introduction of [32] and to Zariski's address to the International Congress of 1950 for the amusing history of these terms and the confusion between them.

Finally, we note that some of the matters we treat can be couched in the language of <u>forms</u>. (See [21].) For instance, one could replace 3.1 of § 3 with the statement

> If k is any field, and $f \in k^{[2]}$ is a line, then for all $\lambda \in k$, $f - \lambda$ is a form of a line.

We leave to the interested reader all other such translations.

0. Curves with one place at infinity - preparatory material

Definition: Let A be an affine domain over k, with quotient field K.

1) We say A has one place at infinity if there is a unique valuation ring V of K over k such that $V \not\supset A$.

2) We say A has one rational place at infinity if 1) holds and V is rational, i. e., the residue field $\kappa(V)$ of V is k.

If A has one place at infinity, then A is necessarily one-dimensional, so V is discrete rank 1. (In 2) above, we prefer the term "rational" to the often-used "residually rational" (as applied to V), because, after all, V is the local ring of a rational point on the complete regular model of K.)

Warning and Remarks: In the literature, the phrase "one place" has always been used with the same meaning as our "one rational place". We are forced to make the distinction in the definition, since we are ~~led~~ very early to work over ground fields which are neither perfect nor separably closed. We have refused to break with tradition in the title of this work, however, which should, to jive with 2) above, really contain the word "rational", since it is with such curves that we are primarily concerned.

We point out here also a fact which we use frequently in the sequel: Suppose V is a place of a function field K in one variable over k, and let $K_{\bar{k}}$ be the field obtained by extending the scalars from k to its algebraic closure $\bar{k}$. Suppose K is a regular extension of k in the sense of Weil. Then:

> There is a unique place $\bar{V}$ of $K_{\bar{k}}/\bar{k}$ lying above V if and only if $\kappa(V)$ is purely inseparable over k.
> ([6], Cor. 4, p. 95)

We shall have considerable dealings with plane curves with one purely inseparable place at infinity, i. e., with plane curves which, <u>generically</u>, have one place at infinity.

<u>Definition</u>: Let A be a domain, and let V be a rank 1 DVR of qt A. Then $\Gamma = \Gamma(A,V) = \{ V(a) \mid a \in A, a \neq 0 \}$ is a subsemigroup of $\mathbb{Z}$, called <u>the value semigroup of</u> V <u>with respect to</u> A. (V(a) denotes the value of a under the valuation induced by V.)

0.1 <u>Lemma</u>: $\Gamma \neq \{0\}$. If Γ contains a positive [negative] integer, then it contains all but finitely many positive [negative] integers.

<u>Proof</u>: Let a, a' $\in$ A be such that t = a/a' generates the maximal ideal of V. Let m = V(a), m' = V(a'). Then m - m' = 1. Suppose $n \in \Gamma$ and n > 0. Suppose $n' \in \mathbb{Z}$

and $n' \geq n(|m| + |m'|)$. Write $n' = qn + r$, $-n' = q'n + r'$, $q, q' \in \mathbb{Z}$, $0 \leq r, r' < n$. We have $n' = n'(m - m') = n'm + (-n')m' = (qm + q'm')n + (rm + r'm')$. Clearly $qm + q'm' > 0$, so $n' \in \Gamma$. The proof of the remaining claim is similar.

0.2 Lemma: Suppose the k - algebra A has one rational place at infinity. Then:

1) k is algebraically closed in $K = \text{qt}A$.
2) A "has trivial units", i. e., $A^* = k^*$.
3) The value semigroup of V with respect to A contains no positive integers.
4) If $x \in A$, $x \notin k$, then A is integral over $k[x]$.
5) If $x \in A$, $x \notin k$, then x is transcendental over k and $[K:k(x)] = -V(x)$.
6) K is separably generated over k.
7) Spec A is geometrically integral, i. e., $A \otimes_k \bar{k}$ is a domain.

Proof: (For more details, see the basic properties of function fields in one variable, given in [6], Ch. 1.) 1): V is rational. 2): $x \in A^* \Rightarrow x$ or $x^{-1} \in k$, so $x \in k^*$. 3): If $0 \neq x \in A$, x has no poles except possibly at V. 4): If V' is a valuation ring of K over k and $V'(x) \geq 0$, then $V' \neq V$, so $V' \supset A$. 5): $[K:k(x)] = \deg\ (x)_\infty = -V(x)$ since V is rational. 6): Let π be the characteristic exponent of k. By 0.1, there exists $x \in A$, $x \notin k$, such that $V(x)$ is prime to π; by 5), x is a separating transcendental for K over k. 7) follows from 1) and 6).

0.3 Lemma: Suppose A has one rational place at infinity, and A_0 is a k - subalgebra of A. Denote by Γ_0 the set of values $\{ V(a) \mid a \neq 0, a \in A_0 \}$.

If $\Gamma(A;V) = \Gamma_0$ then $A = A_0$.

Proof: Let $a \in A$, $a \neq 0$. We do induction on $V(a)$. If $V(a) = 0$, then $a \in k \subset A_0$. Suppose $V(a) < 0$, and every element of A with value greater than $V(a)$ is in A_0. Pick $a_0 \in A_0$ with $V(a_0) = V(a)$. Since V is rational, there is a unique $c \in k^*$ such that $V(a - ca_0) > V(a)$. $a - ca_0 \in A_0 \Rightarrow a \in A_0$.

Convention: If $f \in k^{[2]}$ and $k^{[2]}/fk^{[2]}$ has one [rational] place at infinity, we say that f has one [rational] place at infinity.

0.4 Lemma: Suppose $f \in k^{[2]}$ has one rational place at infinity. Let $x, y \in k^{[2]}$ with $k[x,y] = k^{[2]}$. Suppose f has positive x- and y-degree. (By 0.2 7), this amounts to ruling out the trivial cases: $f \in k[x]$ or $f \in k[x]$ is linear.) Then:

1) f is "monic" in x and y. I. e., writing

$$f = \sum_{i=0}^{m} f_i(y)x^i \in k[y][x], \; f_m(y) \neq 0, \text{ we have}$$

$f_m(y) \in k$ - "the term in f of highest degree in x does not involve y", etc.

2) $f - \lambda \in k^{[2]}$ is irreducible for all $\lambda \in k$.

Proof: 1): Let $\bar{\ }$ denote image modulo f. If $\bar{x}$ or $\bar{y} \in k$ we have a trivial case. So $\bar{x}, \bar{y} \notin k$. $\bar{x}$ is integral over $k[\bar{y}]$ by 0.2 4). Let $\bar{x}^n + \sum_{i=1}^{n} a_i(\bar{y})\bar{x}^{n-i} = 0$. Then f divides $x^n + \sum_{i=1}^{n} a_i(y)x^{n-i}$, so $f_m(y)$ divides 1, etc. 2): Suppose $\lambda \in k^*$ and g divides $f - \lambda$, g nonconstant. Then $\bar{g} \in A^*$, where $A = k^{[2]}/fk^{[2]}$. Since A has trivial units, g is a constant modulo f. So deg g = deg f and $(f-\lambda)/g \in k$.

Remark: Suppose $f \in k[x,y]$ has one place at infinity, and let $A = k[x,y]/(f)$. Then in general 2) fails. But one sees easily that either

i) $A \simeq k'^{[1]}$, where $k' \supset k$ is finite algebraic (and, in general, the k-curve A is geometrically a family of (multiple) parallel lines), or

ii) f is "monic" in x and y.

The proof is straightforward, and is omitted.

Definitions: Let $f \in R \simeq k^{[2]}$. f is a line (in R) if $R/fR \simeq k^{[1]}$. f is a coordinate line, or a variable, if $R = k[f]^{[1]}$, i. e., if there exists $g \in R$ such that $k[f,g] = R$. f is a field generator if there exists $g \in qtR$ such that $k(f,g) = qtR$.

Remarks: Clearly a coordinate line is both a line and a field generator. Also, a line has one rational place at infinity. We have eschewed the frequently used term "embedded line" for a coordinate line; since if $f \in R$ is <u>any</u> line, $R \to R/fR$ always defines an embedding of the line in the plane.

0.5 <u>Lemma</u>: 1) Suppose $f \in k[x,y] = R$ has one rational place V at infinity. If $g \in R$, $0 \neq \bar{g} \in R/fR$, then

$$V(\bar{g}) = -i(f,g)_{\text{fin.}},$$

where $i(f,g)_{\text{fin.}} = \dim_k R/(fR + gR)$ is the intersection index of f and g at finite distance.

2) Suppose $f \in R$ is a line. Let $R \to k[t] \simeq k^{[1]}$ be a surjection, with kernel fR, which carries x to $u(t)$ and y to $v(t)$. Then:

$$\deg_x f = \deg_t v(t) \text{ and } \deg_y f = \deg_t u(t).$$

<u>Proof</u>: 1): Since f has one point at infinity, rational over k, the degree form of f, up to a unit in k, is a power of a linear form in x and y. So, making a linear change of variables, we may assume $f = cy^n +$ (terms of total degree $< n$ in x and y), $c \in k^*$. Then $1/x$, y/x are local coordinates at the point P at infinity of f, and $\tilde{f} = f/x^n$ is a local equation for f at P. $\tilde{f} \in k[[1/x,y/x]]$ is irreducible and, since V is rational, there is a primitive branch representation $(u(t),v(t)) \in (k[[t]])^2$ for $\tilde{f}$. By Bezout's theorem, since f has just one point at infinity, $\text{ord}_t u(t) = i(\tilde{f},1/x)_P = n\cdot 1$. Now let $g \in R$. We have $V(\bar{g}) = \text{ord}_t g(1/u(t), v(t)/u(t))$. If $\deg g = m$, then $\tilde{g} = g/x^m$ is a local equation for g at P. Now suppose $\bar{g} \neq 0$. The intersection multiplicity $i(\tilde{f},\tilde{g})_P$ of f and g at P equals

$\mathrm{ord}_t\tilde{g}(u(t),v(t)) = \mathrm{ord}_t u(t)^m g(1/u(t),v(t)/u(t)) = V(\bar{g}) + mn$.
By Bezout, $i(\tilde{f},\tilde{g})_P + i(f,g)_{\mathrm{fin.}} = mn$, and 1) follows.

2): The case $\deg_x f = 0$ being clear, we assume $\deg_x f > 0$. Let $f_1(Z) = f(Z,v(t))$. $f_1(Z)$ is "monic" in Z (see 0.4.1)) and irreducible in $k[v(t)][Z]$, hence, by the Gauss lemma, irreducible in $k(v(t))[Z]$, and $f_1(u(t)) = 0$. Since $k[t] = k[u(t),v(t)]$, we have $\deg_t v(t) = [k(t):k(v(t))] = \deg_Z f_1(Z) = \deg_x f$, etc.

Suppose $f \in k[x,y]$ is nonconstant. We shall be concerned with properties of the linear pencil of curves

$$\Lambda(f) = \{ V(f - \lambda) \mid \lambda \in k \}$$

on $\mathbb{A}_k^2$. Here V(f) denotes the curve, or effective divisor, defined by f. We shall use the notation V(f) only a few times in the entire sequel; there will be no real danger of confusing the "V" for a valuation on any of these occasions. Suppose t is transcendental over k, and let X,Y be indeterminates over k(t). For $f(x,y) \in k[x,y] - k$, the map

$$(*) \qquad A = \frac{k(t)[X,Y]}{(f(X,Y) - t)} \to k(f(x,y))[x,y] = B,$$

which carries t, X and Y to $f(x,y)$, x and y respectively, is an isomorphism. $f(X,Y) - t \in k(t)[X,Y]$ can be regarded as the generic curve of the pencil $\Lambda(f)$; since B is a regular domain, Spec A $\subset \mathbb{A}^2_{k(t)}$ is a regular integral curve. (For all this, see [22], § 1.)

We collect here some facts about lines:

0.6 Proposition:

1) A is a regular, rational k-algebra with one rational place at infinity if and only if $A \simeq k^{[1]}$.

2) Suppose A has one rational place at infinity. Then $A \simeq k^{[1]}$ if and only if $-1 \in \Gamma(A,V)$.

Let $f \in k^{[2]}$.

3) Suppose f has one place at infinity (not necessarily rational over k), and suppose there is a $g \in k^{[2]}$ whose intersection with f at finite distance is 1. Then f is a line, and conversely.

4) If f is a line and a field generator, then f is a coordinate line. More generally, if f has one rational place at infinity and if f divides a field generator, then f is a coordinate line.

5) f - t is a line over k(t) ⇒ f is a coordinate line over k ⇒ f - t is a coordinate line over k(t).

6) (Absence of nontrivial separable forms of lines and coordinate lines) If f is a [coordinate] line over the separable closure k_s of k, then f is a [coordinate] line over k.

Proof: 1): The "if" part is clear. Let $qtA = k(t)$, t transcendental over k. Suppose V is the rational place at infinity of A. Then $V(t - a) = 1$ for a unique $a \in k$, or $V(t) = -1$. Let $T = \frac{1}{t-a}$ in the first case, $T = t$ in the second. Then $k(t) = k(T)$ and the valuation rings of $k(T)$ over k which contain A are the rings $k[T]_P$, $P = p(T)k[T]$, $p(T)$ irreducible. A is normal, so $A = \bigcap_P k[T]_P = k[T] \simeq k^{[1]}$.

2): The "only if" is clear. Suppose $-1 \in \Gamma$. Pick $t \in A$ with $V(t) = -1$, put $A_0 = k[t]$, and invoke 0.3.

3): Again, only in one direction is an argument called for. (See the "only if" of 2), and 0.5 1).) Let $\dim_k k[x,y]/(f,g) = 1$. f and g meet at just one point of $\mathbb{A}_k^2$, which is rational over k, and simple on f. Let $A = k[x,y]/(f)$, $K = qtA$. Then there is a unique valuation ring V' of K over k such that $V' \supset A$ and $V'(\bar{g}) > 0$, where $\bar{g} = g \bmod f$. V' has residue field k (so k is the field of constants of K/k) and $V'(\bar{g}) = 1$. The divisor of $\bar{g} \in K$ is $(\bar{g}) = (\bar{g})_0 - (\bar{g})_\infty = V' + V(\bar{g})V$, where V is the place at infinity of f. $0 = \deg(\bar{g}) = 1 + V(\bar{g})[\kappa(V):k]$, so V is rational and $V(\bar{g}) = -1$. 3) now follows from 2).

4): Suppose f divides a field generator g. By [22], Theorem 4.5, there exist variables x,y such that either g is linear in x,y - in which case we are done - or such that g has two ordinary points at infinity. If f is linear in x,y we are done, so suppose not. Then by Bezout's theorem f is tangent to the line L at infinity of $k[x,y]$, hence has at least two points, one infinitely near to the other, on L. So g has at least three infinitely near points on L, which contradicts [22], Cor. 3.7.

5): Since f − t is a rational k(t)-curve, f is a field generator by (*) above. As in 4), there exist variables x,y such that either f is linear in x,y or f has two ordinary points at infinity. In the latter case f − t has two ordinary points at infinity (f and f − t have the same degree form), hence at least two places at infinity, which cannot be. This proves the first part of 5), and the second is clear.

6): The absence of nontrivial separable forms of lines has been known for a long time. If f is a coordinate line over k_s then f − t is a line over $k_s(t)$, hence over $k(t)_s$, hence over k(t), and the second part of 6) follows from 5). (See [23], Lemma 1.5, for another proof.)

Undoubtedly the most interesting negative result concerning lines in the plane is that not all such are coordinate lines. We do not know who first found examples of such lines. Certainly Nagata and Abhyankar knew of some by 1970. (See, e. g., [17], p. 154.) In the paper [3] (the sequel to [2], a watershed in the theory of plane curves), the authors give the following family of examples: (Actually, they give parametrizations for these f.)

Let k be of characteristic $p > 0$, $d,n \geq 0$, $a_0, \cdots, a_d \in k$, $a_d \in k^*$ if $d > 0$, and

$$f = y^{p^n} - x - \sum_{i=0}^{d} a_i^{p^n} x^{ip} \in k[x,y].$$

Then f is a line, and, if $p^n \nmid dp$ and $dp \nmid p^n$, f is not a coordinate line.

Leaving aside the question of historical priority, we will refer to such lines as <u>Abhyankar - Moh lines</u>. That such an f (with $p^n \nmid dp$, etc.) is not a coordinate line follows from

0.7 <u>Theorem</u>: Suppose $f \in R \simeq k^{[2]}$ is a coordinate line, and let $x,y \in R$ be such that $R = k[x,y]$. Then $\deg_x f$ divides $\deg_y f$ or $\deg_y f$ divides $\deg_x f$.

This theorem is the main ingredient in Van der Kulk's and Nagata's proofs of the structure theorem for the automorphism group of $k[x,y]$. (See [30], [18]; in the latter work two proofs are provided for 0.7.) We also express the conclusion of 0.7 this way: f has "principal bidegree" (with respect to x and y).

The next theorem is the main positive result concerning lines in the plane.

0.8 <u>Epimorphism Theorem</u>: (Abhyankar - Moh, [3])

Let k be a field of characteristic $p \geq 0$, $f \in k[x,y]$ a line. If $p \nmid$ g.c.d.$(\deg_x f, \deg_y f)$, then f is a coordinate line.

In particular, if char $k = 0$, all lines are coordinate lines. The authors of 0.8 did not state it quite this way, but

0.8 is equivalent to their formulation. (See 0.5 2).) Apparently at least two false proofs of 0.8 were published before Abhyankar and Moh proved it. Since then, Miyanishi has given a proof for characteristic zero. ([14]) The authors of 0.8 had earlier ([2]) published a proof of this result:

0.9 <u>Theorem:</u> Let $f \in k[x,y]$ have one rational place at infinity, and suppose $p \nmid \text{g.c.d.}(\deg_x f, \deg_y f)$, where $p = \text{char } k$. Then

all $f - \lambda$, $\lambda \in \bar{k}$, have one place at infinity over $\bar{k}$, and

the characteristic pairs (equivalently, the multiplicity sequences) at infinity of $f - \lambda$ are independent of λ. (See also [1], § 11.)

We will not attempt to summarize the paper [2], but will just remark that its authors show also that, under the hypotheses of 0.9, all $f - \lambda$ have the same <u>value semigroup</u> at infinity. They are able to do this since they work with <u>meromorphic characteristic pairs</u> (as we do not), and because they prove that the value semigroup is determined by the meromorphic characteristic pairs.

0.8 and 0.9 will be proven anew in § I below, as corollaries of "more general" results to be found there. Our approach does <u>not</u> yield the above result on value semigroups, however.

1. The main theorems. Corollaries

The principal goal of this section is to prove the following:

1.1 MAIN THEOREM Let k be a field, and let $f \in R \approx k^{[2]}$ have one rational place at infinity. Let t be transcendental over R and consider

$$f - t \in k(t) \otimes_k R \approx k(t)^{[2]}.$$

1) f − t has one place $\mathfrak{P}$ at infinity, which is purely inseparable over k(t).

2) If $x, y \in R$ and $R = k[x,y]$, then the residue field degree $[\kappa(\mathfrak{P}):k(t)]$ of $\mathfrak{P}$ divides $g.c.d.(\deg_x f, \deg_y f)$.

The proof of 1.1 is fairly lengthy, involving several steps. While the details are not themselves devoid of interest - we call attention especially to 1.8 (the "Local Lemma") and 1.15 (the "algebraic Kodaira lemma") below - we have thought it best to state at once the central result. (See also 1.16, where the value of the invariant $[\kappa(\mathfrak{P}):k(t)]$ of the curve $f \in R$ is established.)

In the same spirit we mention here another theorem which is a corollary of the proof of 1.1, rather than of the main theorem itself, and whose proof is therefore deferred:

1.2 Theorem: Let k, R, f, t be as in 1.1, and let π be the characteristic exponent of k.

1) For some integer $N \geq 0$, and some finite purely inseparable extension k' of k, $f - t \in k'(s) \otimes_k R \simeq k'(s)^{[2]}$ has one rational place at infinity, where $s = t^{\pi^{-N}}$.

2) For general $\lambda \in \bar{k}$, $f - \lambda \in \bar{k}^{[2]}$ has one place at infinity.

3) Let $x, y \in R$ generate R over k, and let $\nu = \langle \nu_0, \nu_1, \cdots \rangle$ be the resulting multiplicity sequence of the successive infinitely near points at infinity of $f - t$ over k'(s). Then for general $\lambda \in \bar{k}$, 2) holds and, if $\nu(\lambda)$ is the corresponding multiplicity sequence, then $\nu(\lambda) = \nu$.

Following the proofs of 1.1 and 1.2, we give some corollaries, as well as new proofs of the epimorphism theorem, and other results, of Abhyankar and Moh.

If k is a field, by a <u>variety</u> over k we mean a reduced (separated) scheme of finite type over k. If V is an irreducible variety over k, we may tacitly identify V with the collection of local rings $\{\mathcal{O}_{V,P} \mid P \in V\}$ of the function field of V. We begin the proof of 1.1 by recalling for the reader some machinery which figures crucially in it.

1.3 Suppose X is a nonsingular irreducible surface over k and Λ is an n-dimensional linear system of curves (i. e., effective divisors) on X. Suppose Λ has no fixed component. Then we have a corresponding rational map $X \overset{\phi}{\rightarrow} \mathbb{P}^n_k$, regular on the complement of a finite set of closed points of X (the ordinary base points of Λ).

1.4 Now suppose k is algebraically closed, and $P \in X$ is a closed point. Then one can consider the <u>blowing-down</u> (σ-process) centred at P (see [26], p. 208): One has a non-singular irreducible surface X' over k, and a birational morphism $X' \overset{\sigma}{\rightarrow} X$, whose fibre E over P is a projective line over k, called the <u>exceptional fibre</u>. (The rational map $X \overset{\sigma^{-1}}{\rightarrow} X'$ is the <u>blowing-up</u>, or dilatation, of P.)

1.4.1 If X is projective, so is X'.

σ induces a map σ^*: Div $X \rightarrow$ Div X' of the groups of divisors on X and X'. If D is a curve on X, $D^* = \sigma^*(D)$ is a curve on X', called the <u>total transform</u> of D. Let $\mu(D,P)$ be the multiplicity of D at P. Then one has a curve $D' = D^* - \mu(D,P)E$ on X', called the <u>proper transform</u> of D.

1.4.2 If D is a prime divisor, so is D'.

If Λ is a linear system of curves on X, we define $\mu(\Lambda,P)$ to be $\min \{ \mu(D,P) \mid D \in \Lambda \}$. P is called a <u>base point</u> of Λ if $\mu(\Lambda,P) > 0$. The set $\Lambda^* = \{ D^* \mid D \in \Lambda \}$ is a linear system on X', called the total transform of Λ.
The set $\Lambda' = \{ D^* - \mu(\Lambda,P)E \mid D \in \Lambda \}$ is also a linear system

on X', called the proper transform of Λ. If Λ lacks fixed components, so does Λ'. (<u>Note</u>: Λ' is <u>not</u> the set of proper transforms of members of Λ. In general, the member $D^* - \mu(\Lambda,P)E$ of Λ' corresponding to D equals D' + a nonnegative multiple of E.)

We remark that all of the statements in 1.4, as well as the usual formulas relating the intersection theories on X and X', continue to hold when k is an arbitrary field, provided $P \in X$ is a <u>rational</u> point. (See the last paragraph before 1.8 below.)

1.5 Now in the situation described in 1.3, it is well-known that there is a diagram of varieties and rational maps

$$\begin{array}{ccc} X_{s+1} & & \\ \sigma_s \downarrow & \searrow^{\Phi} & \\ \vdots & & \\ \sigma_1 \downarrow & & \\ X_1 & & \\ \sigma_0 \downarrow & & \\ X_0 = X & \xrightarrow{\phi} & \mathbb{P}^n_k \end{array},$$

where each σ_i is a blowing-down (with centre Q_i, say), such that $\Phi = \phi \circ \sigma_0 \circ \cdots \circ \sigma_s$ is <u>regular</u> on X_{s+1}.

1.6 Next suppose X is as in 1.3 (k is any field), and C is an irreducible k-curve lying on X (i. e., a 1-dimensional irreducible closed k-subvariety of X). Suppose $P = P_0$ is a (closed) point of $C = C^{(0)}$, and that there is just one valuation ring of the function field of C which dominates $\mathcal{O}_{C,P}$. Let $Y_1 \xrightarrow{\tau_0} X$ be the blowing-down centred at P, and let $C^{(1)}$ be the proper transform of C. Then there is a unique point $P_1 \in C^{(1)}$ such that $\tau_0(P_1) = P$. Moreover, $\mu(C^{(1)},P_1) \leq \mu(C,P)$. Continuing in this fashion, one has a uniquely determined diagram

1.6.1 $$\cdots \to Y_{i+1} \xrightarrow{\tau_i} Y_i \to \cdots \xrightarrow{\tau_1} Y_1 \xrightarrow{\tau_0} X$$

of surfaces and blowings-down, together with curves $C^{(i)}$ on Y_i and points $P_i \in C^{(i)}$, P_i = centre of τ_i, P_{i+1} = unique point on $C^{(i+1)}$ such that $\tau_i(P_{i+1}) = P_i$, and $\mu(C^{(i+1)},P_{i+1}) \leq \mu(C^{(i)},P_i)$. It is well-known that $\mu(C^{(i)},P_i) = 1$ for $i>>0$. $P_0,P_1,\cdots$ are called the successive infinitely near points on C above P.

1.7 Now let $f \in R \approx k^{[2]}$ have one rational place at infinity. Let $x,y \in R$ and $R = k[x,y]$, $A = R/fR$. The choice of generators x,y for R determines an embedding of Spec A into $\mathbb{A}^2_k$. For $\lambda \in k$, $f - \lambda \in k[x,y]$ determines a prime divisor $V(f - \lambda)$ on $\mathbb{A}^2_k$ (by 0.4 2)), and a prime divisor Λ_λ on $\mathbb{P}^2_k$, whose support is the closure in $\mathbb{P}^2_k$ of the support $|V(f - \lambda)|$ of $V(f - \lambda)$. We set $\Lambda_\infty = dL_\infty$, where d = deg f and $L_\infty = \mathbb{P}^2_k - \mathbb{A}^2_k$ is the prime divisor at infinity of $\mathbb{A}^2_k$. (Note that this notion depends on the choice of x,y.) Then $\Lambda = \Lambda(f) = \{ \Lambda_\lambda \mid \lambda \in k \cup \{\infty\} \}$ is a linear pencil on $\mathbb{P}^2_k$, without fixed component. By the remarks in 1.6, we have diagram 1.6.1 (with $X = \mathbb{P}^2_k$). Here $P_0 \in \mathbb{P}^2_k$ is k-rational, and $L_\infty \cap |\Lambda_0| = \{P_0\}$.

Since, for $\lambda \in k$, $L_\infty \cap |\Lambda_\lambda|$ is in natural bijection with the irreducible factors of the degree form of $f - \lambda$, and since the degree form is independent of λ, P_0 is the only base point of Λ on $\mathbb{P}^2_k$.

We eliminate here, once and for all, a special case: Suppose deg $f = 1$. Then 1.1 and 1.2 become trivial; $f - t$ and all $f - \lambda$, $\lambda \in \bar{k}$, are coordinate lines, and ν and $\nu(\lambda)$ are just sequences of 1's. We assume henceforth that $d = \deg f > 1$. Then

1.7.1 $\quad \mu(\Lambda_\lambda, P_0) < d = \mu(\Lambda_\infty, P_0)$ for all $\lambda \in k$,

so by Bezout's theorem every Λ_λ is tangent to L_∞ at P_0.

1.7.2 The degree form of f is a power of a linear form ξ in x and y. Either x or $y \not| \xi$, say $x \not| \xi$. Then $x' = 1/x$ and $y' = \xi/x$ are local coordinates at P_0, x'^d is a local equation for Λ_∞, and

$$(f - \lambda)/x^d = f/x^d - \lambda x'^d$$

is a local equation for Λ_λ, $\lambda \in k$.

<u>Remark</u>: Even if the place at infinity of f were not rational, we would have 1.7.2, except that ξ is then only an <u>irreducible</u> form, and $y' = \xi/x^{\deg \xi}$.

We have $\mu(\Lambda_\lambda, P_0) = \mu(\Lambda, P_0) =: \mu_0 < \mu(\Lambda_\infty, P_0)$ for all $\lambda \in k$. Let $X_1 \xrightarrow{\sigma_0} X_0 = \mathbb{P}^2_k$ be the blowing-down centred at P_0. Put $E_0 = L_\infty$, $E_1 = \sigma_0^{-1}(P_0)$, $\Lambda^{(0)} = \Lambda$, $\Lambda^{(1)}$ = proper transform of Λ,

$(\Lambda^{(1)})_\lambda$ = member of $\Lambda^{(1)}$ corresponding to Λ_λ, $\Lambda_\lambda^{(1)}$ = proper transform of Λ_λ. Then $\Lambda_\lambda^{(1)} \doteq (\Lambda^{(1)})_\lambda$ for all $\lambda \in k$, and

1.7.3 $(\Lambda^{(1)})_\infty = \Lambda_\infty^{(1)} + (d-\mu_0)E_1 = dE_0^{(1)} + (d-\mu_0)E_1$.

1.7.4 If $\deg_x f < d$ [resp. $\deg_y f < d$] then clearly $\mu_0 = d-\deg_x f$ [resp. $d-\deg_y f$], i. e.,

$$d = \deg_y f, \quad d - \mu_0 = \deg_x f \text{ [resp., switch x and y].}$$

(<u>Proof</u>: Suppose $\deg_x f < d$. Then in the notation of 1.7.2, $\xi = cy$, $c \in k^*$, so $d = \deg_y f$. Also, $x' = 1/x$, $y' = y/x$ are local coordinates at P_0. Let $f = \Sigma f_{ij}x^i y^j$. Then $f' := f/x^d = \Sigma f_{ij}x'^{d-i-j}y'^j \in k[x',y']$. Since $\deg x'^{d-i-j}y'^j = d - i$, if we put $e = \deg_x f$ and recall that f is "monic" in x (0.4 1)), we have $f' = f_{e,0}x'^{d-e}$ + (higher order terms in x',y'). So $\mu_0 = d - \deg_x f$.)

Consider now 1.5, with $n = 1$, $X_0 = \mathbb{P}_k^2$, and $\Lambda = \Lambda(f)$. We shall require several pieces of information concerning the centres Q_i of the σ_i, and the pencils $\Lambda^{(i+1)}$ on X_{i+1}, $0 \le i \le s$. (One fact we shall need is that the column in 1.5 is a subdiagram of 1.6.1.) The information we need can be extracted from a recent paper ([14]) of Miyanishi (with allowances for the fact that his base field is algebraically closed), in which only global methods are used. (See also 1.10 below.) In that paper, the author develops fairly elaborate machinery which allows him, given the structure of $\Lambda^{(i)}$, to determine that of $\Lambda^{(j)}$, where $j > i$ is computed in terms of (or sometimes only bounded by) certain numerical characters associated to $\Lambda^{(i)}$. This machinery gives considerable insight into what is going on in 1.5 ("blowing up the base points of $\Lambda(f)$"), and was in fact the original inspiration for the examples that appear in § 2 below.

However, in 1.7 we were able to get global data about the pencil $\Lambda^{(1)}$ on X_1 in terms of local equations at $P_0 = Q_0$ for the members of the pencil $\Lambda^{(0)}$ on X_0. This situation persists - it turns out that local data concerning $\Lambda^{(i)}$ at Q_i enable one to describe $\Lambda^{(i+1)}$. The main advantage which the local machinery we are about to introduce enjoys over Miyanishi's is simplicity - our machinery involves only one blowing-up, whereas Miyanishi's deals with sequences of blowings-up. The main lemma (1.8) we prove below also suggests that pencils on local rings merit further investigation in their own right.

Definition: Let $(\mathcal{O}, \mathfrak{m}, k)$ be a local ring of Krull dimension 2, and a unique factorization domain. Let $f, g \in \mathfrak{m}$ and suppose $f \nmid g$ and $g \nmid f$. Then $\Lambda = \Lambda(f,g) =$

$$\left\{ \begin{array}{c} \text{ideals } (uf + vg) \text{ such that} \\ u,v \in (\mathcal{O} - \mathfrak{m}) \cup \{0\} \text{ are not both } 0 \end{array} \right\}$$

is the local pencil on $\mathcal{O}$ spanned by f and g. (Note that Λ has at least card $k + 1$ members - there is a surjection from Λ onto the set of rational points of $\mathbb{P}^1_k$.) Λ is without fixed component if g.c.d.(f,g) = 1. Note: If $\alpha\mathcal{O}$, $\beta\mathcal{O} \in \Lambda$ are distinct, then $\Lambda = \Lambda(\alpha,\beta)$. Also, Λ is without fixed component $\Leftrightarrow$ for all such α,β, g.c.d.$(\alpha,\beta) = 1$. So the notions "pencil" and "pencil without fixed component" are independent of the choice of spanning elements.

Remark: We have not found these notions in the literature. At any rate, the definition seems reasonable, and allows us to state and prove the results we need.

We must next recall several definitions and facts.

A local domain A is called unibranch if the integral closure $\bar{A}$ of A in its quotient field is local. ([8], 4.3.6) If A is noetherian and 1-dimensional, this amounts to saying that there is a unique valuation ring of qtA which dominates A (because $\bar{A}$ is a noetherian ring; see [16], THEOREM (33.2), p. 115). We will call a nonzero element f of a local ring $\mathcal{O}$ unibranch if $\mathcal{O}/f\mathcal{O}$ is a domain and is unibranch.

We also need some facts concerning regular local rings $(\mathcal{O},\mathfrak{m},k)$ of dimension 2. If $\alpha,\beta \in \mathcal{O}$ are relatively prime non-units, they generate an ideal primary for $\mathfrak{m}$, and one defines the intersection multiplicity $\alpha\cdot\beta$ to be the length of the artinian $\mathcal{O}$-module $\mathcal{O}/(\alpha\mathcal{O} + \beta\mathcal{O})$. (See [26 bis], p. 83.) We recall also the blowing up of $\mathfrak{m}$: One has a regular scheme X and a birational morphism $X \to \operatorname{Spec}\mathcal{O}$ which gives an isomorphism of $\operatorname{Spec}\mathcal{O} - \{\mathfrak{m}\}$ with $X - E$, where $E \approx \mathbb{P}^1_k$ is the fibre over $\mathfrak{m}$. (See [26 bis], p. 12.) The immediate quadratic transforms (iqt's) $\tilde{\mathcal{O}}$ of $\mathcal{O}$ are the rings $\mathcal{O}_{X,z}$, $z \in E$. If $x,y \in \mathcal{O}$ generate $\mathfrak{m}$, these are just the rings $\mathcal{O}[x/y]_{\mathfrak{x}}$, $y \in \mathfrak{x}$, $\mathcal{O}[y/x]_{\mathfrak{q}}$, $x \in \mathfrak{q}$. If e is an exceptional parameter in $\tilde{\mathcal{O}}$ (i. e., a local equation for E at $\tilde{\mathcal{O}}$), and $\alpha \in \mathfrak{m}$ is of multiplicity μ, the proper transform of α in $\tilde{\mathcal{O}}$ is $\alpha' = \alpha e^{-\mu}$; it is well-defined up to a unit in $\tilde{\mathcal{O}}$. The leading form $\mathrm{lf}(\alpha)$ of α is the image of α in $\mathfrak{m}^{\mu}/\mathfrak{m}^{\mu+1} \subset \sum_{i=0}^{\infty} \mathfrak{m}^{i}/\mathfrak{m}^{i+1} \simeq k[x,y]$. One easily checks that the iqt's $\tilde{\mathcal{O}}$ of $\mathcal{O}$ such that α' is a nonunit in $\tilde{\mathcal{O}}$ are in natural bijection with the distinct irreducible factors of $\mathrm{lf}(\alpha)$, regarded as a form of degree μ in $k[x,y]$.

Suppose $\Lambda = \Lambda(\alpha,\beta)$ is a local pencil on $\mathcal{O}$, without fixed component. Let $\mu = \mu(\Lambda,\mathcal{O}) = \min\{\mu(\alpha), \mu(\beta)\}$. (This is independent of the choice of α,β.) If $\tilde{\mathcal{O}}$ is an iqt of $\mathcal{O}$ and e is an exceptional parameter in $\tilde{\mathcal{O}}$, we say $\tilde{\mathcal{O}}$ is a base point of Λ if $\alpha e^{-\mu}$, $\beta e^{-\mu}$ are nonunits in $\tilde{\mathcal{O}}$. The proper transform $\Lambda' = \Lambda(\alpha e^{-\mu}, \beta e^{-\mu})$ of Λ on $\tilde{\mathcal{O}}$ is then a pencil on $\tilde{\mathcal{O}}$, without fixed component. By induction one defines the set $B(\Lambda)$ of all ("infinitely near") base points of Λ. (See, e. g., [22], § 2.) $B(\Lambda)$ is finite. (Cf. the proof of 1.9 1) below.)

Suppose $f \in \mathcal{O}$ is irreducible of multiplicity μ, and $f\mathcal{O} \neq x\mathcal{O}$. Then the inclusion $\mathcal{O} \to \mathcal{O}[y/x]$ induces a birational inclusion of domains $\mathcal{O}/f\mathcal{O} \to \mathcal{O}[y/x]/(fx^{-\mu})$.

Suppose f is unibranch. Then using the above facts, and the fact that a unique valuation ring of $\mathrm{qt}(\mathcal{O}/f\mathcal{O})$ dominates $\mathcal{O}/f\mathcal{O}$, one sees that there is a unique iqt $\tilde{\mathcal{O}}$ of $\mathcal{O}$ in which f' is a nonunit, and that $f' \in \tilde{\mathcal{O}}$ is unibranch. So $\mathrm{lf}(f)$ is a power of an irreducible form in $k[x,y]$. (If F is the degree of this form, we have $[\kappa(\tilde{\mathcal{O}}):k] = F$.) One consequence is that for unibranch f, $\mu(f) = \min\{f\cdot x, f\cdot y\}$.

Finally, we recall these formulas involving intersection multiplicities in $\mathcal{O}$ and <u>intersection indices</u> on X: For relatively prime $\alpha,\beta \in \mathfrak{M}$, we have

$$\sum_{\tilde{\mathcal{O}} \text{ iqt of } \mathcal{O}} [\kappa(\tilde{\mathcal{O}}):k] \ (\alpha'\cdot\beta')_{\tilde{\mathcal{O}}} = \alpha\cdot\beta - \mu(\alpha)\mu(\beta).$$

(*)

$$\text{Also,} \quad \sum_{\tilde{\mathcal{O}} \text{ iqt of } \mathcal{O}} [\kappa(\tilde{\mathcal{O}}):k] \ (e_{\tilde{\mathcal{O}}}\cdot\alpha')_{\tilde{\mathcal{O}}} = \mu(\alpha),$$

where $e_{\tilde{\mathcal{O}}}$ is an exceptional parameter in $\tilde{\mathcal{O}}$.

We can now state

1.8 <u>Lemma</u>: (Russell) Let $(\mathcal{O},\mathfrak{M},k)$ be a regular local ring of dimension 2. Let x,y be a regular system of parameters, A, B nonnegative integers not both 0, and $g = x^A y^B$. Let $f \in \mathcal{O}$ be unibranch, suppose g.c.d.$(f,g) = 1$, and let $\Lambda = \Lambda(f,g)$ be the corresponding pencil without fixed component. Let $d = f\cdot x$, $\mu = \mu(f,\mathcal{O})$, and put $\delta = \text{g.c.d.}(d,\mu)$. Suppose $A = M\delta$, $B = N\delta$, where M,N are integers and

1) $M\delta \geq \mu$ and $N \geq 0$, and
2) $(M + N)\delta \geq d$.

<u>Case 1</u>: $M + N = 1$. (So that $M = 1$, $N = 0$, and $d = \mu$.)

Then no iqt of $\mathcal{O}$ is a base point of Λ.

<u>Case 2</u>: $M + N > 1$. Let $(\tilde{\mathcal{O}}, \tilde{\mathfrak{m}}, \tilde{k})$ be the unique iqt of $\mathcal{O}$ such that the proper transform f' of f in $\tilde{\mathcal{O}}$ is not a unit. Then $\tilde{\mathcal{O}}$ is a base point of Λ, and is the only iqt of $\mathcal{O}$ which is. The proper transform Λ' of Λ on $\tilde{\mathcal{O}}$ is without fixed component, and of this form:

$\Lambda' = \Lambda(f', \tilde{g} = \tilde{x}^{\tilde{M}\tilde{\delta}}\tilde{y}^{\tilde{N}\tilde{\delta}})$, where $\tilde{x}, \tilde{y}$ generate $\tilde{\mathfrak{m}}$,

either $\tilde{x}$ or $\tilde{y}$ is an exceptional parameter in $\tilde{\mathcal{O}}$

(and appears with positive exponent in $\tilde{g}$),

$\tilde{\mu} = \mu(f', \tilde{\mathcal{O}})$, $\tilde{d} = f' \cdot \tilde{x}$, and $\tilde{\delta} = \text{g.c.d.}(\tilde{d}, \tilde{\mu})$ <u>divides δ</u>.

We have

$\tilde{1})$ $\tilde{M}\tilde{\delta} \geq \tilde{\mu}$ and $\tilde{N} \geq 0$, and

$\tilde{2})$ $(\tilde{M} + \tilde{N})\tilde{\delta} \geq \tilde{d}$.

Moreover, if V is the valuation ring of $\text{qt}(\mathcal{O}/f\mathcal{O})$ which dominates $\mathcal{O}/f\mathcal{O}$, then $\tilde{k} \subset \kappa(V)$.

<u>Proof</u>: First we dispose of Case 1. We have $x \nmid \text{lf}(f)$, and $g = x^{\mu}$. The only iqt of $\mathcal{O}$ in which g' is not a unit is $\mathcal{O}[x/y]_{(y, x/y)}$, and $f' = fy^{-\mu}$ is a unit in this ring.

Next suppose $d \geq 2\mu$. Since $\mu(g, \mathcal{O}) = (M + N)\delta > \mu$ (by 2)), the unique iqt $\tilde{\mathcal{O}}$ of $\mathcal{O}$ which is a base point of Λ is $\mathcal{O}[x/y]_{(x/y, y)}$, because this is the only iqt of $\mathcal{O}$ in which f' is a nonunit. Put $\tilde{x} = x/y$, $\tilde{y} = y$. By (*) above, and since $\kappa(\tilde{\mathcal{O}}) = k$, we have $\tilde{d} = f' \cdot \tilde{x} = d - \mu$ and $f' \cdot \tilde{y} = \mu$. Since $f' \in \tilde{\mathcal{O}}$ is unibranch,

$\tilde{\mu} = \min\{ d - \mu, \mu \} = \mu$, hence $\tilde{\delta} = \delta$. If we put $\tilde{M} = M$ and $\tilde{N} = M+N - \mu/\delta$, Λ' is as claimed. $\tilde{1}$) holds by 1), and $\tilde{2}$) holds by 1) and 2).

Suppose $\mu < d < 2\mu$. Put $\tilde{x} = y$, $\tilde{y} = x/y$. Then $\tilde{d} = \mu$, $\tilde{\mu} = d - \mu$ and $\tilde{\delta} = \delta$. If we put $\tilde{M} = M + N - \mu/\delta$ and $\tilde{N} = M$, Λ' is as claimed. $\tilde{1}$) and $\tilde{2}$) each hold by 1) and 2).

Suppose $d = \mu$. i) $f \cdot y > \mu$. Put $\tilde{x} = x$, $\tilde{y} = y/x$. $\tilde{d} = \mu = \delta$ (so $\tilde{\delta} \mid \delta$), and $\tilde{\mu} \le \mu$. If we put $\tilde{M} = (M + N - 1)\delta/\tilde{\delta}$ and $\tilde{N} = N\delta/\tilde{\delta}$, Λ' is as claimed. $\tilde{1}$) and $\tilde{2}$) each hold by 1) and the fact that $M + N > 1$. ii) $f \cdot y = \mu$.* Let $[\tilde{k}:k] = F$. Put $\tilde{x} = x$, and let $\tilde{y}$ be any complementary parameter for $\tilde{\mathcal{O}}$. $\tilde{d} = \mu/F = \delta/F$, so $\tilde{\delta} \mid \delta$: y/x is a unit in $\tilde{\mathcal{O}}$. If we put $\tilde{M} = (M + N - 1)\delta/\tilde{\delta}$ and $\tilde{N} = 0$, Λ' is as claimed. $\tilde{2}$) holds since $M + N > 1$ and $F \ge 1$; a fortiori, $\tilde{1}$) holds. (* $\tilde{\mathcal{O}}$ is of the form $\mathcal{O}[y/x]_{(x,p)}$, where $p \in k[y/x]$ is irreducible and $p(0) \neq 0$.)

The remaining statements in 1.8 are clear, by the above proof and the discussion preceding 1.8.

1.9 <u>Corollaries</u>: Let $\mathcal{O} = \mathcal{O}_0$, $g = g_0$, $f = f^{(0)}$, $\Lambda = \Lambda^{(0)}$, d be as in 1.8. Suppose $\mu(g) > \mu(f)$. Then

1) The base points of Λ form a sequence $\mathcal{O}_0 < \mathcal{O}_1 < \cdots < \mathcal{O}_s$ of $s + 1$ local rings. $(s \ge 1)$

2) Let $\mu_i = \mu(f^{(i)}, \mathcal{O}_i)$ and e_{i+1} be an exceptional parameter in $\mathcal{O}_{i+1}$. If $\Lambda^{(i+1)}$ is the proper transform of $\Lambda^{(i)}$ on $\mathcal{O}_{i+1}$, then $\Lambda^{(i+1)}$ is spanned by $f^{(i+1)}$ and g_{i+1},

where $f^{(i+1)}$ $(= f^{(i)} e_{i+1}^{-\mu_i})$ is the proper transform of $f^{(i)}$ on $\mathcal{O}_{i+1}$, and $g_{i+1} = g_i e_{i+1}^{-\mu_i}$.

3) Consider the nonnegative integers n_i defined by

$$e_i^{n_i} \mid g_i, \quad e_i^{n_i+1} \nmid g_i.$$

Then $n_i > 0$ and μ_s divides d and n_i, $1 \le i \le s$.

4) $\Lambda^{(s)}$ is spanned by $f^{(s)}$ and $e_s^{\mu_s}$.

Proof: 1): $\mu(g) > \mu(f) \Rightarrow M + N > 1$, in the notation of the Lemma. If $\mathcal{O}_1$ is the last base point of Λ, stop; otherwise, continue. Note that if $e_1 \in \mathcal{O}_1$ is exceptional, $g_1 = g e_1^{-\mu(f)}$, $g^{(1)}$ is the proper transform of g in $\mathcal{O}_1$ and $F = [\kappa(\mathcal{O}_1):k]$, then $1 \le (\Lambda^{(1)})^2 := f^{(1)} \cdot g_1 = f^{(1)} \cdot (g^{(1)} e_1^{\mu(g)-\mu(f)}) =$
$f^{(1)} \cdot g^{(1)} + (\mu(g) - \mu(f)) f^{(1)} \cdot e_1 =$
$\frac{1}{F} [f \cdot g - \mu(f)\mu(g) + (\mu(g) - \mu(f))\mu(f)] =$
$\frac{1}{F} [f \cdot g - \mu(f)^2] < f \cdot g$. We see that $f \cdot g =: (\Lambda^{(0)})^2 > (\Lambda^{(1)})^2 > \cdots \ge 1$, and Λ has s + 1 base points, where $s + 1 \le f \cdot g$.
2) was established in the proof of 1.8. 3) follows by induction on the " $\tilde{\delta}$ divides δ" and "appears with positive exponent" of 1.8, and 4) follows from the fact that Case 1 of 1.8 holds for $\Lambda^{(s)}$.

We return now to 1.5 and 1.6.1 and use our results on local pencils to describe the global situations there. We assume that $f \in R = k[x,y]$ has one rational place at infinity, $\Lambda = \Lambda(f)$, $d = \deg f > 1$, etc., as in 1.7. In the notation of 1.5 and 1.6, we have, by 1.9 1) and 2), and remarks in 1.7, that $X_{i+1} = Y_{i+1}$, $P_i = Q_i$, and $\sigma_i = \tau_i$ for $0 \le i \le s$, i. e., the base points of Λ consist of a certain number of the successive infinitely near points on Λ_0 above P_0. The pencil $\Lambda^{(i)}$ on X_i is spanned by $(\Lambda_0)^{(i)}$ and $(\Lambda^{(i)})_\infty$, $0 \le i \le s$. For $1 \le i \le s$, let E_i be the exceptional fibre on X_i and, for a curve D on X_i and $i < j \le s + 1$, denote by $D^{(j)}$ the proper transform of D on X_j. Then

1.10 (See LEMMA 2.8 of the cited paper [14] of Miyanishi)

$$(\Lambda^{(i)})_\infty = \sum_{\ell=0}^{i-1} n_\ell E_\ell^{(i)} + n_i E_i, \; n_0, \cdots, n_i > 0,$$

$n_0 = d$, $n_s = \mu(\Lambda_0^{(s)}, P_s) =: \mu_s$, and μ_s divides n_i, $0 \le i < s$. The divisor

$$(\Lambda^{(s+1)})_\infty = (\Lambda^{(s)})_\infty^{(s+1)} = \sum_{i=0}^{s-1} n_i E_i^{(s+1)} + \mu_s E_s^{(s+1)} = \mu_s \Gamma$$

is globally μ_s-fold, where Γ is an effective divisor on X_{s+1} and μ_s is the multiplicity of the last base point of Λ.

1.11 Remarks: If $\mu_i = \mu(\Lambda^{(i)}, P_i)$, then, in the terminology of Nagata ([18], pp. 14,15), not only does Λ_λ

"go through" $\sum_{i=0}^{s} \mu_i P_i$, but P_i "lies on" Λ_λ, and the "effective multiplicity" $\mu(\Lambda_\lambda^{(i)}, P_i)$ of P_i on Λ_λ equals μ_i. $(0 \le i \le s,\ \lambda \in k)$ We leave it to the interested reader to determine what form 1.10, 1.11 and the ensuing results take in case the place at infinity of f is not necessarily rational, but only, say, purely inseparable over k.

The first consequence to be drawn now is that, by the last statement in 1.8,

1.12 $P_1, \cdots, P_s$ are rational over k, hence $E_1, \cdots, E_s$ are all projective lines over k, and

$$|\Gamma| = \bigcup_{i=0}^{s} |E_i^{(s+1)}|$$ is a curve with s + 1 irreducible components, each isomorphic to $\mathbb{P}_k^1$. (E_i, being non-singular, is isomorphic to $E_i^{(s+1)}$.)

We point out a very fruitful consequence of the rationality of the place at infinity of f, which we shall have occasion to use several times: Let K be any field containing k. Suppose $f = f(x,y) \in k[x,y]$, X, Y are indeterminates over K, and consider $f_K = f(X,Y) \in K[X,Y]$. Then f_K has one rational place at infinity. If $\Lambda_K = \Lambda(f_K)$, then the counterparts of 1.10, 11 and 12 are gotten by simply replacing k by K - in particular the sequence of base point multiplicities and the structure of $(\Lambda_K^{(s+1)})_\infty$ remain unchanged. In the sequel we will drop the subscripts "K".

Our next objective is to determine the value of μ_s.

Let $\Delta_i = |E_i^{(s+1)}|$, $0 \le i \le s$. We need to describe $|\Gamma| = \bigcup_{i=0}^{s} \Delta_i$ topologically; in order to use the known theorems, we must pass to the algebraic closure $\bar{k}$ of k. The following fact is no doubt very well-known; having no reference for it we give a proof.

1.13 <u>Lemma</u>: Suppose $k = \bar{k}$ and Z is a k-variety which is the union of two closed subvarieties Z_1, Z_2 which intersect in a single point P. If Z_1, Z_2 are connected and simply connected, so is Z.

<u>Proof</u>: Z is clearly connected. Suppose $\sigma: Y \to Z$ is a connected étale covering. The restrictions $Y_i = \sigma^{-1}(Z_i) \to Z_i$ are étale coverings, so for i = 1,2,

$$Y_i \simeq \coprod_{j=1}^{m_i} Z_{i,j}, \quad Z_{i,j} \simeq Z_i, \text{ and } Y_i \to Z_i \text{ is the canonical map.}$$

Since $Y = \cup Z_{i,j}$ and Y is connected, each $Z_{1,j}$ meets some $Z_{2,j'}$ and vice versa. Any point of such an intersection is a preimage of P, so $Z_{1,j}$ and $Z_{2,j'}$ meet in a <u>unique</u> point. So $m_1 = m_2$ and, relabelling if necessary, Y is the disjoint union of the m_1 closed sets $Z_{1,j} \cup Z_{2,j}$. So $m_1 = 1$, $Y_i \to Z_i$ are isomorphisms, and $\sigma^{-1}(P) = Y_1 \cap Y_2$ consists of a single point Q. So σ, being a closed bijection, is a homeomorphism, and, if $Q \neq Q' \in Y$,

$\sigma^*: \mathcal{O}_{Z,\sigma(Q')} \to \mathcal{O}_{Y,Q'}$ is an isomorphism. But $\mathcal{O} = \mathcal{O}_{Z,P} \to \mathcal{O}_{Y,Q} = \mathcal{O}'$ is also an isomorphism. (See [7], I, 4.4, and use the fact that $\hat{\mathcal{O}}$ is faithfully flat over $\mathcal{O}$, and that $\mathcal{O}'$ is a finite module over the noetherian local ring $\mathcal{O}$.)

1.14 Corollary: Let Γ be as in 1.12. $(k = \bar{k})$ Then $|\Gamma|$ is connected and simply connected.

Proof: Actually, we remark that, starting with a projective line over an algebraically closed field k, the variety Δ gotten by blowing up, successively, s points, is connected and simply connected. (No ambient variety is needed.) This variety will be a union $\bigcup_{i=0}^{s} \Delta_i$ of closed subvarieties, each a copy of $\mathbb{P}_k^1$. Its connectedness is evident. One shows by induction on s that, combinatorially, Δ is a tree of $\mathbb{P}_k^1$'s (there are no loops); such a configuration is simply connected, by 1.13 and induction on s.

1.15 Proposition: Let k be algebraically closed, and let h: W → C be a morphism from a nonsingular irreducible k-variety W to an irreducible curve C. Suppose

i) h is proper and surjective,

ii) the function field $k(C)$ of C is separably algebraically closed in $k(W)$,

iii) $z \in C$ is a simple point,

iv) the image $h^*(\{z\})$ of the divisor $\{z\}$ under $h^*: \text{Div } C \to \text{Div } W$ is $\mu\Gamma$, where μ is a positive integer and Γ is an effective divisor on W, and

v) $|\Gamma|$ is simply connected.

Let π be the characteristic exponent of k. Then

μ is a power of π.

<u>Proof</u>: Let $t \in k(C)$ be a local parameter at z. There is an open neighbourhood U of z such that on U, t is everywhere defined and vanishes only at z, and such that U consists only of normal points. $h^{-1}(U) \to U$ satisfies the conditions of the proposition. In other words, we can assume that t is regular on C and vanishes only at $z \in C$, and that C is a non-singular curve.

Let $\mu = q\pi^r$, with $r \geq 0$ and q prime to π. We must show $q=1$.

First note that $q(\pi^r\Gamma)$ is the principal divisor div $h^*(t)$. Given any $w \in W$, there is an open affine neighbourhood $U_w = \text{Spec } A_w$ of w, and an $h_w \in A_w$, such that the restriction of the divisor $\pi^r\Gamma$ to Div U_w is the principal divisor div h_w. (See [26], p. 131.) Note that $h_{w'}/h_w$ is a unit on $U_w \cap U_{w'}$, for all w, w'. Also, since div $(h^*(t)/h_w{}^q) = 0$ in Div U_w, $u_w = h^*(t)/h_w{}^q$ is a unit in A_w.

Let $B_w = A_w[T]/(T^q - u_w)$, T an indeterminate. B_w is a finite étale A_w-algebra. If $x \in U_w \cap U_{w'} = \text{Spec } A$, then there is a canonical A-isomorphism $A[T]/(T^q - u_w) \to A[T]/(T^q - u_{w'})$, which sends T to $(h_{w'}/h_w)T$. Thus the B_w define a coherent $\mathcal{O}_W$-Algebra, hence a scheme W' and an étale covering ρ: W' → W.

Now W', which is reduced and nonsingular, is also irreducible. To see this, reduce the cover $\{U_w\}$ of W to a finite one - $W = \bigcup_w U_w$. Each A_w is a domain. Fix a w, and suppose $T^q - u_w$ is reducible in $k(W)[T]$. Then (e. g. by [11], p. 214, Theorem 10. (b)), there is a $v \in k(W)$, and a q' dividing q, $1 < q'$, such that $u_w = v^{q'}$. We have $h^*(t) = (vh_w{}^{q/q'})^{q'}$. But t is a local parameter at z, hence has no q'^{th} root in k(C). Since q' is prime to π, this contradicts ii). One checks now directly that $B_w \to k(W)[T]/(T^q - u_w)$ is an <u>injection</u>, so W' is a finite union of irreducible spaces which intersect pairwise, hence is irreducible. We have also established that $[k(W'):k(W)]$ equals q, hence $k(W') = k(W)(h^*(t)^{1/q})$, and one easily checks that W' is just the normalization of W in k(W'). Making a similar construction on C, one gets a variety C', which is the normalization of C in $k(C)(t^{1/q})$, and a commutative picture

$$\begin{array}{ccc} W' & \xrightarrow{h'} & C' \\ \rho \downarrow & & \downarrow \psi \\ W & \xrightarrow[h]{} & C \end{array}$$

Since C' is separated, ψ is. Since ρ is finite, $h \circ \rho$ is proper, so h' is proper, and clearly dominant, hence surjective. $|\psi^{-1}(z)|$ consists of a single point z'. (z is the unique ramification point of ψ.)

We claim k(C') is separably algebraically closed in k(W'). If not, there is a finite separable extension L of k(C') contained in k(W'), with $[L:k(C')] > 1$. (We identify k(C') with $h'^*(k(C'))$, etc., so that all function fields are subfields of k(W').) Let s be a primitive element of L/k(C), and let F be the minimal polynomial of s over k(C). If $F_1 \in k(W)[X]$ is a monic irreducible factor of F in k(W)[X], then the coefficients of F_1 lie in a splitting field for F over k(C), hence are separable algebraic over k(C). By ii), $F_1 = F$. So $q = [k(W'):k(W)] \geq [k(W)(s):k(W)] = \deg F > q$, an absurdity.

Now by one version of the Connectedness Theorem of Zariski, $|\rho^{-1}(|\Gamma|)| = |\rho^{-1}(h^{-1}(z))| = |h'^{-1}(z')|$ is connected. ([8], Cor. 4.3.7)

On the other hand, ρ is everywhere q-to-1 on closed points. (This is easily seen for Spec $B_w \to$ Spec A_w, for all w.) Since ρ is finite étale, so is $\rho^{-1}(|\Gamma|) \to |\Gamma|$. By v), $\rho^{-1}(|\Gamma|)$ is a disjoint union $\coprod |\Gamma|$ of copies of $|\Gamma|$, and $\coprod |\Gamma| \to |\Gamma|$ is the canonical map. So $|\rho^{-1}(|\Gamma|)|$ has q connected components, and q = 1.

1.16 <u>Theorem</u>: In the notation of 1.10, $\mu_s = \pi^r$, $r \geq 0$.

Proof: Blow up the base points of $\Lambda(f)$ over $\bar{k}$. Denoting again by X_{s+1} the surface thus obtained, we have a morphism $h: W = X_{s+1} \to \mathbb{P}^1_{\bar{k}} = C$. h is surjective, and $h^*(\infty) = (\Lambda^{(s+1)})_\infty = \mu_s\Gamma$ has simply connected support. Since $\mathbb{P}^2_{\bar{k}}$ is projective, W is projective by s + 1 applications of 1.4.1. So h is proper. If the function field of C were not separably algebraically closed in that of W, no member of $\Lambda^{(s+1)}$ would be a prime divisor. (See [12], [31], p. 60, [33], p. 50.) But $(\Lambda^{(s+1)})_\lambda = \Lambda_\lambda^{(s+1)}$ is a prime divisor for all $\lambda \in \bar{k}$. (See 0.4 2), with $\bar{k}$ in place of k, and apply 1.4.2 s + 1 times.) Another way to see that k(C) is algebraically closed in k(W) is to note that $f - t \in \overline{k(t)}^{[2]}$ is irreducible – 0.4 2) again – so the function field of f – t over k(t) is a regular extension of k(t). 1.16 now follows from 1.15.

Proof of 1.1: Blow up the base points of $\Lambda(f)$ over k(t). By 1.9 4) and 1.16, we can choose local coordinates ξ,η at the last base point P_s of $\Lambda(f)$ such that ξ^{π^r} is the leading form of a local equation ℓ for $\Lambda_0^{(s)}$ at P_s, η^{π^r} is a local equation for $(\Lambda^{(s)})_\infty$ at P_s, and $\ell_t = \ell - t\eta^{\pi^r}$ is a local equation for $\Lambda_t^{(s)}$ at P_s. If $\mathfrak{O}_s = \mathcal{O}_{X_s,P_s}$, the unique iqt $\mathfrak{O}'$ of $\mathfrak{O}_s$ in which the proper transform ℓ_t' of ℓ_t is not a unit is $\mathfrak{O}_s[\xi/\eta]_{(\eta,\zeta = (\xi/\eta)^{\pi^r} - t)}$, and $\ell_t' = \ell_t/\eta^{\pi^r} = \zeta + \zeta'$, where $\zeta' \in \eta\mathfrak{O}'$. Since ζ,η are a regular system of parameters at $\mathfrak{O}'$, $\mu(\ell_t',\mathfrak{O}') = 1$, and $\mathfrak{O} = \mathfrak{O}'/\ell_t'\mathfrak{O}'$ is the unique place at infinity of $f - t \in k(t)^{[2]}$. The residue field $\kappa(\mathfrak{P}) = \kappa(\mathfrak{P}') = \kappa(\mathfrak{P}_s)(t^{\pi^{-r}}) = k(t^{\pi^{-r}})$ since P_s is rational over $k(t)$. So the place $\mathfrak{P}$ at infinity of $f - t$ is purely inseparable of degree π^r over $k(t)$. 2) of 1.1 follows from 1.7.3, 1.7.4, and 1.10.

Proof of 1.2 1): There are "fancy" ways to prove 1.2 1); we give a relatively elementary argument:

First, we note that $f - t \in \overline{k(t)}^{[2]}$ has one place at infinity by 1.1 1) and [6], Cor. 4, p. 95. Let $Q_t^{(0)} = P_s$ be the last base point of $\Lambda(f)$. As in the proof of 1.1, we have a local equation $\ell_t^{(0)}$ for $f - t$ at $Q_t^{(0)}$ with leading form $\xi^{\pi^r} - t\eta^{\pi^r}$. Passing to $k(t^{\pi^{-r}}) = k_1$, we see that there is a unique iqt $Q_t^{(1)}$ of $Q_t^{(0)}$ "along $\ell_t^{(0)}$", and that $Q_t^{(1)}$ is rational over k_1. Suppose now $i \geq 1$, let k_i be a field of the form $L_i(s_i)$, where L_i is a finite purely inseparable extension of k and $s_i = t^{\pi^{-M_i}}$ for some $M_i \geq 0$, and suppose the first $i + 1$ i. n. points above $Q_t^{(0)}$ "along $\ell_t^{(0)}$" form a sequence $Q_t^{(0)}$, $Q_t^{(1)}, \ldots, Q_t^{(i)}$ of points rational over k_i. ($Q_t^{(j+1)}$ = unique iqt of $Q_t^{(j)}$ "along $\ell_t^{(0)}$", $0 \leq j < i$.) If $\ell_t^{(i)}$ is a local equation for the proper transform of $\ell_t^{(0)}$ at $Q_t^{(i)}$ and $x^{(i)}$, $y^{(i)}$ are local parameters at $Q_t^{(i)}$, we consider the leading form $\mathrm{lf}(\ell_t^{(i)}) = (a_i x^{(i)} - b_i y^{(i)})^{\nu_{s+i}}$, where ν_{s+i} is the multiplicity of $\ell_t^{(i)}$ at $Q_t^{(i)}$ over $\overline{k(t)} = \overline{k_i}$ and $a_i, b_i \in \overline{k_i}$. We may assume

$lf(\wp_t^{(i)}) = p^{\nu'}$, where $p \in k_i[x^{(i)}, y^{(i)}]$ is an irreducible form. Assuming as we may that $a_i \neq 0$, we see that the irreducible polynomial $p(x^{(i)},1)$ in $x^{(i)}$ has only one root ξ_i. The unique iqt $Q_t^{(i+1)}$ of $Q_t^{(i)}$ "along $\ell_t^{(i)}$" is rational over $k_i(\xi_i)$. Clearly $k_i(\xi_i) \subset L_{i+1}(s_{i+1}) = k_{i+1}$, where L_{i+1}/L_i is finite purely inseparable and $s_{i+1} = t^{\pi^{-M_{i+1}}}$ for some $M_{i+1} \geq 0$. The process stops when we reach a simple i. n. point over $\overline{k(t)}$, and then we have 1.2 1).

Before considering the rest of 1.2, we make a

Remark: Suppose $\mathfrak{O}$ is a regular local ring over a field K, x, y form a system of regular parameters for $\mathfrak{O}$, $\alpha \in K[x,y] \subset \mathfrak{O}$ and the leading form of α is $(ax - by)^{\mu}$, $\mu > 0$, $a, b \in K$, $b \neq 0$. Then $\tilde{\mathfrak{O}} = \mathfrak{O}[y/x]_{(\tilde{x} = x, \tilde{y} = y/x - a/b)}$ is the unique iqt of $\mathfrak{O}$ "along α", $\tilde{x}, \tilde{y}$ are regular parameters for $\tilde{\mathfrak{O}}$, and the proper transform $\alpha' = \alpha/x^{\mu} \in K[\tilde{x},\tilde{y}] \subset \tilde{\mathfrak{O}}$.

Proof of 1.2 2) and 3): Let N be as in 1.2 1) and consider the pencil $\Lambda(f)$ over $\bar{k}(s)$, $s^{\pi^N} = t$. At the last base point $Q^{(0)}$ of Λ, choose local coordinates $x^{(0)}, y^{(0)}$ such that for all $\lambda \in \bar{k}(s)$, a local equation $\ell_{\lambda}^{(0)}$ for Λ_{λ} at $Q^{(0)}$ is $\ell_0^{(0)} - \lambda y^{(0)\,\pi^r}$. By the remark we can require that $\ell_0^{(0)} \in k[x^{(0)}, y^{(0)}]$, so that $\ell_{\lambda}^{(0)} \in \bar{k}(s)[x^{(0)}, y^{(0)}]$ for all $\lambda \in \bar{k}(s)$. For $\lambda \in \bar{k}$, $\ell_{\lambda}^{(0)}$ is the specialization of $\ell_t^{(0)}$ under $s \to \lambda^{\pi^{-N}}$. Let $Q_t^{(1)}, \ldots, Q_t^{(M)}, \ldots$ be the successive points infinitely near $Q^{(0)}$ "along $\ell_t^{(0)}$", and suppose the multiplicity of the proper transform of $\ell_t^{(0)}$ at $Q_t^{(M)}$ is 1. We fix explicitly local coordinates $x_t^{(i)}, y_t^{(i)}$ at $Q_t^{(i)}$ and proper transforms $\ell_t^{(i)} \in \bar{k}(s)[x_t^{(i)}, y_t^{(i)}]$ of $\ell_t^{(0)}$:

Put $x_t^{(0)} = x^{(0)}$, $y_t^{(0)} = y^{(0)}$, $Q_t^{(0)} = Q^{(0)}$. Suppose the leading form of $\ell_t^{(i)}$ at $Q_t^{(i)}$ is $(a^{(i)}x_t^{(i)} - b^{(i)}y_t^{(i)})^{\mu_i}$, $a^{(i)}, b^{(i)} \in \bar{k}(s)$. If $b^{(i)} \neq 0$, we set

$$\ell_t^{(i+1)} = \ell_t^{(i)}/x_t^{(i)\,\mu_i},$$

$$x_t^{(i+1)} = x_t^{(i)}, \quad y_t^{(i+1)} = y_t^{(i)}/x_t^{(i)} - a^{(i)}/b^{(i)}.$$

If $b^{(i)} = 0$, we reverse $x_t^{(i)}$ and $y_t^{(i)}$, $a^{(i)}$ and $b^{(i)}$ in the above. Note that $\ell_t^{(i+1)} \in \bar{k}(s)[x_t^{(i+1)}, y_t^{(i+1)}]$.

Now for $0 \le i \le M$, let T_i be the finite set of nonzero coefficients $c \in \bar{k}(s)$ of $\ell_t^{(i)}$, and let $S_i =$ $\{ \lambda \in \bar{k} \mid$ for some $c \in T_i$, c has a zero or a pole at $s = \lambda^{\pi^{-N}} \}$. (One can get away with a bit less than this.) Let $S =$ $\bar{k} - \bigcup_{i=0}^{M} S_i$. S is the complement in $\bar{k}$ of a finite set. Now for $\lambda \in S$, we claim 1.2 2) and 3) hold. We sketch the induction:

Fix a $\lambda \in S$. If c is a coefficient appearing in $\ell_t^{(i)}$, put $c_\lambda = c_{|s=\lambda^{\pi^{-N}}}$. Now suppose $0 \le i \le M$ and the infinitely near points above $Q^{(0)}$ "along $\ell_\lambda^{(0)}$" form a sequence $Q^{(0)}, Q_\lambda^{(1)}, \cdots, Q_\lambda^{(i)}$. Suppose also that $\mu(\ell_\lambda^{(j)}) = \mu(\ell_t^{(j)})$, $0 \le j \le i$, and that there are local coordinates $x_\lambda^{(i)}, y_\lambda^{(i)}$ at $Q_\lambda^{(i)}$ such that if $\ell_t^{(i)} = (a^{(i)}x_t^{(i)} - b^{(i)}y_t^{(i)})^{\mu_i} + \sum_{\gamma+\delta>\mu_i} c_{\gamma\delta}x_t^{(i)\gamma} y_t^{(i)\delta}$, then $\ell_\lambda^{(i)} :=$

$(a^{(i)}{}_\lambda x_\lambda^{(i)} - b^{(i)}{}_\lambda y_\lambda^{(i)})^{\mu_i} + \Sigma\, c_{\gamma\delta\lambda} x_\lambda^{(i)\gamma} y_\lambda^{(i)\delta} \in \bar{k}[x_\lambda^{(i)}, y_\lambda^{(i)}]$ is a local equation for Λ_λ at $Q_\lambda^{(i)}$. (Note that $a^{(i)}{}_\lambda$, $b^{(i)}{}_\lambda$ make sense, and that $a^{(i)}{}_\lambda = 0 \Leftrightarrow a^{(i)} = 0$, $b^{(i)}{}_\lambda = 0 \Leftrightarrow b^{(i)} = 0$.) One sees straightforwardly that there is a unique iqt $Q_\lambda^{(i+1)}$ of $Q_\lambda^{(i)}$ "along Λ_λ", that there are local coordinates $x_\lambda^{(i+1)}, y_\lambda^{(i+1)}$ at $Q_\lambda^{(i+1)}$ such that a local equation $\ell_\lambda^{(i+1)}$ for Λ_λ at $Q_\lambda^{(i+1)}$ is gotten by replacing $s, x_t^{(i+1)}$ and $y_t^{(i+1)}$ in $\ell_t^{(i+1)}$ by $\lambda^{\pi^{-N}}, x_\lambda^{(i+1)}, y_\lambda^{(i+1)}$, and that $\mu(\ell_\lambda^{(i+1)}) = \mu(\ell_t^{(i+1)})$.

We next mention two applications of the results in this section. The first will see use in the second, and is a strengthening of part of 0.6 6) (with a slight additional hypothesis).

1.17 Proposition: Suppose $f \in k^{[2]}$ has one rational place at infinity, and $f \in \bar{k}^{[2]}$ is a coordinate line. Then f is a coordinate line over k.

Proof: $f - t \in \bar{k}(t)^{[2]}$ is a line, hence has a rational place at infinity. So the multiplicity of the last base point of $\Lambda(f)$ (with respect to any generators x, y for $k^{[2]}$) is 1. So the order $\sum_{i=0}^{s} \mu_i(\mu_i - 1)$ of the divisor of singularities at infinity of $f - t \in k(t)^{[2]}$ equals $(d-1)(d-2)$, where $d = \deg f$. Hence $f - t$, having genus zero and a rational place at infinity, is rational. Since $f - t$ is regular at finite distance, it is a line. Now use 0.6 5).

The next result is already known - one proof, shown the writer by Peter Russell, uses Hamburger-Noether expansion. We mention that in [5], Question 4.7, p. 97, the authors ask, given two retracts of an augmented algebra which satisfy certain compatibility conditions, whether a ⊗ - decomposition of the algebra is induced. They remark that the first nontrivial case is the following:

Suppose $f, g \in R \simeq k^{[2]}$ are lines, and that f generates R mod g and vice versa. (These are precisely the hypotheses of 1.18 below.) Do f and g generate R? Using the epimorphism theorem, they give an affirmative answer in characteristic 0. 1.18 removes the restriction on char k.

1.18 Proposition: Suppose $f,g \in k[x,y] \simeq k^{[2]}$ each have one rational place at infinity, and $i(f,g)_{\text{fin.}}$ equals 1. Then $k[x,y] = k[f,g]$.

Proof: By 1.2 2), $f - \lambda$ [resp. $g - \mu$] has one place V_λ [resp. W_μ] at infinity for general λ [μ] $\in \bar{k}$. By 0.5 1), $W_0(\bar{f}) = -1$, where $\bar{f} = f \bmod g$. So for all $\lambda \in \bar{k}$, $1 = -W_0(\overline{f - \lambda}) = i(f - \lambda, g)_{\text{fin.}}$. If $f - \lambda$ has one place at infinity, then $1 = i(f - \lambda, g)_{\text{fin.}} = -V_\lambda(\bar{g}) = -V_\lambda(\overline{g - \mu}) = i(f - \lambda, g - \mu)_{\text{fin.}}$ for all $\mu \in \bar{k}$. Similarly if $g - \mu$ has one place at infinity, then $i(f - \lambda, g - \mu)_{\text{fin.}} = 1$ for all $\lambda \in \bar{k}$. Hence we have

$$(*) \qquad i(f - \lambda, g - \mu)_{\text{fin.}} = 1 \quad \text{for all but finitely many } (\lambda,\mu) \in \bar{k}^2.$$

Now the proof told the writer by A. Sathaye continues roughly thus: Conclude that $k(f,g) = k(x,y)$, so that f and g, being lines and field generators, are coordinate lines, and argue 1.18 from this. We give another proof.

Suppose $\bar{k}(f,g)$ were not separably closed in $\bar{k}(x,y)$. Then the map $X = \operatorname{Spec} \bar{k}[x,y] \to \operatorname{Spec} \bar{k}[f,g] = Y$ would factor through $Z = \operatorname{Spec} B$, B = integral closure of $\bar{k}[f,g]$ in L, L = separable closure of $\bar{k}(f,g)$ in $\bar{k}(x,y)$. $Z \to Y$ is a finite separable map, which is therefore unramified over the general point of Y, and has $n > 1$ points in its general fibre. One easily sees that this contradicts $(*)$. So $\bar{k}(x,y)/\bar{k}(f,g)$ is purely inseparable.

Suppose $x^{\pi^m} = \alpha(f,g)/\beta(f,g)$, $\alpha,\beta \in \bar{k}[f,g]$ relatively prime, π as usual the characteristic exponent of k. If β is nonconstant, there exist infinitely many $(\lambda,\mu) \in \bar{k}^2$ such that $\beta(\lambda,\mu) = 0 \neq \alpha(\lambda,\mu)$. Pick such a (λ,μ), such that $f - \lambda$, $g - \mu$ meet at $x = a$, $y = b$. Then $a^{\pi^m} \cdot 0 = \alpha(\lambda,\mu) \neq 0$. So assume $\beta(f,g) = 1$. Let $\alpha(f,g) = \sum_{i=0}^{d} \alpha_i(g)f^i$. We know g generates $\bar{k}[x,y]$ mod f. Say $x = h(g)$ mod f. Then $h(g)^{\pi^m} = \alpha_0(g)$ mod f. Since g is a variable mod f, $\alpha_0(g) \in (\bar{k}[g])^{\pi^m}$. Also, f^{π^m} divides $(x - h(g))^{\pi^m} = \sum_{i=1}^{d} \alpha_i(g)f^i$. Going mod f repeatedly, one sees that $\alpha_1 = \cdots = \alpha_{\pi^m-1} = 0$, $\alpha_{\pi^m}(g) \in (\bar{k}[g])^{\pi^m}$, etc. So x (similarly, y) $\in \bar{k}[f,g]$. By 1.17, f is a coordinate line over k. Say $k[f,z] = k[x,y]$. Let $g = \sum_{i=0}^{N} \delta_i(z)f^i$. Using the fact that $(f - \lambda, g)_{fin.}$ equals 1 for <u>all</u> $\lambda \in \bar{k}$, one sees that $\delta_0(z)$ is linear, $\delta_i(z)$ constant, $i > 0$. So $k[x,y] = k[f,g]$.

Finally, we turn to the results of Abhyankar and Moh already mentioned (0.8, 0.9). The hypotheses of these theorems imply that the last base point of $\Lambda(f)$ is simple on all members of $\Lambda(f)$, by 1.7.3, 1.7.4, 1.10 and 1.16. 0.9 follows at once. To see the Epimorphism Theorem (0.8), note that the linear system of curves of degree deg f which " go through the multiple points " of f at infinity has dimension ≥ 1. Now [18], Theorem 4.1, p. 32, should give the result, but, finding both [18] and [17] somewhat unclear on this point, we invoke another argument - the proof of 1.17.

2. Counterexamples

Let A be an affine k-domain of dimension 1. Following Abhyankar ([1]) and other writers, we call a surjective k-algebra map $k^{[2]} \to A$ an <u>embedding</u> of the curve (Spec) A in the affine plane. We say that two embeddings α, β of A in the plane are <u>equivalent</u> if there is a $\gamma \in \mathrm{Aut}_k k^{[2]}$ such that $\alpha = \beta \circ \gamma$. If k is of characteristic zero, the epimorphism theorem of Abhyankar and Moh says precisely that all embeddings of the affine line in the plane are equivalent. In [4], the authors prove (assuming that char k = 0 or that one considers only "non-wild" embeddings) the

FINITENESS THEOREM: Suppose A has one rational place at infinity. Then the number of inequivalent embeddings of A in the plane is finite.

We remark that if A has more than one place at infinity, the conclusion of the theorem may not hold. This example appears in [24]:

Let $A = k[t, t^{-1}] = k[t, t^{-n}]$, $n \geq 1$. Define embeddings $\alpha_n : k[x,y] \to A$ by $x \rightsquigarrow t$, $y \rightsquigarrow t^{-n}$. The kernel of α_n is generated by $f_n = x^n y - 1$. Suppose $\gamma \in \mathrm{Aut}_k k[x,y]$ and $\alpha_n = \alpha_m \circ \gamma$. Then $\gamma(f_n) = cf_m$ for some $c \in k^*$. Now $\gamma(x)$, $\gamma(y)$ are each of degree at least 1. So $\gamma(f_n)$ has degree at least $n + 1$. So $m \geq n$. Similarly $n \geq m$, so $m = n$.

We have seen (1.16, end of proof of 1.1), for char $k = p > 0$, that one invariant (under automorphisms of $k[x,y]$) of a curve $f \in k[x,y]$ with one rational place at infinity is the residue field degree p^r of the place at infinity of $f - t \in k(t)[x,y]$. One can easily write down Abhyankar-Moh lines f having, for each positive integer r, prescribed invariant p^r. (Let $1 < A < p^{r+1}$ and $p \nmid A$. Then $f = y^{p^{2r+1}} - x - x^{Ap^r}$ does the job - see 2.13 below.) So the finiteness theorem breaks down in positive characteristic for even the most well-behaved curve with one place at infinity - of course we are dealing here with "wild" embeddings.

We suppose, in the remainder of this section, that k is of characteristic $p > 0$, and algebraically closed. (Since the results we prove will be mainly <u>negative</u> in character, this latter assumption is, in most of the sequel, no restriction. We also observe that 2.8 through 2.14 and the proofs of them given here are independent of char k.)

The next example provides negative answers to the following questions about lines $f \in k^{[2]}$:

2.1 1) Suppose all $f - \lambda$ are lines. We have embeddings $\alpha_\lambda : k^{[2]} \to k^{[1]}$ for all $\lambda \in k$, such that $\ker \alpha_\lambda$ is generated by $f - \lambda$. Is α_0 equivalent to α_λ for all λ?

2) Is every line equivalent to an Abhyankar-Moh line?

Before this example was found, the answers to 1) and 2) seem not to have been known. Even the version of 1), where f is only presumed to have one place at infinity, was not known to have a negative answer. For instance, all such f exhibited in [9], 1.5.3 have the property that all α_λ are equivalent, and Abhyankar-Moh lines also enjoy this property. We thank Peter Russell for the idea of looking at the ring $k[x,y^p]$, and for pointing out the Lemma below. We also thank A. Sathaye for showing us the line $(y^3 - x^2)^6 - x$ in characteristic 3, in response to our request for a nonprincipal (cf. § 0, comment before 0.8) line neither of whose degrees is a power of p.

2.2 Example: Let a,b be integers > 1, not divisible by p. Consider α: $k[x,y] \to k[t]$ defined by $x \rightsquigarrow t^{ap^2}$, $y \rightsquigarrow t^{abp} + t$. Let $u = y^p - x^b$. Then $\alpha(y - u^{ab}) = t$. Let $f = u^{ap} - x$. Clearly f generates ker α. Also $f \in k[x,y^p] = k[f,u] = A$. So f and all $f - \lambda$ are (coordinate) lines in A.

2.3 Lemma: Let $h \in k[x,y]$ and $f = h^p - x \in k[x,y^p]$ be a line in $k[x,y^p]$. Then f is a line in $k[x,y]$.

Proof: Suppose f generates the kernel of the surjection α: $k[x,y^p] \to k[t^p]$. Let $\alpha(x) = u(t)^p$, $\alpha(y^p) = v(t)^p$. (Recall that k is perfect.) Then α is the restriction of β: $k[x,y] \to k[t]$ to $k[x,y^p]$, where $\beta(x) = u(t)^p$, $\beta(y) = v(t)$. Since $\beta(f) = 0$, $u(t)^p = h(u(t)^p, v(t))^p$, so $u(t) \in k[u(t)^p, v(t)]$, so this ring equals $k[u(t),v(t)]$, so β is surjective, since

$t \in k[u(t),v(t)]$. Suppose g generates ker β. By 0.5 2), $\deg_x g = \deg_t v(t)$, and $\deg_y g = \deg_t u(t)^p$; also $\deg_x f = \deg_{t^p} v(t)^p = \deg_t v(t)$, and $\deg_{y^p} f = \deg_{t^p} u(t)^p = \deg_t u(t)$, so $\deg_y f = \deg_t u(t)^p$. So f generates ker β. (<u>Remark</u>: When we have 3.3, we can conclude from $\frac{d}{dt}u(t)^p = 0$ that $g_y = 0$, so that $g \in k[x,y^p]$. So g divides f in $k[x,y]$, and f divides g in $k[x,y^p]$, etc.)

By the Lemma, all $f - \lambda$ in example 2.2 are lines in $k[x,y]$ – this can also be seen directly. Now suppose $\phi \in \mathrm{Aut}_k k[x,y]$, $\lambda \in k$, $c \in k^*$, and $\phi(f) = c(f - \lambda)$. Since $\phi(y^p)$, $\phi^{-1}(y^p) \in k[x^p,y^p] \subset A$ and $\phi(x) = \phi(u)^{ap} - c(f - \lambda)$ (similarly $\phi^{-1}(x)) \in A$, ϕ restricts to an automorphism of A. Moreover $\phi(u)$ generates A over $k[f]$. So $\phi(u) = du + h(f)$ for some $d \in k^*$, $h(f) \in k[f]$. So $\phi(x) = (du + h(f))^{ap} - c(f - \lambda) =$

$$(d^{ap} - c)u^{ap} + \sum_{i=0}^{a-1} \binom{a}{i} (du)^{ip} h(f)^{p(a-i)} + cx + c\lambda.$$

Suppose $\deg h = m \geq 1$. Then $\deg_x \phi(x) = \deg_x h(f)^{ap} = b(a^2mp^2)$ (since $a > 1$), and $\deg_y \phi(x) = p(a^2mp^2)$. But $p \nmid b$ and $b \nmid p$, which contradicts the Automorphism Theorem (for $k[x,y]$, not A). So $h(f) = h \in k$, and $\phi(x) =$

$$(d^{ap} - c)u^{ap} + \sum_{i=0}^{a-1} \binom{a}{i} h^{p(a-i)} (du)^{ip} + cx + c\lambda.$$

$\deg_x u^{ip} = ibp$, $\deg_y u^{ip} = ip^2$ implies (repeating the above argument twice, and noting that $a > 1$, $p \nmid a$ and $d \neq 0$) that $d^{ap} = c$

and $h = 0$. So $\phi(x) = c(x + \lambda)$, and $\phi(u) = du = d(y^p - x^b)$. But $\phi(u) = \phi(y)^p - (c(x + \lambda))^b$. So, $\frac{\partial}{\partial x}\phi(u) = -bdx^{b-1} = -b(c(x + \lambda))^{b-1}$. Setting x equal to 0, we get $\lambda = 0$ and $d = c^{b-1}$. Also, $(d - c^b)x^b = dy^p - \phi(y)^p$ and $p \nmid b$, so $d = c^b$ and $c = 1$. From this it follows that

$$\{f - \lambda\}_{\lambda \in k}$$

is a set of pairwise inequivalent lines, of cardinality card k. In particular, f is not equivalent to an Abhyankar-Moh line. The proof also shows that if $\lambda \in k$, then there is no nontrivial automorphism of the plane which fixes the curve $f - \lambda$. So example 2.2 exhibits a sort of family of "totally inert" curves.

The next example shows that 1.2 3) cannot be strengthened. Before finding it, the writer had hoped that for f having one rational place at infinity, those $f - \lambda$ having one place at infinity would have the same multiplicity sequence at infinity as does f.

2.4 <u>Example</u>: Let char k = 17. Let $f = y^{119} + x^{85} + xy^2$.

We leave to the reader the verification of these claims:

For all $\lambda \in k$, $f - \lambda$ has one place at infinity. The base points of $\Lambda(f)$ consist of

3 points of multiplicity 34 and

37 points of multiplicity 17.

The multiplicity sequence of f at infinity continues

$$17, \underbrace{2, \cdots, 2}_{8}, 1.$$

That of $f - \lambda$, $\lambda \in k^*$, continues

$$17, 1.$$

f is a rational curve, and $f - \lambda$ has genus 12, for $\lambda \in k^*$.

1.2 2) cannot be strengthened either, as the next example shows. It provides a negative answer to Abhyankar's question

2.5 If f has one place at infinity, does $f - \lambda$ have one place at infinity for all $\lambda \in k$? ($k = \bar{k}$, of course.)

(This is QUESTION (11.18) on p. 91 of [1].)

2.6 Example: Let char k = 2, and let $f = x^6 + xy^{227} + y^{682}$. For $\lambda \neq 1$, $f - \lambda$ has one place at infinity. $f - 1$ has two. (f is again a rational curve; $f - \lambda$, $\lambda \neq 0,1$, has genus 340, $f - 1$ has genus 339.)

Professor Sathaye has, upon analyzing this example, explained to the writer how to construct many such, and has kindly suggested that the construction be included here. Both to help the reader understand Sathaye's examples and the others of this section, then, and to remind him of the flavor of the mathematics involved in blowing up a point on a plane curve, we first collect a few elementary facts. The following statements about positive integers must appear in many textbooks (although the writer has not seen them), and are in any case easy to prove:

2.7 Let A, B be integers, $1 \le A \le B$, g.c.d.$(A,B) = 1$. Then there exist unique integers ℓ, m, ℓ', m' such that

1) $0 < \ell \le B$, $0 \le \ell' < B$, $0 \le m < A$, $0 < m' \le A$ and
2) $\ell A - mB = 1 = m'B - \ell' A$. $(m' = A - m, \ \ell' = B - \ell.)$

Moreover,

3) the above inequalities are all strict if $1 < A$,
4) a) $\ell m' - \ell' m = 1$,
   b) $0 < \ell - m$, and
   c) if $A < B$, then $\ell - m \le B - A$, hence $0 \le \ell' - m' < B - A$.

2.8 We next describe a process for obtaining, given a power series ring $\mathfrak{O}$ in two variables x, y, and a nonunit $f \in \mathfrak{O}$ satisfying certain conditions, another power series ring $\tilde{\mathfrak{O}}$, variables $\tilde{x}, \tilde{y}$ for $\tilde{\mathfrak{O}}$, and a nonunit $\tilde{f} \in \tilde{\mathfrak{O}}$.

Let $f = f(x,y) = \Sigma f_{ij} x^i y^j \in k[[x,y]] = \mathfrak{O}$ be of order $d_1 > 0$.

i) Suppose that the leading form of f is cx^{d_1}, $c \in k^*$, and that $d_1 < d_0 = \text{ord } f(0,y) < \infty$. Then we put

$$\tilde{x} = x/y,\ \tilde{y} = y,\ \tilde{\mathcal{O}} = k[[\tilde{x},\tilde{y}]], \text{ and}$$

$$\tilde{f} = \Sigma\, \tilde{f}_{ij}\tilde{x}^i\tilde{y}^j, \text{ where}$$

$$\tilde{f}_{ij} = \begin{cases} f_{i,j-i+d_1}, & \text{if } j + d_1 \geq i \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\tilde{f} = f/y^{d_1}$.

ii) Suppose x and y are interchanged in the hypotheses of i). Then we put

$$\tilde{x} = x,\ \tilde{y} = y/x,\ \tilde{\mathcal{O}} = k[[\tilde{x},\tilde{y}]],$$

$$\tilde{f} = \Sigma\, \tilde{f}_{ij}\tilde{x}^i\tilde{y}^j, \text{ where}$$

$$\tilde{f}_{ij} = \begin{cases} f_{i-j+d_1,j}, & \text{if } i + d_1 \geq j \\ 0, & \text{otherwise.} \end{cases}$$

Observe that $\tilde{f} = f/x^{d_1}$.

<u>Remarks</u>: 1) We emphasize that here we are concerned with setting up a precise process $(\mathcal{O},x,y,f) \rightsquigarrow (\tilde{\mathcal{O}},\tilde{x},\tilde{y},\tilde{f})$, defined only under assumption i) or ii) on f above, which provides us not only with a new element $\tilde{f}$ of a power series ring $\tilde{\mathcal{O}}$, but also with specified variables $\tilde{x},\tilde{y}$ for $\tilde{\mathcal{O}}$.

2) We call $\tilde{f}$ the proper transform of f. Referring to the discussion in § 1 above Lemma 1.8, we clarify the relationship between what we are doing here and what was done there. Assume i) above. Then the unique iqt $\mathcal{O}'$ of $\mathcal{O}$ in which f/y^{d_1} is a nonunit is $\mathcal{O}[x/y]_{(x/y,y)}$; the $\tilde{\mathcal{O}}$ defined in i) is the <u>completion</u>

of $\mathcal{O}'$. In § 1, we could have shown that under certain conditions on $f \in \mathcal{O}$, we have

$$(*) \qquad f \in \mathcal{O} \text{ unibranch} \Leftrightarrow f' \in \mathcal{O}' \text{ unibranch, where}$$

f' is the proper transform of f in $\mathcal{O}'$. The next lemma essentially establishes the counterpart of (*), where "unibranch" is replaced by "analytically irreducible". One way to do this would be to prove the chain of implications f analytically irreducible ⇔ f unibranch ⇔ f' unibranch ⇔ f' analytically irreducible; we have instead given a direct proof.

Given any $f = \sum_{i \geq d_1} F_i(x,y) \in k[[x,y]]$, $F_i \in k[x,y]$ forms of degree i, $F_{d_1} \neq 0$, we put $\hat{f} := f/y^{d_1} \in k[[x/y,y]] =: \hat{\mathcal{O}}$. (This notation is in force only for the next lemma.)

2.9 <u>Lemma</u>: 1) If $lf(f) = cx^{d_1}$, $c \in k^*$, and $\hat{f} \in \hat{\mathcal{O}}$ is irreducible, then $f \in \mathcal{O}$ is irreducible.

2) If ord $f(0,y) > d_1$ and $f \in \mathcal{O}$ is irreducible, then $\hat{f} \in \hat{\mathcal{O}}$ is irreducible.

<u>Proof</u>: $\hat{f} = \sum_{i \geq d_1} y^{i-d_1} F_i(x/y,1)$. If $\hat{f} \in \hat{\mathcal{O}}$ is irreducible, then $\hat{f} \neq 0$ and $F_{d_1}(0,1) = \hat{f}(0,0) = 0$, so $f = \hat{f}y^{d_1} \neq 0$ and $d_1 > 0$, so $f \notin \mathcal{O}^*$. Suppose $f = gh$, $g,h \in \mathcal{O}$ of positive order, and $lf(f) = cx^{d_1}$, $c \in k^*$. Then $lf(g) = c_1x^{\mu}$, $lf(h) = c_2x^{\nu}$, $c_i \in k^*$, $\mu,\nu > 0$. $\hat{g} = g/y^{\mu}$, $\hat{h} = h/y^{\nu}$ are nonunits in $\hat{\mathcal{O}}$ and

$\hat{f} = \hat{g}\hat{h}$. So 1) holds.

Next suppose $y \nmid lf(f)$. (f is "regular in x, of order d_1".) By the Weierstrass Preparation Theorem, there is a unique $u \in \mathcal{O}^*$ such that $uf = x^{d_1} + \sum_{i=1}^{d_1} a_i(y)x^{d_1-i}$, $a_i(y) \in k[[y]]$; also, $\text{ord } a_i > 0$. (uf is "special in x".) $f \in \mathcal{O}$ is irreducible $\Leftrightarrow$ $uf \in k[[y]][x]$ is irreducible. (See, e. g., [34], p. 146, COROLLARY 2.) Since $\widehat{uf} = u\hat{f}$, we can assume $u = 1$. Suppose $\text{ord } f(0,y) > d_1$. f irreducible in $\mathcal{O} \Rightarrow lf(f) = x^{d_1}$, by Hensel's Lemma. (Recall that we are assuming k is algebraically closed.) So $\text{ord } a_i > i$, $\hat{f} = (x/y)^{d_1} + \sum_{i=1}^{d_1} y^{-i}a_i(y)(x/y)^{d_1-i} \in k[[y]][x/y] = S$ is special in x/y, and we must show $\hat{f}$ is irreducible in S. Suppose $\hat{f} = gh$, $g,h \in S$, $g,h \notin S^*$. Since $\hat{f}$ is monic in x/y, g and h have positive degree in x/y. So $f = y^{d_1}gh$, $g,h \in k((y))[x] - k((y))$, which contradicts the Gauss Lemma.

As a consequence of 2.9 we have

2.10 For f as in 2.8, $f \in \mathcal{O}$ is irreducible $\Leftrightarrow$ $\tilde{f} \in \tilde{\mathcal{O}}$ is irreducible.

2.11 Now let $f \in \mathfrak{O}$, $\mathrm{lf}(f) = cx^{d_1}$, $0 < d_1 = \mathrm{ord}\, f(x,0) < d_0$, etc., as in 2.8.
[Resp. $\mathrm{lf}(f) = cy^{d_1}$, $0 < d_1 < d_0 = \mathrm{ord}\, f(x,0)$, etc.]
Let

$$d_0 = q_1 d_1 + d_2$$
$$d_1 = q_2 d_2 + d_3$$
$$\cdots$$
$$d_{\alpha-1} = q_\alpha d_\alpha$$

be the Euclidean algorithm, $q = \Sigma q_i$. Let $f = \Sigma f_{ij} x^i y^j$ and suppose $f_{ij} = 0$ for $id_0 + jd_1 < d_0 d_1$. [Resp., $f_{ij} = 0$ for $jd_0 + id_1 < d_0 d_1$.] Then we have a unique sequence $\mathfrak{O}_1 < \mathfrak{O}_2 < \cdots < \mathfrak{O}_q$ and elements $f_i \in \mathfrak{O}_i = k[[x_i, y_i]]$ such that $f_1 = f$, $\mathfrak{O}_1 = \mathfrak{O}$, $f_{i+1} = \tilde{f}_i$, $\mathfrak{O}_{i+1} = \tilde{\mathfrak{O}}_i$, $x_1 = x$, $y_1 = y$, $x_{i+1} = \tilde{x}_i$, $y_{i+1} = \tilde{y}_i$. Moreover, $\mathrm{ord}\, f_q = \mathrm{ord}\, f_q(x_q, 0) = \mathrm{ord}\, f_q(0, y_q) = d_\alpha$.

<u>Proof</u>: We do induction on q. Suppose $q = 2$. Then $d_0 = 2d_1$ and we are done. Suppose $q > 2$, and suppose 2.11 holds for $q' < q$.

$\tilde{f} = \Sigma f_{ij} (x/y)^i y^{i+j-d_1} = \Sigma \tilde{f}_{ij} \tilde{x}^i \tilde{y}^j$, where $\tilde{f}_{ij} = f_{i,j-i+d_1}$.

i) $d_0 > 2d_1$. Put $\tilde{d}_0 = d_0 - d_1$, $\tilde{d}_1 = d_1$. We have

$\tilde{d}_0 = (q_1 - 1)\tilde{d}_1 + d_2, \cdots, d_{\alpha-1} = q_\alpha d_\alpha$, and $q' = q_1 - 1 + \sum_{i=2}^{\alpha} q_i = q - 1$. $i\tilde{d}_0 + j\tilde{d}_1 < \tilde{d}_0\tilde{d}_1 \Rightarrow i(d_0 - d_1) + jd_1 < (d_0 - d_1)d_1 \Rightarrow id_0 + (j - i + d_1)d_1 < d_0 d_1 \Rightarrow \tilde{f}_{ij} = 0$. 2.11 follows by the induction hypothesis on q'.

ii) $d_0 < 2d_1$. Put $\tilde{d}_0 = d_1$, $\tilde{d}_1 = d_0 - d_1$. $\tilde{y}$ is the tangent to $\tilde{f}$. $j\tilde{d}_0 + i\tilde{d}_1 < \tilde{d}_0\tilde{d}_1 \Rightarrow jd_1 + i(d_0 - d_1) < d_1(d_0 - d_1) \Rightarrow$ $id_0 + (j - i + d_1)d_1 < d_0 d_1 \Rightarrow \tilde{f}_{ij} = 0$.

2.12 <u>Corollary</u>: Suppose f is as in 2.11. If g.c.d.$(d_0, d_1) = 1$, then f is irreducible.

<u>Proof</u>: $f_q \in \mathfrak{O}_q$ has order 1; apply the "if" part of 2.10 $q - 1$ times.

2.12.1 <u>Remark</u>: In the general case $(d_\alpha \geq 1)$, 2.11 may be regarded as saying that the branches of f stay together through the points $\mathfrak{O}_1, \cdots, \mathfrak{O}_q$.

2.13 <u>Theorem</u>: Let $f = \sum f_{ij}x^i y^j \in \mathfrak{O} = \mathfrak{O}_1$ be as in 2.11. Recall that successive applications of ~ determine not only the power series rings $\mathfrak{O}_i$, $1 \leq i \leq q$, but also generators for them. Let x', y' be the generators for $\mathfrak{O}_q$ so determined, and let $f' = \sum f'_{ij}x'^i y'^j \in \mathfrak{O}_q$ be the proper transform of f. Let $A = d_1/d_\alpha$, $B = d_0/d_\alpha$, $\ell A - mB = 1 = m'B - \ell' A$ as in 2.7 1) and 2).

Then

1) $f'_{ij} = f_{i'j'}$, where
$i' = m'i - mj + md_\alpha$, $j' = -\ell'i + \ell j + \ell'd_\alpha$, hence

2) $f_{ij} = f'_{i''j''}$, where
$i'' = \ell i + mj - md_0$, $j'' = \ell'i + m'j - \ell'd_1$.

3) $x' = x^{m'}y^{-\ell'}$, $y' = y^\ell x^{-m}$, hence
$x = x'^\ell y'^{\ell'}$, $y = x'^m y'^{m'}$.

<u>Proof</u>: 2) follows from 1) upon solving for i,j in terms of i',j', recalling that $\ell + \ell' = B$, $m + m' = A$, and $\ell m' - \ell'm = 1$. Similarly we need only prove the first part of 3).

Let $d_0 = q_1d_1 + d_2, \ldots, d_{\alpha-1} = q_\alpha d_\alpha$, $q = \Sigma q_i$ as in 2.11. Suppose $q = 2$. Then $f' = \Sigma f_{i,j-i+d_1}x'^iy'^j$. We have $d_0 = 2d_1$, $d_1 = d_\alpha$, $\ell = \ell' = m' = 1$ and $m = 0$, so 1) holds. Now suppose $q > 2$. $\tilde{f} = \Sigma f_{i,j-i+d_1}\tilde{x}^i\tilde{y}^j$.

i) $d_0 > 2d_1$. Put $\tilde{d}_0 = d_0 - d_1$, $\tilde{d}_1 = d_1$, $\tilde{A} = A$, $\tilde{B} = B - A$. By 2.7, 4) b) and c), if $\tilde{\ell}, \ldots$ are defined as were $\ell, \ldots$, then $\tilde{\ell} = \ell - m$, $\tilde{m} = m$, $\tilde{\ell}' = \ell' - m'$, $\tilde{m}' = m'$. We have $f'_{ij} = \tilde{f}_{m'i-mj+md_\alpha,\ -(\ell'-m')i+(\ell-m)j+(\ell'-m')d_\alpha} = f_{i'j'}$.

ii) $d_0 < 2d_1$. Put $\tilde{d}_0 = d_1$, $\tilde{d}_1 = d_0 - d_1$, $\tilde{A} = B - A$, $\tilde{B} = A$. $\tilde{\ell} = m'$, $\tilde{m} = \ell' - m'$, $\tilde{\ell}' = m$, $\tilde{m}' = \ell - m$. $\tilde{f} = \Sigma \tilde{f}_{ij}\tilde{x}^i\tilde{y}^j = \Sigma \tilde{f}_{ji}\tilde{y}^i\tilde{x}^j$, $f' = \Sigma f'_{ij}x'^iy'^j = \Sigma f'_{ji}y'^ix'^j$. We have

$f'_{ji} = \tilde{f}_{-mi+m'j+md_\alpha,\ (\ell-m)i-(\ell'-m')j+(\ell'-m')d_\alpha} =$

$f_{-mi+m'j+md_\alpha,\ \ell i-\ell' j+\ell' d_\alpha}$, and 1) follows. The same consideration of cases gives 3).

2.14 <u>Corollary</u>: Suppose $f = \Sigma f_{ij}x^iy^j$ is irreducible in $k[[x,y]]$, and $d_1 = \text{ord } f(x,0) < d_0 = \text{ord } f(0,y) < \infty$. Then

1) $f_{ij} = 0$ for $id_0 + jd_1 < d_0d_1$, and

2) $$\sum_{id_0+jd_1=d_0d_1} f_{ij}x^iy^j =$$

$$\left(f_{d_1,0}^{1/d_\alpha} x^{d_1/d_\alpha} + f_{0,d_0}^{1/d_\alpha} y^{d_0/d_\alpha} \right)^{d_\alpha},$$

where $d_\alpha = \text{g.c.d.}(d_0,d_1)$.

Suppose $f = \Sigma f_{ij}x^iy^j \in k[x,y]$ has one place at infinity, and let $D = \deg_y f$, $d = \deg_x f$. Then

i) $f_{ij} = 0$ for $iD + jd > Dd$, and

ii) $$\sum_{iD+jd=Dd} f_{ij}x^iy^j =$$

$$\left(f_{d,0}^{1/d_\alpha} x^{d/d_\alpha} + f_{0,D}^{1/d_\alpha} y^{D/d_\alpha} \right)^{d_\alpha},$$

where $d_\alpha = \text{g.c.d.}(D,d)$.

Proof: Recall that if $g \in k[[x,y]]$ is irreducible, then $\mathrm{lf}(g)$ is a power of a linear form, hence ord $g =$ min { ord $g(x,0)$, ord $g(0,y)$ }. Define q as before. If $q = 2$, $d_0 = 2d_1$. One checks that ord $\tilde{f}(\tilde{x},0) =$ ord $\tilde{f}(0,\tilde{y}) = d_1$. By the "only if" part of 2.10, $\tilde{f}$ is irreducible. So ord $\tilde{f} = d_1$, and 1) holds. The proof of 1) for $q > 2$ is the reverse of steps i), ii) in the proof of 2.11, together with the cited part of 2.10.

Now by $q - 1$ applications of the "only if" part of 2.10, $f' = f_q \in \mathcal{O}_q$ is irreducible, so ord $f' =$ ord $f'(x',0) =$ ord $f'(0,y') = d_\alpha$. So $f'_{ij} = 0$ for $i + j < d_\alpha$, and

$$\sum_{i+j=d_\alpha} f'_{ij}x'^i y'^j = \left(f'^{1/d_\alpha}_{d_\alpha,0}\, x' + f'^{1/d_\alpha}_{0,d_\alpha}\, y' \right)^{d_\alpha} .$$

If $f'_{ij} = f_{i'j'}$, then $i + j = d_\alpha \Leftrightarrow i'd_0 + j'd_1 = d_0d_1$, as follows from 1) and 2) of 2.13. By 2.13 3), $\sum_{id_0+jd_1=d_0d_1} f_{ij}x^iy^j =$

$$\sum f_{ij}x'^{i\ell+jm}y'^{i\ell'+jm'} = x'^{md_0}y'^{\ell'd_1} \sum_{i+j=d_\alpha} f'_{ij}x'^iy'^j =$$

$$\left(f'^{1/d_\alpha}_{d_\alpha,0}\, x'^{md_0/d_\alpha+1}\, y'^{\ell'd_1/d_\alpha} + f'^{1/d_\alpha}_{0,d_\alpha}\, x'^{md_0/d_\alpha}\, y'^{\ell'd_1/d_\alpha+1} \right)^{d_\alpha} =$$

$$\left(f^{1/d_\alpha}_{d_1,0}\, x^{d_1/d_\alpha} + f^{1/d_\alpha}_{0,d_0}\, y^{d_0/d_\alpha} \right)^{d_\alpha} .$$

To see i) and ii), first suppose $D = d$. Then i) and ii) just amount to saying that f has one point at infinity, which is the case. So we may assume $D > d$. Then $\hat{x} = 1/x$, $\hat{y} = y/x$ are local parameters at the point at infinity of f, and $\hat{f} = \sum f_{ij}\hat{x}^{D-i-j}\hat{y}^j \in k[[\hat{x},\hat{y}]]$ is a local equation for f there, hence is irreducible. Put $\hat{f}_{ij} = f_{D-i-j,j}$. By 1), $f_{D-i-j,j} = 0$ for $iD + j(D - d) < D(D - d)$, i. e., $\hat{f}_{ij} = 0$ for $(D - i - j)D + j(D - d) < D(D - d)$, which is i). Similarly ii) follows from 2).

Remarks: 2.11 through 2.14 were discovered independently by the writer; some of these results may have been known for some time. 2.14 i), for instance, can be found, stated and proved differently, in [1], (11.19), under the assumption that char k does not divide g.c.d.(D,d). Note that the second half (i), ii)) of 2.14 holds for f with one rational place at infinity over any field k, simply by passing to $\bar{k}$.

We are now ready to describe, in stages, Sathaye's family of examples providing negative answers to 2.5.

2.15 Let a, b, α, β, n be positive integers such that $0 < \text{char } k = p \nmid a$, $\alpha + \beta = p + p \sum_{i=0}^{n} p^i$, and $a + b = p(1 + p^n)$. Let $f_\lambda = y^p + x^p + x^a y^b - \lambda x^\alpha y^\beta \in k[[x,y]]$, and $\lambda_0 = a^n(-1)^{\alpha+a(n-1)}$. Then

f_λ is irreducible if and only if $\lambda \neq \lambda_0$.

(Note: In the following long computations we have tacitly assumed that a,n > 1; if a or n = 1 the reader is invited to strike out the superfluous terms.)

<u>Proof</u>: Let $F = \alpha + \beta + 1$. Consider the automorphism α_λ of $k[[x,y]]$ defined by $y \rightsquigarrow y$,

$$x \rightsquigarrow x - y - \sum_{i=0}^{n-1} a^i (-1)^{a(i+1)} y^{1+m_i} - c_\lambda y^{1+m_n},$$

where

$$m_i = \frac{1}{p-1} p^{n-i} (p^{i+1} - 1) \text{ and}$$

$$c_\lambda{}^p = a^n (-1)^{a(n-1)} - \lambda (-1)^\alpha .$$

We have

$$\alpha_\lambda (y^p + x^p) = x^p - \sum_{i=0}^{n-1} a^i (-1)^{a(i+1)} y^{p(1+m_i)} - c_\lambda{}^p y^{p(1+m_n)}$$

since a and -1 are in the prime field of k.

$$\alpha_\lambda (x^a y^b) = y^b \Bigg(x^a + \sum_{j=1}^{a-2} \binom{a}{j} (-1)^j x^{a-j} \Big[y + \sum_{i=0}^{n-1} a^i (-1)^{a(i+1)} y^{1+m_i} + c_\lambda y^{1+m_n} \Big]^j +$$

$$a(-1)^{a-1} x \sum_{j=1}^{a-1} \binom{a-1}{j} y^{a-1-j} \Big[\sum_{i=0}^{n-1} a^i (-1)^{a(i+1)} y^{1+m_i} + c_\lambda y^{1+m_n} \Big]^j + a(-1)^{a-1} x y^{a-1} +$$

$$(-1)^a y^a \big(1 + \sum_{i=0}^{n-1} a^i (-1)^{a(i+1)} y^{m_i} + c_\lambda y^{m_n}\big)^a \Bigg) .$$

Now $(1 + \sum_{i=0}^{n-1} a^i (-1)^{a(i+1)} y^{m_i} + c_\lambda y^{m_n})^a = 1 + a c_\lambda y^{m_n} +$

$$a\sum_{i=0}^{n-2} a^i(-1)^{a(i+1)}y^{m_i} + a^n(-1)^{an}y^{m_{n-1}} +$$

$$\sum_{j=2}^{a}\binom{a}{j}\left(\sum_{i=0}^{n-1} a^i(-1)^{a(i+1)}y^{m_i} + c_\lambda y^{m_n}\right)^j$$, so, since

$a + b + m_{n-1} = \alpha + \beta$ and $F = a + b + m_n$,

$$\alpha_\lambda(x^a y^b) = a(-1)^{a-1}xy^{a+b-1} + a^n(-1)^{a(n-1)}y^{\alpha+\beta} + a(-1)^a c_\lambda y^F +$$

$$(-1)^a y^{a+b} + \sum_{i=0}^{n-2} a^{i+1}(-1)^{ai}y^{m_i+a+b} + \Sigma_1 \text{, where}$$

$$\Sigma_1 = y^b\Bigg((-1)^a y^a \sum_{j=2}^{a}\binom{a}{j}\left(\sum_{i=0}^{n-1} a^i(-1)^{a(i+1)}y^{m_i} + c_\lambda y^{m_n}\right)^j +$$

$$a(-1)^{a-1}x\sum_{j=1}^{a-1}\binom{a-1}{j}y^{a-1-j}\left[\sum_{i=0}^{n-1} a^i(-1)^{a(i+1)}y^{1+m_i} +\right.$$

$$\left.c_\lambda y^{1+m_n}\right]^j +$$

$$\sum_{j=1}^{a-2}\binom{a}{j}(-1)^j x^{a-j}\left(y+\sum_{i=0}^{n-1} a^i(-1)^{a(i+1)}y^{1+m_i}+c_\lambda y^{1+m_n}\right)^j +$$

$$x^a\Bigg).$$ Since $p(1 + m_0) = a + b$, $p(1 + m_n) = \alpha + \beta$ and $p(1 + m_{i+1}) = m_i + a + b$ for $0 \le i \le n - 2$,

$$\alpha_\lambda(x^p+y^p+x^a y^b) = x^p + a(-1)^{a-1}xy^{a+b-1} + a(-1)^a c_\lambda y^F +$$

$$\left(a^n(-1)^{a(n-1)} - c_\lambda{}^p \right) y^{\alpha+\beta} + \Sigma_1.$$

$\alpha_\lambda(-\lambda x^\alpha y^\beta) = -\lambda(-1)^\alpha y^{\alpha+\beta} + \Sigma_2$, where $\Sigma_2 =$

$$-\lambda y^{\beta}\left((-1)^{\alpha}\sum_{j=1}^{\alpha}\binom{\alpha}{j}y^{\alpha-j}\left(\sum_{i=0}^{n-1}a^{i}(-1)^{a(i+1)}y^{1+m_i}+c_\lambda y^{1+m_n}\right)^{j}+\right.$$

$$\left.\sum_{j=0}^{\alpha-1}\binom{\alpha}{j}(-1)^{j}x^{\alpha-j}\left(y+\sum_{i=0}^{n-1}a^{i}(-1)^{a(i+1)}y^{1+m_i}+c_\lambda y^{1+m_n}\right)^{j}\right).$$

So $\alpha_\lambda(f_\lambda) = x^p + a(-1)^{a-1}xy^{a+b-1} + a(-1)^a c_\lambda y^F + \Sigma_1 + \Sigma_2$.

We make a few observations:

i) p and F are relatively prime.

ii) $c_\lambda = 0$ if and only if $\lambda = \lambda_0$.

iii) $(1)F + (a+b-1)p > Fp$, and

$(1)(F+1) + (a+b-1)p \leq (F+1)p$.

iv) If $x^r y^s$ is any monomial occurring in Σ_1 or Σ_2, then $rF + sp > Fp$, and $(r,s) \neq (1,a+b-1)$.

(The proofs are omitted, involving as they do arithmetic only.)

If $\lambda \neq \lambda_0$, then by i), the "only if" in ii), the first inequalities in iii) and iv), and 2.12, f_λ is irreducible.

If $\lambda = \lambda_0$, then by the "if" in ii), the second inequalities in iii) and iv), the fact that $t = \text{ord } f_\lambda(0,y) \geq F + 1$ (if $t = +\infty$, note that $p > 1$), and 2.14 1) and 2), f_λ is reducible.

2.16 <u>Observation</u>: Let p be a prime. Then:

1) There exist $n \geq 1$, and coprime integers B, C such that

$$1 < C < B \text{ and } BC = \sum_{i=0}^{n} p^i = \frac{p^{n+1} - 1}{p - 1} .$$

Put $A = B - C$. Then $1 \leq A < B - 1$. Let ℓ, m be the unique integers such that $0 < \ell < B$, $0 \leq m < A$, and $\ell A - mB = 1$. (See 2.7.) Put

$$a = 1 + (\ell - m)(p - 1)B = p + \ell C(p - 1) .$$

We can arrange that

2) p does not divide a.

Proof: If $p = 2$, take e. g. $B = 5$, $C = n = 3$. ($\ell = 3$, $m = 1$.)

Suppose p is odd. If p is not of the form $2^r - 1$, then there exist coprime B, C, $1 < C < B$, such that $BC = 1 + p$. Now $0 < \ell < p$ and $0 < C < p$, so $p \nmid a$. Suppose now $p = 2^r - 1$. Let $B = p^4 + p^2 + 1$, $C = p + 1 = 2^r$, $n = 5$. B and C are coprime since B is odd. If $p \mid a$, then $(\ell - m)B \equiv 1 \pmod{p}$, so $\ell - m \equiv 1 \pmod{p}$. But $A < B - 1 \Rightarrow \ell - m < C$, so $\ell - m = 1$. So $\ell C + 1 = (\ell - m)B = B$, so $p+1 \mid p^4+p^2$, which is false.

2.17 Lemma: Let A, B, a, ℓ, m, n be as in 2.16. Put

$\ell' = B - \ell$, $m' = A - m$, $b = p(1 + p^n) - a$,

$a^* = am' - bm + pm$, $b^* = -a\ell' + b\ell + p\ell'$. Then

1) a^* and b^* are positive, and

2) $a^* + b^* < pB$ and $a^*B + b^*A > ABp$.

Proof: $0 < 1 = \ell A - mB = (\ell - m)A - mC \Rightarrow \frac{\ell-m}{C} > \frac{m}{A} \Rightarrow$

$$(\ell - m)B(p - 1) + \frac{\ell-m}{C} = \frac{\ell-m}{C}((p-1)BC + 1) > \frac{m}{A}\, p^{n+1} .$$

Since $\ell - m < C$, we have $a > mp^{n+1}/A$, so $a^* = aA - mp^{n+1} > 0$. $\ell > 0 \Rightarrow 0 > -\ell \Rightarrow \ell p(1 + p^n) > \ell p + (p^{n+1} - 1)\ell =$ $\ell p + B(\ell C(p - 1)) = \ell p + B(a - p) \Rightarrow b^* = \ell p(1 + p^n) + B(p - a) -$ $\ell p > 0$. Using the first expression for a in 2.16 and the fact that $\ell - m < C$, one checks that $a^* + b^* < pB$. The last inequality in 2) follows from the fact that $\ell A - mB = 1$.

We can now exhibit Sathaye's family of examples. We call a curve f as in 2.18 a curve <u>of Sathaye type</u>.

2.18 Proposition: Keeping the preceding notation, put

$M = Bp - a^* - b^*$, $N = b^*$, and let

$$f = y^{Bp} + x^{Cp} + x^M y^N .$$

$f \in k[x,y]$. Put $\lambda_0 = a^n(-1)^{a(n-1)+Bp(\ell-m)}$. Then

$f - \lambda$ has one place at infinity if and only if $\lambda \neq \lambda_0$.

Proof: By 2.17, $f \in k[x,y]$ and the degree form of f is y^{Bp}. Hence a local equation for $f - \lambda$ at its point at infinity is $g_\lambda = \frac{f-\lambda}{x^{Bp}} = y'^{Bp} + x'^{Ap} + x'^{a^*}y'^{b^*} - \lambda x'^{Bp}$, where $x' = 1/x$, $y' = y/x$. We must show $g_\lambda \in k[[x',y']]$ is irreducible $\Leftrightarrow \lambda \neq \lambda_0$.

By the second part of 2.17 2), and since $A < B$ and $p =$ g.c.d.(Ap, Bp), this amounts to proving the same for

$f_\lambda = y''^p + x''^p + x''^a y''^b - \lambda x''^\alpha y''^\beta \in k[[x'', y'']]$, where

$\alpha = Bp(\ell - m)$ and $\beta = (B - \ell)Cp$. (See 2.10, 2.13 1) and 2), and 2.16.) But this is just 2.15.

Remarks: 1) By 2.16, curves of Sathaye type exist for k of any characteristic $p > 0$.

2) Example 2.6 is of Sathaye type. ($p = 2$, $B = 341$, $C = 3$, $n = 9$, $\ell = 227$, $m = 225$, $a = 683$, $b = 343$, $\lambda_0 = 1$.) A specific example easier to check is $f = x^6 + xy^3 + y^{10}$ ($p = 2$, $B = 5$, $C = n = 3$); as with 2.6, f is a rational curve.

The reader may have noticed that, in 2.4, 2.6 and remark 2) above, the pencil $\Lambda(f)$ has a movable singularity, i. e., f_x and f_y have a common factor. When k is of characteristic zero, Bertini's theorem on the variable singular points of a linear system rules out this eventuality. In view of the uniform behavior (0.9) at infinity of the members of the pencil $\Lambda(f)$ when char $k = 0$, one might therefore hope that if f has one place at infinity and the pencil $\Lambda(f)$ has no movable singularity, then $f - \lambda$ has one place at infinity for all λ. The matter is not so simple:

2.19 Let char $k = 2$, $f = y^8 + x^6 + y + xy^2 + 1$. For $\lambda \neq 1$,

$f + \lambda$ has one place at infinity. $f + 1$ has two. All members of $\Lambda(f)$ are nonsingular at finite distance.

We briefly explain a way to construct such examples:

Suppose char $k = p > 0$ and $p + 1$ has a factor A, $1 \le A < p$, such that $A + 1$ and $p + 1$ are relatively prime. (We note that for $p = 5, 41, \cdots$, there is no such A.) There exist unique C,D, $0 < C < A + 1$, $0 < D < p + 1$, such that $D(A + 1) - C(p + 1) = 1$. Put $B = \frac{p+1}{A}$ and $M = D - C$. One easily sees that $M > 0$, $D > 1$, $pD > B$, and g.c.d.$(pD,B) = 1$. There exist unique ℓ,m, $0 < \ell < pD$, $0 \le m < pD - B$, such that $\ell(pD - B) - mpD = 1$. Let $F = \ell(-1)^{(\ell-m)(p^2D-1)}$. Using the above facts and most of the techniques of this section, one has

2.20 Proposition: Let $f = y^{p^2D} + x^{pB} + xy^{pM} + Fy + 1$. Then

$f - \lambda$ has one place at infinity $\Leftrightarrow \lambda \neq 1$.

All $f - \lambda$ are nonsingular at finite distance, and the genus of $f - 1$ is less than that of $f - \lambda$, for all $\lambda \neq 1$.

Remarks: 1) In 2.19, we have $A = 1$, $B = 3$, $C = 1$, $D = 2$, $\ell = 1$, $m = 0$, $F = 1$. $f - 1$ has genus 3, $f - \lambda$, $\lambda \neq 1$, has genus 4.

2) Suppose $f = y^{Bp} + x^{Cp} + x^My^N$ is an irreducible curve and $\Lambda(f)$ has no movable singularity. One sees that then $\max\{M,N\} = 1$. If f is of Sathaye

type, M and N are positive. So M and N must equal 1. So $b^* = 1$ and $a^* = Bp - 2$. Hence $(p - 1)BC + 1 = p^{n+1} = a + b - p = Ba^* + Ab^* - ABp = BCp - B - C$, so $0 = BC - B - C - 1$ and $(B - 1)(C - 1) = 2$, so $B = 3$, $C = 2$. Since $6 = BC = 1 + \cdots + p^n$, we must have $p = 5$, $n = 1$ and $f = y^{15} + x^{10} + xy$. Then $f - \lambda$ has one place at infinity $\Leftrightarrow \lambda \neq 2$, and $\Lambda(f)$ has no movable singularity. ($f - \lambda$ is nonsingular at finite distance if and only if $\lambda \neq 0$.) f and $f - 2$ are rational, and $f - \lambda$ has genus 1 for $\lambda \neq 0,2$.

3) The following has apparently been an open question for some time:

> Suppose char $k = 0$, $f \in k[x,y]$, and $f - \lambda$ is irreducible and nonsingular at finite distance for all $\lambda \in k$. Are then all $f - \lambda$ lines?

2.20 gives counterexamples in <u>positive</u> characteristic.

3. Lines in the plane in positive characteristic

(Unless otherwise stated, k will, throughout this section, be assumed to be of characteristic $p > 0$.)

Most of the results in the preceding two sections arose in the course of attempts to prove or disprove the following statement:

3.1 If k is algebraically closed, and $f \in k^{[2]}$ is a line, then $f - \lambda$ is a line for all $\lambda \in k$.

(See [28], Q(1,2).) At the time of writing, the truth of 3.1 has not been established. We remark first that none of the examples of § 2 disproves 3.1. For, by 3.4 below, if f is a line, $\Lambda(f)$ has no movable singularity at finite distance. So of the examples cited, only 2.20 could possibly contradict 3.1. And in 2.20 no member of $\Lambda(f)$ has one place at infinity and genus 0, i. e., no member is a line.

Now we describe some of the ways one might try to prove 3.1. We begin by remarking that the most difficult step in the proof of the epimorphism theorem in characteristic zero is the following fact:

3.1.1 If $f \in k^{[2]}$ has one rational place at infinity, then so does $f - \lambda$, for all $\lambda \in k$.

For, suppose this is granted. One sees fairly easily that for general $\lambda \in k$, the geometric number of places at infinity of $f - \lambda$ equals the degree over $k(t)$ of the residue field of the place at infinity of $f - t$, so this place must be rational. Since all members of $\Lambda(f)$ "go through each other", the orders of the divisors of singularities at infinity of f and $f - t$ are the same, so $f - t$ is a line, etc.

Now in positive characteristic one has no such argument (whatever is left we have used in 3.14), but one might have hoped to salvage 3.1.1. ($k = \bar{k}$, of course.) 2.6 shatters this hope. 2.4 scuttles another possible approach to 3.1. If the hope expressed just before 2.4 were not in vain, then 3.1 would follow. (For, $f - \lambda$ would then clearly be a line for <u>general</u> $\lambda \in k$, and one could invoke 3.14 below.)

The hypothesis of 3.1 concerns only one member of the pencil $\Lambda(f)$; the conclusion concerns every member. If one, perhaps not unnaturally, emphasizes the hypothesis on the <u>pencil</u>, that it have no movable singularity at finite distance (again, see 3.4), then 2.20 is discouraging.

The idea that, given a line f and $\lambda \in k$, there is an automorphism of the plane which carries the curve $V(f)$ to $V(f - \lambda)$, is mistaken, by 2.2.

Another possible approach to 3.1 is to use the characterization " $-1 \in \Gamma$. " (Γ the value semigroup at infinity of f) of a line f. The problem with this is that, while one has very complete information indeed on how to compute the value semigroup under the assumption that the characteristic does not divide both the x- and y-degree of f ([2], § 7; [1], § 8), one seems to have no hold on the semigroup when this assumption is <u>not</u> met, as is the case with all lines of interest here (i. e., non-coordinate lines). The writer's <u>guess</u> is that an invariance theorem like 1.2 holds for the value semigroups of $f - \lambda$, for general $\lambda \in k$ (f a curve with one rational place at infinity), and that "for general $\lambda \in k$" cannot be replaced by "for all $\lambda \in k$ such that $f - \lambda$ has one rational place at infinity".

In view, therefore, both of the limited state of knowledge concerning the structure of value semigroups in "interesting" cases, and of the existence of pathological situations like those in § 2, one is led to consider 3.1 <u>ab initio</u>, rather than as a statement about a certain kind of plane curve with one rational place at infinity. But from this point of view, proving 3.1 may not be appreciably easier than achieving this goal:

3.2 Find a "recipe" for all lines in the plane.
(The terminology is Abhyankar's.)

In this section, we present some partial results on lines, with an eye to attacking both 3.1 and 3.2. We will be guided by the known examples of lines, and by the analogies they suggest with the simple situation which prevails in characteristic 0.

3.3 <u>Lemma</u>: Let k be any field, α: $k[x,y] \to k[t]$ a surjective k-algebra map carrying x and y to u(t) and v(t), with kernel $f(x,y)k[x,y]$. Then there exists $C \in k^*$ such that

$$\alpha(f_x) = Cv'(t), \quad \alpha(f_y) = -Cu'(t).$$

(See [3], COROLLARY (2.6).)

<u>Proof</u>: Let $g = g(x,y) \in k[x,y]$, with $t = \alpha(g) = g(u(t),v(t))$. We have $0 = \alpha(f) = f(u(t),v(t))$. By the Chain Rule,

$$1 = \alpha(g_x)u'(t) + \alpha(g_y)v'(t) \quad \text{and}$$

$$0 = \alpha(f_x)u'(t) + \alpha(f_y)v'(t) \ .$$

So u', $v' \in k[t]$ are relatively prime. We need only show that

1) $\alpha(f_x)$, $\alpha(f_y) \in k[t]$ are relatively prime.

Suppose $d \in k[t]$ is a nonconstant common factor of $\alpha(f_x)$, $\alpha(f_y)$, and let $c \in \bar{k}$ be a root of d. Put $a = u(c)$, $b = v(c)$. Then $f(a,b) = 0$, and $f_x(a,b) = 0 = f_y(a,b)$. But $V(f) \subset \mathbb{A}^2_{\bar{k}}$ is biregular to $\mathbb{A}^1_{\bar{k}}$, hence is nonsingular, and we have contradicted the Jacobian criterion. So 1) holds.

3.3.1 <u>Remark</u>: Given x, y, $f = f(x,y)$ as in 3.3, sending x to $\tilde{u} = u(Ct)$, y to $\tilde{v} = v(Ct)$ gives a surjection $\tilde{\alpha}: k[x,y] \to k[t]$ with kernel (f). $\tilde{\alpha}(f_x) = \tilde{v}'$, $\tilde{\alpha}(f_y) = -\tilde{u}'$. In other words, given a line f, we can "normalize" the parametrization α so that $C = 1$ in 3.3.

3.4 Corollary: Let k be any field, $f \in k[x,y]$ a line. Then for general $\lambda \in \bar{k}$, $f - \lambda$ is nonsingular.

Proof: $f \in \bar{k}[x,y]$ is a line. By 0.4 2), $f - \lambda \in \bar{k}[x,y]$ is irreducible for all $\lambda \in \bar{k}$. Suppose $f - \lambda$ is singular at $P = (a,b) \in \bar{k}^2$. Then $f_x(P) = f_y(P) = 0$. So it suffices, by Bezout's theorem, to show that f_x, $f_y \in \bar{k}[x,y]$ are relatively prime. Suppose $h \in \bar{k}[x,y]$ is a nonconstant common factor of f_x, f_y. Then (we have 3.3 with $\bar{k}$ in place of k, and in the notation there) $\alpha(h)$ divides u' and v' in $\bar{k}[t]$. So $\alpha(h) = c \in \bar{k}$. So f divides $h - c \neq 0$, hence $\deg_x h \geq \deg_x f$ and $\deg_y h \geq \deg_y f$. Since $f \in \bar{k}[x,y]$ is irreducible, either f_x or f_y, say f_x, is nonzero. Then $\deg_x f \leq \deg_x h \leq \deg_x f_x \leq \deg_x f - 1$, an absurdity.

We are about to define, given an $f \in R = k^{[2]}$, a derivation D_f of R associated to f. Now in zero characteristic, given a derivation of R one has a canonical way of constructing a corresponding "iterative higher derivation" (i. e., Hasse-Schmidt derivative) - if D is the derivation, $\left(\frac{1}{i!} D^i \right)_{i=0}^{\infty}$ is the corresponding higher derivation. In the papers [20], [13], [19], derivations and higher derivations are used to some advantage in investigations about planes and lines on them. If one looks at some of the proofs in, e. g., [20], however, one finds that they break down utterly in positive characteristic. Moreover, it is not even clear that, having defined the derivation D_f, one can construct a Hasse-Schmidt derivative whose first order term is D_f. Nonetheless, one can get some results in positive characteristic by studying derivations, and this we proceed to do.

<u>Definition</u>: Let $f \in R = k[x,y]$. For $h \in R$, $J(f,h) = J_{x,y}(f,h) = f_x h_y - f_y h_x$ is the <u>Jacobian</u> of f and h. We define $D_f: R \to R$ by $D_f(h) = J(f,h)$. Clearly $D_f \in \mathrm{Der}_k(R,R)$. ($D_f = f_x\, \partial/\partial y - f_y\, \partial/\partial x$.)

<u>Remark</u>: Suppose $R = k[x',y']$, and define $D'_f: R \to R$ by $D'_f(h) = f_{x'}h_{y'} - f_{y'}h_{x'}$. One sees easily that $D'_f = cD_f$, where $c = J_{x,y}(x',y') \in k^*$. So D_f does depend on the choice of variables for R, but not heavily. Every statement we prove about D_f will hold for cD_f, $c \in k^*$.

<u>Notation</u>: Let char $k = p$, $R = k[x,y] \simeq k^{[2]}$. For any positive integer n, we define $R^{(p^n)}$ to be $k[x^{p^n}, y^{p^n}] = k[\{r^{p^n} \mid r \in R\}]$. $R^{(p)}$ is the set of p^{th} powers of elements of $R \Leftrightarrow k$ is perfect.

Now let $f \in R$, and let D_f be the corresponding derivation. We denote also by D_f the extension of D_f to a derivation on qtR. Let $A_f = \{h \in R \mid D_f(h) = 0\}$ be the ring of D_f-constants, $K_f = \{h \in \mathrm{qt}R \mid D_f(h) = 0\}$ the field of D_f-constants. We have the diagram

$$(*)\qquad \begin{array}{ccccccc} L = \mathrm{qt}R^{(p)} & \subset & L(f) & \subset & K_f & \subset & \mathrm{qt}R \\ \cup & & \cup & & \cup & & \cup \\ R^{(p)} & \subset & R^{(p)}[f] & \subset & A_f & \subset & R \end{array}$$

3.5 A_f is normal, for any $f \in R$.

One of the ways to see this is to note that $A_f = R \cap qtA_f$. (If $D \in Der(B,B)$, B a normal domain, then the ring B' of D-constants is normal, since $B' = B \cap qtB'$.)

Now suppose $f \notin R^{(p)}$. Then $L \subsetneqq L(f) \subset K_f \subsetneqq qtR$, so

3.6 $K_f = L(f) = k(x^p, y^p, f)$ and $[qtR:K_f] = p$.

3.7 <u>Lemma</u>: Suppose $f \in R$ is nonconstant, and $\Lambda(f)$ has no movable singularity. Then the ring of D_f-constants $A_f = R^{(p)}[f] = k[x^p, y^p, f]$.

<u>Proof</u>: Since $f \notin R^{(p)}$, A_f and $R^{(p)}[f]$ are birational by 3.6 and $(*)$. Let $f = f(x,y) = \Sigma f_{ij}x^iy^j$. We have $R^{(p)}[f] = k[x^p, y^p, f] \cong k[X,Y,Z]/(Z^p - f^{(p)}(X,Y))$, where $f^{(p)}(X,Y) = \Sigma f_{ij}{}^pX^iY^j$. Now if the irreducible hypersurface $Z^p = f^{(p)}(X,Y)$ in $\mathbb{A}_{\bar{k}}^3$ has a singularity at $(a,b,c) \in \bar{k}^3$, with $c = f(a^{1/p}, b^{1/p})$, then $(f^{(p)})_x(a,b) = (f^{(p)})_y(a,b) = 0$. But $(f^{(p)})_x = (f_x)^{(p)}$, etc., so the condition on a,b is that $(a^{1/p}, b^{1/p})$ be a point common to f_x and f_y. Since there are only finitely many such, $R^{(p)}[f]$ has isolated singularities. By [29], Proposition 9, p. III-13, $R^{(p)}[f]$ is normal. (It is Cohen-Macaulay, being a hypersurface.)

Since A_f is integral over $R^{(p)}[f]$, 3.7 follows.

3.7.1 Remark: Under the hypotheses of 3.7, the proof shows that $R^{(p^n)}[f]$ is normal for all $n > 0$.

Now suppose $f \in R = k[x,y]$ is a line, and let $D = D_f$. Pick a surjection α: $R \to k[t]$ with kernel fR, such that α is normalized, in the sense of 3.3.1. Then

3.8
$$\begin{array}{ccc} R & \xrightarrow{\alpha} & k[t] \\ D \downarrow & & \downarrow d/dt \\ R & \xrightarrow[\alpha]{} & k[t] \end{array}$$

commutes. 3.8 seems to be the key to studying the derivation D, relating it as it does to the very manageable d/dt. As one illustration of its usefulness, we prove the following

3.9 Proposition: If $f \in R$ is a line, then $R^{(p)}[f]$ is factorial.

Proof: All the conditions of [27], Theorem 2.1(a), p. 62, are met: If $\bar{j}$ is the map from the divisor class group of A_f (which equals $R^{(p)}[f]$ by 3.4, 3.7) to that of R, and

$$\mathcal{L} = \{ \text{ logarithmic derivatives } \frac{D(q)}{q} \mid \frac{D(q)}{q} \in R,\ 0 \neq q \in qtR \},$$

then we have a canonical injection of ker $\bar{j}$ into $\mathcal{L}$. Let $q = a/b$, $a, b \in R$ relatively prime, and suppose $D(q)/q = (bD(a) - aD(b))/ab = h \in R$. Then a divides D(a) and b divides

D(b) in R. Suppose $a = f^m a'$, $m \geq 0$, $f \nmid a'$. Then $D(a) = f^m D(a')$, so a' divides D(a') in R. By 3.8, $\alpha(a')$ divides $\alpha(D(a')) = d/dt\ (\alpha(a'))$. So $\alpha(D(a')) = d/dt\ (\alpha(a')) = 0$, and f divides D(a'). Since f,a' are relatively prime, fa' divides D(a'). But $D(a') = f_x a'_y - f_y a'_x$, if nonzero, has degree $\leq$ deg f + deg a' − 2. Since fa' divides D(a'), we have D(a) = D(a') = 0. Similarly D(b) = 0. Hence $\mathcal{L} = \{0\}$. Since the divisor class group of R is {0}, we have 3.9.

It seems certain that, given a line $f \in R$, the ring $R^{(p)}[f]$ is a very important object of study. (Note that $R^{(p)}[f]$ is just the <u>Frobenius</u> of the k[f]-algebra R; see e. g. [21], p. 529.) We next formulate a statement involving this subring, due to Peter Russell.

3.10 Suppose k is any field of characteristic $p > 0$, and $f \in R \simeq k^{[2]}$ is a line. Then $R^{(p)}[f]$ is a plane.

We do not know if this is true. However, something apparently stronger is true of all known lines f: f is a coordinate line plus a p^{th} power in R. (One sees easily that then $R^{(p)}[f]$ is a plane.) Also, 3.9 is a step in the right direction if one tries to prove 3.10. If 3.10 is true, it should be of considerable assistance in finding a recipe for all lines in the plane. In this connection we ask another question:

If $A \subset R \simeq k^{[2]}$ is a plane and R is purely inseparable of degree p over A, is R just the result of adjoining to A the p^{th} root of a variable in A?

3.11 Proposition: 3.10 implies 3.1.

Before proving 3.11, we note first that by 1.2 1) we have

3.12 If $f \in k^{[2]}$ has one rational place at infinity, then for some $N \geq 0$, $f - t \in (k(t))^{p^{-N}}\ {}^{[2]}$ has one rational place at infinity.

We point out the following more intrinsic version of 3.12, which is an immediate consequence of the isomorphism (*) mentioned just before 0.6:

3.12.1 Suppose $f \in k[x,y]$ has one rational place at infinity. Let $K = k(f)$. Then for some $N \geq 0$, the $K^{p^{-N}}$- curve $K^{p^{-N}}[x,y]$ has one rational place at infinity.

We note that for any $n \geq 0$, the map $K^{p^{-n}}[x,y] \to K[x^{p^n}, y^{p^n}]$ ($z \rightsquigarrow z^{p^n}$) is an isomorphism, and restricts to an isomorphism of the base fields. Thus

3.12.2 Under the conditions of 3.12.1, the k(f)-curve $k(f)[x^{p^N}, y^{p^N}]$ has one rational place at infinity, for some $N \geq 0$.

3.13 Lemma: 1) Suppose $f \in R \simeq k^{[2]}$ is a line. Then for any $n \geq 0$, f is a line in $R^{(p^n)}[f]$. (I. e., $R^{(p^n)}[f]/fR^{(p^n)}[f] \simeq k^{[1]}$.)

2) Conversely, suppose $f \in R \simeq k^{[2]}$ is irreducible in $k^{p^{-n}}{}^{[2]}$, and f is a line in $R^{(p^n)}[f]$. Then $f \in k^{p^{-n}}{}^{[2]}$ is a line.

Proof: Suppose $R \to k[t]$ is a surjection with kernel fR. Then one easily checks that its restriction to $R^{(p^n)}[f]$ has image $k[t^{p^n}]$ and kernel $fR^{(p^n)}[f]$. Conversely, given a surjection $R^{(p^n)}[f] \to k[t^{p^n}]$ which carries x^{p^n} to $u(t^{p^n})$, y^{p^n} to $v(t^{p^n})$, and has kernel $fR^{(p^n)}[f]$, one sees easily that $k^{p^{-n}}[x,y] \to k^{p^{-n}}[t]$, $x \rightsquigarrow u^{(p^{-n})}(t)$, $y \rightsquigarrow v^{(p^{-n})}(t)$, is a surjection with kernel generated by f (under the assumption that f is irreducible in $k^{p^{-n}}[x,y]$).

Proof of 3.11: Suppose $f \in R_0 = k[x,y]$ is a line. Letting N be as in 3.12.2, we have, by N applications each of 3.10 and 3.13 1) (with $n = 1$), that $f \in R_N = R^{(p^N)}[f] = k[f, x^{p^N}, y^{p^N}] = k[x_N, y_N] \simeq k^{[2]}$ is a line, and the $k(f)$-curve $k(f)[x_N, y_N]$ has one rational place at infinity. By the argument in the proof of 1.17, it follows that $f \in R_N$ is a coordinate line. So for all $\lambda \in k$, $f - \lambda \in R_N$ is a line. By 0.4 2),

$f - \lambda \in k[x,y]$ is irreducible over $k^{p^{-N}}$. By 3.13 2) (and since $R_N = R^{(p^N)}[f - \lambda])$, $f - \lambda \in k^{p^{-N}}[x,y]$ is a line for all $\lambda \in k$. In particular, if k is algebraically closed, we have 3.1.

In closing this section, we prove the

3.14 Proposition: Suppose $f \in k[x,y] \approx k^{[2]}$ is a line, and assume for simplicity that k is algebraically closed. Then

the following three statements are equivalent:

1) $f - \lambda \in k[x,y]$ is a line for all $\lambda \in k$.
2) $f - \lambda \in k[x,y]$ is a line for infinitely many $\lambda \in k$.
3) For all $n >> 0$, $k[f,x^{p^n},y^{p^n}]$ is a plane.

Moreover,

4) f is then also a variable in $k[f,x^{p^n},y^{p^n}]$ for all $n >> 0$.

Proof: 2) $\Rightarrow$ 3) and 4): By 1.2 1) and 3), and a by now familiar argument involving genus, $f - t \in k(t^{p^{-n}})[x,y]$ is a line for all $n >> 0$. I. e., the k(f)-curve $k(f)[x^{p^n},y^{p^n}]$ is a line. We refer now to Theorem 2.3.1 of [25], and replace the S,k,K,A there by our $k[f] - \{0\}$, $k[f]$, $k[f]$, $k[f,x^{p^n},y^{p^n}]$ respectively. We have just verified condition (i) of the theorem. Of the other (six or so) conditions, we check only the one requiring that $f - \lambda$ generate a prime ideal in $k[f,x^{p^n},y^{p^n}]$, for all $\lambda \in k$: By 0.4 2), $f - \lambda$ is irreducible in $k[x,y]$. But if $h \in k[x,y]$ is irreducible and $B = k[h,x^{p^n},y^{p^n}]$, then $hB \subset B$ is prime. For, $hB \subset B \cap hk[x,y]$, and one checks the reverse inclusion by direct computation.

Since all the conditions of the cited theorem are satisfied, we conclude that $k[f,x^{p^n},y^{p^n}] = k[f]^{[1]}$, and 3) and 4) follow.

3) $\Rightarrow$ 1): Take N big enough so that $k[f,x^{p^N},y^{p^N}]$ is a plane and the $k(f)$-curve $k(f)[x^{p^N},y^{p^N}]$ has a rational place at infinity. By 3.13 1), f is a variable in $k[f,x^{p^N},y^{p^N}]$, and we proceed as in the proof of 3.11.

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