ABSTRACT

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A Spectral Model of Bubble Convection

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The motion of a warm bubble in neutral surroundings is studied using the anelastic equations for shallow convection. The equations are developed in cylindrical coordinates and the spectral technique of integration is used throughout. Two models are constructed - an axisymmetric model and a quasi-three-dimensional model.

The non-dissipative form of the axisymmetric model is used to study the rapid cascade of potential temperature variance in the formation of the bubble "cap". With the addition of a dissipative mechanism the results of this model are compared with those obtained from similarity theory as well as from other numerical models.

The quasi-three-dimensional model is used to study the growth of small non-axisymmetric perturbations on an initially axisymmetric thermal.

A SPECTRAL MODEL OF BUBBLE CONVECTION

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by

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The motion of a warm bubble in neutral surroundings is studied using the anelastic equations for shallow convection. The equations are developed in cylindrical coordinates and the spectral technique of integration is used throughout. Two models are constructed - an axisymmetric model and a quasi-three-dimensional model.

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The quasi-three-dimensional model is used to study the growth of small non-axisymmetric perturbations on an initially axisymmetric thermal.

RESUME

Le déplacement d'une ampoule chaude dans une atmosphère neutre est étudié au moyen d'équations non-élastiques pour la convection peu profonde. Les équations sont développées dans les coordonnées cylindriques et la méthode spectrale est utilisée partout. Deux modèles sont construits - un modèle de la symétrie axiale et un modèle quasi-tridimensionnel.

La forme non-dissipative du modèle de la symétrie axiale est utilisée pour étudier la cascade rapide de la variance de la température potentielle dans la formation du " capuchon " de l'ampoule. Après l'addition d'un mécanisme dissipatif, les résultats de ce modèle sont comparés avec ceux qui sont obtenus de la théorie de la similitude en plus de ceux des autres modèles numériques.

Le modèle quasi-tridimensionnel est utilisé pour étudier la croissance des perturbations non-symétriques sur un thermique qui a initialement la symétrie axiale.

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I INTRODUCTION

Convective phenomena occur on all scales and at all intensities, from the rising plume of a cigarette to the most destructive hailstorm. The collective effect of many separate convective systems is significant in hurricanes and even in the general circulation.

Due to its complexity, the study of convection has been rather sporadic until recent times. Most convective phenomena are highly non-linear and only fairly simple results can be obtained from linear theory. Lately, however, mathematical and numerical models have been developed for convective phenomena of all complexities, from relatively simple Bénard convection, to bubble and plume convection in the atmosphere and even for the thunderstorm itself.

Historically, the most important triumph in the early study of convection was Rayleigh's derivation of a criterion to determine the onset of Bénard convection. Bénard's experiment (1900) consisted of a thin, horizontal fluid layer heated from below by a metal plate maintained at a constant temperature. Bénard found that when the temperature difference between the warm plate and the cooler upper surface became great enough, convection cells developed throughout the fluid. The shape of the cells depended somewhat on the lateral boundaries but at distances

far removed from these boundaries, hexagonal cells eventually predominated.

The equations appropriate for the study of Benard convection are the incompressible Boussinesq equations. It can be shown by dimensional analysis that this type of convection depends only on the Rayleigh number and the Prandtl number. The Rayleigh number is a function of the layer thickness, the temperature difference, the thermal diffusivity and kinematic viscosity. Rayleigh (1916) was able to find the critical value of the Rayleigh number for the onset of cellular convection. His linear study was extended by Jeffries (1926, 28) and Pellew and Southwell (1940) to include more realistic boundary conditions.

More recent investigators Malkus (1954), Veronis (1959), Kuo (1961), Herring (1963, 64), Deardorff (1965) and Segel (1965) have attacked various problems concerning the cellular structure of Benard convection at Rayleigh numbers higher than the critical value. Theoretical work in this Rayleigh number regime is very difficult because of the non-linear nature of the problem, but many important results have been obtained in the past ten years.

Attempts have been made to apply some of the results of Rayleigh convection theory to the atmosphere; but with little success. There are cellular cloud patterns in the atmosphere as can be seen from satellite pictures, but the ratio of cell width to depth is much greater than the values characteristic of small amplitude Benard convection. The atmospheric cells are obviously very complicated and it appears necessary to take into account both cloud physics and the radiative properties of clouds in order to explain them.

In recent years observations of all scales and intensities of atmospheric convection phenomena have become available. These phenomena are all much more complicated than the convection cells in parallel plate convection, but the largest-scale convective systems, thunderstorms, are so complex as to almost defy any analysis other than empirical. The reason for the complexity is the interdependence of cloud physical, dynamic

and thermodynamic processes. The atmospheric convection problem can be made much more tractable by the neglect of one or more of these effects. In particular, if cloud physics is neglected, a whole subset of dry convection phenomena can be considered. These phenomena, which are mostly smaller scale and confined to the atmospheric boundary layer, include buoyant plumes, oil flares, thermals and even small cumulus clouds.

In the absence of cloud physical processes, atmospheric convection still differs substantially from classical Benard convection because the atmosphere is fully turbulent and the Rayleigh number is many orders of magnitude greater than the critical value. However, there are methods of analysis used in the study of fully turbulent flow which can be applied to the study of dry atmospheric convection. In addition, this type of convection can be modelled numerically and studied experimentally in the laboratory.

During the early 1950's many observations of convection in the lower troposphere had been obtained from glider pilots. In England, two schools of thought sprang up to explain these observations.

The plume school postulated a maintained, point heat source at the earth's surface which resulted in steady-state plumes or jets. Plume solutions obtained by Rouse, Yih and Humphreys (1952), Priestly (1954), Morton (1959) and Telford (1966) were compared with atmospheric observations with varying success. However, there are two objections to this theory. Firstly, no well-defined source regions can usually be identified in nature. Secondly, the theory neglects any time evolution of the convection and does not allow mixing or entrainment at the top of the cloud.

The second idea was the bubble theory put forward by Batchelor (1954) and Scorer (1953, 57). They asked the question: what happens when a finite quantity of heat is suddenly released at a point in the atmosphere? Clearly the heated air will rise and set up a turbulent motion rather like a ring vortex and cold air will be entrained into the moving bubble. Dimensional analysis, combined with a shape-preserving assumption, can be used to obtain the time evolution of certain gross features

of the rising thermal. For example, the theory gives the expansion angle and the maximum vertical velocity as a function of time.

The bubble theory has its shortcomings - in particular, it is necessary to explain how the finite quantity of heat came to be concentrated at one point initially. The conflict between bubble and plume theory has not been resolved and in fact actual atmospheric convection seems to combine features from both models. However, because bubble convection is fairly easy to examine in the laboratory and can be simulated numerically as an initial value problem, it has become quite widely studied.

In order to confirm bubble convection theory and to evaluate certain numerical constants, Scorer (1957) performed a number of laboratory experiments. He released a small quantity of marked fluid from a hemispherical cup into a tank of lighter ambient fluid. The axisymmetric blob of white fluid very quickly developed marked asymmetries as it accelerated downward. A turbulent ring vortex was set up which caused entrainment of lighter fluid into the thermal resulting in expansion and deceleration. Comparisons with shape-preserving bubble theory were quite successful. Similar laboratory experiments were performed later by Woodward (1959) and Richards (1963) generally confirming Scorer's findings.

The ultimate test of Scorer's bubble theory lies in its ability to predict convective behavior in the lower atmosphere. It was known that towers developing on cumulus cloud tended to behave like buoyant thermals, but true confirmation seemed to require an almost prohibitive amount of instrumentation.

There was another possibility. The techniques of numerical simulation have been applied successfully to many meteorological problems. In principle, a buoyant thermal rising from rest through a stratified or neutral atmosphere could be tackled as an initial value problem. Such a bubble would, of course, be governed by the Navier-Stokes and thermodynamic equations and with the specification of boundary and initial conditions, a solution could be obtained. The Navier-Stokes equations are unnecessarily general, admitting high frequency soundwave solutions as

well as the gravity modes being studied. A so-called "filtered" numerical model, which does not admit soundwave solutions, can be run with a much longer timestep, thus cutting down appreciably on computation time. To this end, the "sound-proof" or anelastic equations for convection were first obtained by Ogura and Phillips in 1961 and have been widely used ever since.

Thermal convection is very much a three-dimensional problem as Scorer's (1957) photographs show. However, the magnitude of the necessary calculations has precluded until recently three-dimensional numerical simulations of this phenomenon. Most experiments have had to be content with two-dimensional models using either radial or slab symmetry. The first such experiment was performed in 1959 by Malkus and Witt with only partial success.

In their model, the convection equations were expressed in finite difference form and integrated over a slab-symmetric domain 640 metres square. Their integration was started with a perturbation in the thermal field with a scale of about 160 metres. As the model was integrated forward in time, many features of Scorer's tank thermal were reproduced. The initial thermal perturbation started to rise as a vortex flow was set up. The bubble began to expand and a "cap" or region of intense thermal gradient appeared at the top of the bubble. Unfortunately the experiment could not be continued until the bubble reached the top of the box because of the appearance of serious non-linear computational instability.

This problem, which has to be faced in every numerical model of bubble convection, is due to the intense thermal gradient set up in the "cap" of the bubble. Necessarily, spatial derivatives of potential temperature become very large in the "cap" and there is a cascade of potential temperature variance to smaller and smaller scales. Ultimately, the potential temperature variance is dissipated by molecular diffusion. However, numerical models are not able to model this cascade properly because they have a minimum resolvable scale. Serious aliasing errors quickly develop which can result in the non-linear computational instability observed by Malkus and Witt.

Later investigators in this field have resorted to various numerical techniques to get around this problem. Usually the cascade to scales smaller than the minimum resolvable scale is modelled by some form of numerical dissipation. Ogura (1962) and Arnason et al (1968) used numerical procedures which were inherently diffusive, while Lilly (1962) used an explicit non-linear diffusion. Both these remedies are not without realism and have led to reasonable agreement with laboratory experiment. In particular, the integrations of Ogura and Lilly both approached the shape-preserving thermals of Scorer's tank experiments.

Another technique for preventing non-linear computational instability was used by Lilly (1964). He used the quasi-conservative Arakawa Jacobian for the advective terms, coupled with a Fickian dissipation. This method keeps the computation stable, but does not prevent aliasing.

The methods of Ogura and Lilly have been widely used in the numerical simulation of more complicated forms of atmospheric convection. Ogura (1963) included latent heat release to investigate the growth of a cumulus cloud in a conditionally unstable atmosphere. Takeda (1966) has formulated numerical thunderstorm models including cloud physical processes.

However, these techniques are not completely satisfactory. It is clear that the central problem in modelling buoyant convection is still the proper formulation of the exchange of energy from the source scales to the turbulent scales where it is dissipated. For this reason, it is important that numerical models use techniques that minimize the fictitious generation of energy, particularly in the smallest scales of motion.

In recent years there have been many workers who have attempted to model certain aspects of large-scale atmospheric flow by using the so-called spectral method rather than the grid-point method, (e.g. Robert (1966), Baer (1964)). This technique completely eliminates aliasing errors and is thus free of non-linear computational instability, but provides as well for the explicit study of the exchange of energy between scales of motion. There is also the possibility of using highly truncated series to

represent the variables and thus obtain insight into the non-linear exchanges. The spectral method does suffer from a serious practical drawback, in that the amount of computation increases with a higher power of the resolution that it does using the grid point method.

It was thought that a spectral model might be able to shed some light on the cascade of energy in two-dimensional buoyant convection. Such a non-dissipating model could be run for some time without serious numerical difficulties. Of course the results would not remain physically realistic indefinitely because of the minimum resolvable scale; but they would be unaliased and the accelerating stage would be reproduced faithfully. With the addition of some form of diffusion, an attempt could be made to simulate the decelerating shape-preserving stage as well.

Lilly (1962) and Ogura (1962) have both suggested that a two-dimensional numerical model, by its very nature, might be incapable of simulating bubble convection correctly. Three-dimensional turbulence behaves quite differently from two-dimensional turbulence. Lilly pointed out that the kinetic energy cascade in two-dimensions is inhibited because of the blocking theorem. On the other hand, the cascade of potential energy is not inhibited and this cascade is the one that gives rise to the numerical difficulties associated with convection models. It is not clear if the same results will be true for convection in three-dimensions, but certainly there is the possibility that the kinetic energy cascade will be much less inhibited when the flow can advect and distort patterns which are not part of itself.

There is another problem associated with three-dimensional bubble convection. In Scorer's experiment, the originally axisymmetric bubble rapidly developed marked asymmetries. It would appear that the axisymmetric solutions to the governing equations of bubble convection can be unstable to small non-axisymmetric perturbations.

It has recently become possible to simulate bubble convection with a three-dimensional numerical model (Fox (1970)). Such models, unfortunately, do not yet have the spatial resolution necessary for the proper

investigation of the three-dimensional energy cascade. However, the second problem, the growth of three-dimensional asymmetries can be investigated using a fairly simple quasi-three-dimensional model. For instance, a small-amplitude non-axisymmetric perturbation could be imposed on a large-amplitude axisymmetric bubble and its rate of growth or decay determined. The "linearized" model based on this approach could not simulate properly a three-dimensional energy cascade, but it would have degrees of freedom not available to two-dimensional simulations.

The purpose of the present study is to investigate two and three-dimensional bubble convection using the spectral technique. Three separate aspects of the problem are investigated.

1. - The two-dimensional energy cascade.
2. - The approach to the shape-preserving two-dimensional thermal.
3. - The growth of three-dimensional asymmetries.

Briefly, the plan of attack is the following. The first two aspects are investigated with an axisymmetric model developed in Chapter 3. The model is run first without dissipation yielding kinetic energy, enstrophy and potential temperature variance spectra. With Fickian dissipation added the model results are compared with Scorer's tank thermals and the numerical simulations of Ogura (1962).

The three-dimensional aspect (3) involves a slightly more complicated approach. The fully three-dimensional equations are developed in cylindrical coordinates and the variables expanded into axisymmetric and non-axisymmetric parts. It is assumed that the axisymmetric bubble is much larger in amplitude than the non-axisymmetric perturbation. Thus, interactions between the axisymmetric bubble and the non-axisymmetric perturbation would only be allowed to affect the perturbation. This implies that the axisymmetric equations would remain unchanged and the only addition would be the set of non-axisymmetric equations. This quasi-three-dimensional model is used to investigate the growth of asymmetries on the "cap" of an axisymmetric bubble.

II THE CONVECTION SYSTEM

1. The Anelastic Equation

The starting point for the development of an atmospheric convection model is the set of equations known as the anelastic or "sound-proof" equations. Developed by Ogura and Phillips (1962) from the Navier-Stokes and thermodynamics equations, the anelastic set has widespread application in the study of thunderstorms, heat-island circulations and sea-breezes. Basically, the technique of Ogura and Phillips involves the expansion of the fundamental equations in terms of a small parameter related to the deviation of the potential temperature from a constant basic state. In 1968, Dutton and Fichtl generalized the theory to allow for vertical variation in the basic state.

Rather than re-derive the anelastic equations, we will simply discuss the assumptions made and the restrictions these impose upon the phenomena under investigation.

Assumption 1.

There exists a basic state which is in hydrostatic equilibrium and at rest. The basic state potential temperature θ is at most a function of the vertical coordinate Z .

Assumption 2.

Upon this basic state is imposed a small perturbation in the potential temperature field, Θ such that $\Theta/\Theta \ll 1$. Θ can vary with time and the three spatial coordinates.

Assumption 3.

All modes of the Navier-Stokes equations which have frequencies greater than the Brunt-Vaisala frequency will not be considered. This cut-off frequency, which is the frequency of gravity waves, is given by

$$f_{BV} = \sqrt{\frac{g}{\Theta} \frac{\partial \Theta}{\partial z}}$$

where g is the gravitational constant. This assumption implies the filtering of sound waves, thus allowing numerical integrations to use a much longer timestep.

Assumption 4.

The vertical scale of the motion is much less than the scale height of an adiabatic atmosphere. (≈ 30 Km) This assumption, known as the shallow convection assumption, is valid if the vertical scale of the phenomena is less than about 3 kilometres. In order to study thunderstorms it is necessary to use the less restrictive deep-convection equations. In this study, however, only the growth of small convective elements is considered and thus the shallow convection assumption is valid.

When these four assumptions are applied to the Navier-Stokes, thermodynamic and continuity equations, the following anelastic set results

$$\frac{dV}{dt} = -c_p \nabla \Theta \pi + \frac{g}{\Theta} \vec{\Theta} \cdot \vec{K} + F_v$$

$$\frac{d\Theta}{dt} = -w \frac{\partial \Theta}{\partial z} + D_\Theta \quad (2.1)$$

$$\nabla \cdot V = 0$$

where $\mathcal{W}$ represents the velocity vector with vertical component W .

Θ is the deviation of the potential temperature from the basic state Θ_0 .

$\vec{k}$ is the unit vector in the vertical direction

$\Pi = (P/P_0)^{R/\rho}$ is a non-dimensional perturbation pressure expressed in terms of a reference pressure P_0

F_W represents a frictional force

D_Θ represents some diffusion mechanism for potential temperature.

It should be noted, that though the variables $\mathcal{W}$, Θ , and Π are all essentially deviations from basic state quantities, the traditional primes have been dropped. In form, these equations are very similar to the incompressible Boussinesq equations; that is, the perturbation temperature (Θ) is neglected except when multiplying the gravitational constant and the system is non-divergent. However, the equations are still highly non-linear, all advective terms being retained.

Following Ogura (1962) we develop the equations in cylindrical coordinates. The fundamental domain is given in terms of Z (the vertical coordinate), r (the radial coordinate), and ϕ (the tangential coordinate). The corresponding velocity components are W , U , and V .

Figure 1 - The Fundamental Domain

r varies between 0 and R and z varies between 0 and H

Equations (2.1) in cylindrical coordinates are

$$\begin{aligned}
 \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{v}{r} \frac{\partial u}{\partial \phi} + w \frac{\partial u}{\partial z} &= -c_p \frac{\partial(\Theta \Pi)}{\partial r} + \frac{v^2}{r} + F_u \\
 \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + \frac{v}{r} \frac{\partial v}{\partial \phi} + w \frac{\partial v}{\partial z} &= -\frac{c_p}{r} \frac{\partial(\Theta \Pi)}{\partial \phi} - \frac{uv}{r} + F_v \\
 \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + \frac{v}{r} \frac{\partial w}{\partial \phi} + w \frac{\partial w}{\partial z} &= -c_p \frac{\partial(\Theta \Pi)}{\partial z} + g \frac{\Theta}{\Theta} + F_w \\
 \frac{\partial \Theta}{\partial t} + u \frac{\partial \Theta}{\partial r} + \frac{v}{r} \frac{\partial \Theta}{\partial \phi} + w \frac{\partial \Theta}{\partial z} &= -w \frac{\partial \Theta}{\partial z} + D_\Theta \\
 \frac{1}{r} \frac{\partial(ru)}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \phi} + \frac{\partial w}{\partial z} &= 0
 \end{aligned} \tag{2.2}$$

where F_u , F_v , F_w and D_Θ represent dissipation terms.

Before proceeding to non-dimensionalize the equations, we will make two further assumptions.

$$N \equiv \frac{1}{\Theta} \frac{\partial \Theta}{\partial z} = \text{constant}$$

F_u , F_v , F_w and D_Θ are, for the time being, given the usual Fickian form, so that

$$\begin{aligned}
 F_u &= \nu_Q (\nabla^2 u - \frac{u}{r^2} - \frac{2}{r} \frac{\partial v}{\partial \phi}) \\
 F_v &= \nu_Q (\nabla^2 v - \frac{v}{r^2} + \frac{2}{r} \frac{\partial u}{\partial \phi}) \\
 F_w &= \nu_Q \nabla^2 w \\
 D_\Theta &= \nu_\Theta \nabla^2 \Theta
 \end{aligned} \tag{2.3}$$

where ν_Q and ν_Θ are suitably chosen viscosity and heat diffusion coefficients and ∇^2 is the three-dimensional Laplacian in cylindrical coordinates defined by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

It is convenient to non-dimensionalize the equations in the following way. Let the non-dimensional independent variables be defined as follows: $z^* = zH^{-1}$, $r^* = rH^{-1}$, $t^* = t\tau^{-1}$, $\phi^* = \phi$. The corresponding dependent non-dimensional variables will be $u^* = u\tau H^{-1}$, $w^* = w\tau H^{-1}$, $v^* = v\tau H^{-1}$, $\theta^* = \theta\theta^{-1}$, $\pi^* = \pi\theta C_p \tau H^{-2}$. Then if we choose $\tau^2 = Hg^{-1}$ and denote R/H by $\mathcal{L}$ equations (2.2) take the following non-dimensional form.

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{v}{r} \frac{\partial u}{\partial \phi} + w \frac{\partial u}{\partial z} = - \frac{\partial \pi}{\partial r} + \frac{v^2}{r} + \gamma_Q (\nabla^2 u - \frac{u}{r^2} - \frac{2}{r} \frac{\partial v}{\partial \phi}) \quad (2.4)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + \frac{v}{r} \frac{\partial v}{\partial \phi} + w \frac{\partial v}{\partial z} = - \frac{1}{r} \frac{\partial \pi}{\partial \phi} - \frac{uv}{r} + \gamma_Q (\nabla^2 v - \frac{v}{r^2} + \frac{2}{r} \frac{\partial u}{\partial \phi}) \quad (2.5)$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + \frac{v}{r} \frac{\partial w}{\partial \phi} + w \frac{\partial w}{\partial z} = - \frac{\partial \pi}{\partial z} + \theta + \gamma_Q \nabla^2 w \quad (2.6)$$

$$\frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial r} + \frac{v}{r} \frac{\partial \theta}{\partial \phi} + w \frac{\partial \theta}{\partial z} = - Nw + \gamma_\theta \nabla^2 \theta \quad (2.7)$$

$$\frac{1}{r} \frac{\partial(ru)}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \phi} + \frac{\partial w}{\partial z} = 0 \quad (2.8)$$

where we have used the usual technique of dropping the asterisks once the non-dimensional variables have been defined. We also note that γ_Q , γ_θ and N have also been appropriately non-dimensionalized. The extent of the fundamental domain is now defined as

$$0 \leq z \leq 1$$

$$0 \leq r \leq \mathcal{L}$$

$$0 \leq \phi \leq 2\pi$$

2. The Second Perturbation Expansion

The non-dimensionalized anelastic equations in cylindrical coordinates (2.4 to 2.8) were obtained from the Navier-Stokes and thermodynamic equations by an expansion technique. In accord with the plan of

attack discussed in Chapter I we will now go a step further and make a second perturbation expansion. The 5 dependent variables u , v , w , θ , and π will each be split up into two parts - a tangentially averaged part and a ϕ -dependent part. Thus

$$\begin{aligned} u(r, z, \phi, t) &= \hat{u}(r, z, t) + u'(r, z, \phi, t) \\ v(r, z, \phi, t) &= \hat{v}(r, z, t) + v'(r, z, \phi, t) \\ w(r, z, \phi, t) &= \hat{w}(r, z, t) + w'(r, z, \phi, t) \\ \theta(r, z, \phi, t) &= \hat{\theta}(r, z, t) + \theta'(r, z, \phi, t) \\ \pi(r, z, \phi, t) &= \hat{\pi}(r, z, t) + \pi'(r, z, \phi, t) \end{aligned} \quad (2.9)$$

The "hat" and "prime" operators are formally defined as follows

$$(\hat{}) = \frac{1}{2\pi} \int_0^{2\pi} () d\phi \quad \text{and} \quad \frac{1}{2\pi} \int_0^{2\pi} ()' d\phi = 0$$

From now on the term "averaged" equation or "averaged" variable will refer to the average over ϕ unless otherwise specified. Similarly, the term "perturbation" equation or "perturbation" variable will refer to the deviation from the tangential average.

When we substitute the expressions (2.9) into equations (2.4 to 2.8) and average over ϕ we obtain the "average" equations. We then subtract these "average" equations from (2.4 to 2.8) to obtain the "perturbation" equations. This process will be illustrated for the case of the thermodynamic equation. Thus equation (2.7) becomes

$$\frac{\partial(\hat{\theta} + \theta')}{\partial t} + (\hat{v} + v') \cdot \nabla(\hat{\theta} + \theta') = -N(\hat{w} + w') + D_{\hat{\theta}} + D_{\theta'} \quad (2.10)$$

where the operators $\hat{v} \cdot \nabla$ and $v' \cdot \nabla$ are defined as follows

$$\hat{V} \cdot \nabla F = \hat{U} \frac{\partial F}{\partial r} + \hat{W} \frac{\partial F}{\partial z}$$

$$\hat{V}' \cdot \nabla F = u' \frac{\partial F}{\partial r} + \frac{v'}{r} \frac{\partial F}{\partial \phi} + w' \frac{\partial F}{\partial z}$$

and the dissipative terms $D\hat{\theta}$ and $D\theta'$ are Fickian as in (2.3).

Upon averaging over ϕ and subtracting, equation (2.10) becomes

$$\frac{\partial \hat{\theta}}{\partial t} + \hat{V} \cdot \nabla \hat{\theta} + \widehat{V' \cdot \nabla \theta'} = -N\hat{W} + D\hat{\theta} \quad (2.11)$$

$$\frac{\partial \theta'}{\partial t} + \hat{V} \cdot \nabla \theta' + \hat{V}' \cdot \nabla \hat{\theta} + \boxed{\hat{V}' \cdot \nabla \theta' - \widehat{V' \cdot \nabla \theta'}} = -Nw' + D\theta' \quad (2.12)$$

The terms in the dotted rectangle are the so-called fluctuating self-interactions (see Herring 1963), or in a spectral sense the wave-wave interactions amongst the components in the ϕ direction. We will make the assumption that the amplitude of the ϕ -dependent "perturbation" is small with respect to the amplitude of the "average" state, in which case these fluctuating self-interactions are of higher order and will consequently be dropped.

If we perform the same operations on the other equations in the set (2.4 to 2.8) we obtain the non-self-interacting set.

$$\frac{\partial \hat{u}}{\partial t} + \hat{V} \cdot \nabla \hat{u} + \widehat{V' \cdot \nabla u} = -\frac{\partial \hat{\Pi}}{\partial r} + \frac{\hat{u}^2}{r} + \frac{\hat{u}^2}{r} + F\hat{u} \quad (2.13)$$

$$\frac{\partial \hat{v}}{\partial t} + \hat{V} \cdot \nabla \hat{v} + \widehat{V' \cdot \nabla v} = -\frac{\hat{u}\hat{v}}{r} - \frac{\hat{u}'v'}{r} + F\hat{v} \quad (2.14)$$

$$\frac{\partial \hat{w}}{\partial t} + \hat{V} \cdot \nabla \hat{w} + \widehat{V' \cdot \nabla w} = -\frac{\partial \hat{\Pi}}{\partial z} + \hat{\theta} + F\hat{w} \quad (2.15)$$

$$\frac{\partial \hat{\theta}}{\partial t} + \hat{V} \cdot \nabla \hat{\theta} + \widehat{V' \cdot \nabla \theta'} = -N\hat{W} + D\hat{\theta} \quad (2.16)$$

$$\frac{1}{r} \frac{\partial(r\hat{u})}{\partial r} + \frac{\partial\hat{w}}{\partial z} = 0 \quad (2.17)$$

$$\frac{\partial u'}{\partial t} + \hat{v} \cdot \nabla u' + \hat{v}' \cdot \nabla \hat{u} = - \frac{\partial \pi'}{\partial r} + \frac{\partial \hat{v} v'}{r} + F_u' \quad (2.18)$$

$$\frac{\partial v'}{\partial t} + \hat{v} \cdot \nabla v' + \hat{v}' \cdot \nabla \hat{v} = - \frac{1}{r} \frac{\partial \pi'}{\partial \phi} - \frac{\hat{u} v'}{r} - \frac{u' \hat{v}}{r} + F_v' \quad (2.19)$$

$$\frac{\partial w'}{\partial t} + \hat{v} \cdot \nabla w' + \hat{v}' \cdot \nabla \hat{w} = - \frac{\partial \pi}{\partial z} + \Theta' + F_w' \quad (2.20)$$

$$\frac{\partial \theta'}{\partial t} + \hat{v} \cdot \nabla \theta' + \hat{v}' \cdot \nabla \hat{\theta} = - N_w' + D_\theta' \quad (2.21)$$

$$\frac{1}{r} \frac{\partial(r u')}{\partial r} + \frac{1}{r} \frac{\partial v'}{\partial \phi} + \frac{\partial w'}{\partial z} = 0 \quad (2.22)$$

where $F_{\hat{u}}$, $F_{\hat{v}}$, $F_{\hat{w}}$, $D_{\hat{\theta}}$ etc. are defined as Fickian dissipation terms as in (2.3).

Equations (2.13 to 2.22) we shall herein refer to as the "convection system" and they will serve as the governing equations for the remainder of this study. We will consider two subsets of the "convection system" - the axisymmetric equations in Chapter 3 and the "linearized" three-dimensional set in Chapter 5.

The development of the "convection system" would be incomplete without a discussion of appropriate boundary conditions and energetics. However, differences between the two subsets of the "convection system" preclude this until chapters 3 and 5.

III DEVELOPMENT OF THE AXISYMMETRIC SPECTRAL MODEL

Bubble convection in two dimensions has been modelled by several authors, notably Malkus and Witt (1959), Lilly (1962) and Ogura (1962). As noted in Chapter 1, all these simulations were plagued with aliasing problems and some became computationally unstable. Despite the somewhat artificial means used to counteract these problems the numerical solutions showed some quite promising atmospheric realism.

An axisymmetric spectral model, with its non-aliasing properties seemed an ideal way to investigate the potential temperature cascade involved in formation of the bubble cap. Because Ogura's model was also developed in cylindrical coordinates we shall compare our results directly with his. In addition, the output of this model will be compared with the laboratory experiments and similarity theory of Scorer.

1. The Axisymmetric Set

The axisymmetric equations are a subset of the " convection system " - equations (2.13 to 2.22). If we set all prime terms identically to zero we see that the $\hat{v}$ equation (2.14) becomes decoupled from the other four equations. That is, if $\hat{v}$ is initially everywhere zero then it will remain so. Consequently we will drop this equation from the axisymmetric set.

The fundamental domain is the meridional (r, z) plane defined between $z = 0$ and 1 and $r = 0$ and α .

Figure 2. The meridional plane

The "convection system" reduces to the following axisymmetric equations on this domain

$$\frac{\partial \hat{u}}{\partial t} + \hat{u} \frac{\partial \hat{u}}{\partial r} + \hat{w} \frac{\partial \hat{u}}{\partial z} = - \frac{\partial \hat{\pi}}{\partial r} + \nu_a (\nabla^2 \hat{u} - \frac{\hat{u}}{r^2}) \quad (3.1)$$

$$\frac{\partial \hat{w}}{\partial t} + \hat{u} \frac{\partial \hat{w}}{\partial r} + \hat{w} \frac{\partial \hat{w}}{\partial z} = - \frac{\partial \hat{\pi}}{\partial z} + \hat{\theta} + \nu_a \nabla^2 \hat{w} \quad (3.2)$$

$$\frac{\partial \hat{\theta}}{\partial t} + \hat{u} \frac{\partial \hat{\theta}}{\partial r} + \hat{w} \frac{\partial \hat{\theta}}{\partial z} = - \hat{w} N + \nu_\theta \nabla^2 \hat{\theta} \quad (3.3)$$

$$\frac{1}{r} \frac{\partial (r \hat{u})}{\partial r} + \frac{\partial \hat{w}}{\partial z} = 0 \quad (3.4)$$

where $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$ is the two-dimensional Laplacian operator in the (r, z) plane.

In view of equation (3.4) we may define a stream function $\hat{\psi}$ such that

$$\hat{u} = - \frac{1}{r} \frac{\partial \hat{\psi}}{\partial z} \quad \hat{w} = \frac{1}{r} \frac{\partial \hat{\psi}}{\partial r} \quad (3.5)$$

This in turn permits the elimination of the pressure term by cross-differentiation to obtain a meridional vorticity equation

$$\frac{\partial \hat{\eta}}{\partial t} = - \frac{1}{r} \mathcal{J}(\hat{\psi}, \hat{\eta}) - \frac{\hat{\eta}}{r^2} \frac{\partial \hat{\psi}}{\partial z} + \frac{\partial \hat{\theta}}{\partial r} + F \hat{\eta} \quad (3.6)$$

where
$$\hat{\eta} = \frac{\partial \hat{w}}{\partial r} - \frac{\partial \hat{u}}{\partial z}$$

which we can re-write as

$$\frac{\partial \hat{Q}}{\partial t} = -\frac{1}{r} J(\hat{\psi}, \hat{Q}) + \frac{1}{r} \frac{\partial \hat{\theta}}{\partial r} + \nu_{\hat{Q}} \Delta^2 \hat{Q} \quad (3.7)$$

where $\Delta^2 \hat{Q} = \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{2}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right] \hat{Q}$

and $\hat{Q} = \frac{\hat{h}}{r} = \frac{1}{r^2} \left(\frac{\partial^2 \hat{\psi}}{\partial r^2} - \frac{1}{r} \frac{\partial \hat{\psi}}{\partial r} + \frac{\partial^2 \hat{\psi}}{\partial z^2} \right)$

and J denotes the usual Jacobian operator. In this way equation (3.3) may be re-written as

$$\frac{\partial \hat{\theta}}{\partial t} = -\frac{1}{r} J(\hat{\psi}, \hat{\theta}) - \frac{N}{r} \frac{\partial \hat{\psi}}{\partial r} + \nu_{\hat{\theta}} \nabla^2 \hat{\theta} \quad (3.8)$$

The boundary conditions we impose are the same as Ogura (1962), that is, $\hat{\psi} = 0$ at $z=0, 1$ and $r=0, \alpha$; and $\hat{h} = \hat{Q}r$ also vanish at $z=0, 1$ and at $r=0, \alpha$. These boundary conditions imply that the normal velocities and the normal derivatives of the tangential velocities are zero everywhere on the boundaries.

In summary, the axisymmetric equations and boundary conditions are as follows

$$\frac{\partial \hat{Q}}{\partial t} = -\frac{1}{r} J(\hat{\psi}, \hat{Q}) + \frac{1}{r} \frac{\partial \hat{\theta}}{\partial r} + \nu_{\hat{Q}} \Delta^2 \hat{Q} \quad (3.9)$$

$$\frac{\partial \hat{\theta}}{\partial t} = -\frac{1}{r} J(\hat{\psi}, \hat{\theta}) - \frac{N}{r} \frac{\partial \hat{\psi}}{\partial r} + \nu_{\hat{\theta}} \nabla^2 \hat{\theta} \quad (3.10)$$

$$\hat{Q} = \frac{1}{r^2} \left(\frac{\partial^2 \hat{\psi}}{\partial r^2} - \frac{1}{r} \frac{\partial \hat{\psi}}{\partial r} + \frac{\partial^2 \hat{\psi}}{\partial z^2} \right) \quad (3.11)$$

$$\hat{\psi} = 0 \quad \text{at} \quad r=0, \alpha; \quad \text{and} \quad z=0, 1 \quad (3.12)$$

$$\hat{Q}r = 0 \quad \text{at} \quad r=0, \alpha; \quad \text{and} \quad z=0, 1 \quad (3.13)$$

$$\frac{\partial \hat{\theta}}{\partial r} = 0 \quad \text{at} \quad r=0, \alpha \quad (3.14)$$

$$\hat{\theta} = 0 \quad \text{at} \quad z=0, 1 \quad (3.15)$$

2. The Spectral Transformations

While it is not necessary to use orthogonal functions to represent the dependent variables, nor functions which satisfy the boundary conditions, it makes life much easier if we do. Further, such functions are convenient if we wish to consider simplified versions of the equations. As in most applications of the spectral method, the suitable functions may be determined through the solution of an eigenvalue problem. In this case, consider the eigenvalue problem,

$$D^2 \hat{\psi} \equiv \left(\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right) \hat{\psi} = -\lambda^2 \hat{\psi} \quad (3.16)$$

subject to the boundary conditions that $\hat{\psi} = 0$ at $r = 0, 1$ and $z = 0, 1$. The eigensolutions of this problem are of the form $\hat{\psi} = r J_1(\hat{k}r) \sin n\pi z$ where J_1 is Bessel's function of the first kind of order one, $\hat{k}$ is a root of the equation $J_1(\hat{k}) = 0$, and n is a positive integer. The eigenvalues λ^2 are then given as

$$\lambda^2 = (\hat{k}^2 + n^2 \pi^2)$$

The functions which are appropriate for the expansion of potential temperature $\hat{\theta}$ are not the same as those for the stream function $\hat{\psi}$, since $\hat{\theta}$ does not satisfy the same boundary conditions. However, we may obtain these functions by noting that the buoyancy term in the vorticity equation is linear, and therefore it would be convenient if that term had the same form as the left hand side of the equation. In other words, we wish to determine a function F such that,

$$\frac{1}{r} \frac{\partial F}{\partial r} = -\frac{1}{r^2} D^2 (r J_1(\hat{k}r) \sin n\pi z)$$

Since $J_0' = -J_1$, then it follows that the functions suitable for the expansions of potential temperature are of the form,

$$\hat{\theta} = J_0(\hat{k}r) \sin n\pi z$$

All that remains to be shown is that the functions satisfy the further kinematic condition on $\hat{Q}r$. Now, a component of $\hat{Q}r$ will have the form,

$$\frac{1}{r} D^2 \hat{\psi} = - \frac{\lambda^2}{r} r J_1(\hat{k}r) \sin n\pi z$$

from which the required result follows.

We now define the following set of spectral functions:

$$F_A \equiv F(i_A, n_A) = N_A r J_1(\hat{k}_A r) \sin n_A \pi z \quad (3.17)$$

$$G_A \equiv G(i_A, n_A) = N_A J_0(\hat{k}_A r) \sin n_A \pi z \quad (3.18)$$

where $\hat{k}_A$ is the i_A th root of $J_1(\hat{k}_A r) = 0$, n_A is a positive integer, and

$$N_A = 2 / (\lambda J_0(\hat{k}_A \lambda))$$

These functions have the following orthonormal properties,

$$\int_0^\lambda \int_0^1 F_A F_B \frac{dr dz}{r} = \delta(i_A, i_B) \cdot \delta(n_A, n_B) \quad (3.19)$$

$$\int_0^\lambda \int_0^1 G_A G_B r dr dz = \delta(i_A, i_B) \cdot \delta(n_A, n_B) \quad (3.20)$$

where $\delta(m, n)$ equals one if m equals n , and equals zero otherwise. Further, we have the following differential relationship.

$$D^2 F_A = -(\hat{k}_A^2 + n_A^2 \pi^2) F_A = -\lambda_A^2 F_A$$

We now expand the variables $\hat{\psi}$ and $\hat{\theta}$ in terms of these functions, so that

$$\hat{\psi} = \sum_A \hat{\psi}_A F_A \quad ; \quad \hat{\theta}_A = \sum_A \hat{\theta}_A G_A$$

where the summation takes place over all possible values of i_A and n_A . If we substitute these expansions in equation (3.9) multiply by $F_A r dr dz$ and integrate over the (r, z) domain we obtain,

$$\begin{aligned} \lambda_A^2 \dot{\hat{\psi}}_A &= \hat{k}_A \hat{\theta}_A - \int_0^1 \int_0^{\alpha} \mathcal{J}(\xi_B \hat{\psi}_B F_B, \frac{1}{F^2} \xi_C \hat{\psi}_C \lambda_C^2 F_C) F_A dr dz \\ &\quad - \nu_{\hat{Q}} \lambda_A^4 \hat{\psi}_A \end{aligned} \quad (3.21)$$

Similarly we may obtain the spectral version of the thermodynamic equation,

$$\begin{aligned} \dot{\hat{\theta}}_A &= - \int_0^1 \int_0^{\alpha} \mathcal{J}(\xi_B \hat{\psi}_B F_B, \xi_C \hat{\theta}_C G_C) G_A dr dz \\ &\quad - \hat{k}_A N \hat{\psi}_A - \nu_{\hat{\theta}} \lambda_A^2 \hat{\theta}_A \end{aligned} \quad (3.22)$$

In the above equations it is to be understood that the summations are in fact double summations over all waves in the radial coordinate and all waves in the vertical coordinate.

The Jacobian terms in equations (3.21) and (3.22) give rise to interaction coefficients which may be conveniently defined as follows,

$$\begin{aligned} S_{BC}^A &\equiv \int_0^1 \cos \eta_A \pi z \sin \eta_B \pi z \sin \eta_C \pi z dz \\ &= -1/4 \quad \text{if} \quad \eta_A = \eta_B + \eta_C \\ &= 1/4 \quad \text{if} \quad \eta_A = |\eta_B - \eta_C| \\ &= 0 \quad \text{otherwise} \end{aligned} \quad (3.23)$$

and

$$T_{BC}^A \equiv \int_0^{\alpha} J_0(\hat{k}_A r) J_1(\hat{k}_B r) J_1(\hat{k}_C r) r dr \quad (3.24)$$

If we further define,

$$R_{ABC} \equiv 8\pi / [\alpha^3 J_0(\hat{k}_A \alpha) J_0(\hat{k}_B \alpha) J_0(\hat{k}_C \alpha)] \quad (3.25)$$

then the Jacobian term in equation (3.21) may be written as,

$$\begin{aligned} & \int_0^1 \int_0^\infty J \left(\sum_B \hat{\psi}_B F_B, \frac{1}{r^2} \sum_C \hat{\psi}_C \lambda_C^2 F_C \right) F_A dr dz \\ &= - \sum_B \sum_C R_{ABC} \lambda_B^2 \hat{\psi}_B \hat{\psi}_C [\hat{k}_C n_A S_{BC}^A T_{AB}^C - \hat{k}_A n_C S_{AB}^C T_{BC}^A] \end{aligned} \quad (3.26)$$

while the Jacobian term in equation (3.22) becomes,

$$\begin{aligned} & \int_0^1 \int_0^\infty J \left(\sum_B \hat{\psi}_B \hat{F}_B, \sum_C \hat{\theta}_C G_C \right) G_A dr dz \\ &= - \sum_B \sum_C R_{ABC} \hat{\psi}_B \hat{\theta}_C [\hat{k}_C n_A S_{BC}^A T_{BC}^A - \hat{k}_A n_C S_{AB}^C T_{AB}^C] \end{aligned} \quad (3.27)$$

In order to derive equations (3.26) and (3.27) we have also made use of the following identities:

$$\begin{aligned} n_A S_{BC}^A + n_B S_{AC}^B + n_C S_{AB}^C &= 0 \\ \hat{k}_B \int_0^\infty r J_0(\hat{k}_A r) J_0(\hat{k}_B r) J_0(\hat{k}_C r) dr &= \hat{k}_C T_{BC}^A + \hat{k}_A T_{AB}^C \\ \hat{k}_B \int_0^\infty r J_1(\hat{k}_A r) J_1(\hat{k}_C r) J_2(\hat{k}_B r) dr &= \hat{k}_A T_{BC}^A + \hat{k}_C T_{AB}^C \end{aligned}$$

3. Energetics and Spectrum Truncation

In this section we consider some integral properties of the axisymmetric equations and their behavior under spectral truncation.

The total kinetic energy per unit mass is defined as

$$\begin{aligned} E_K &= \frac{1}{2} \int_0^1 \int_0^\infty (\hat{u}^2 + \hat{w}^2) r dr dz \\ &= - \frac{1}{2} \int_0^1 \int_0^\infty \hat{\psi} \nabla^2 \hat{\psi} \frac{dr}{r} dz \end{aligned} \quad (3.28)$$

where we have used the definition of the stream function and the boundary conditions to derive the second expression. The potential energy per unit mass is defined by

$$E_p = - \int_0^1 \int_0^1 \hat{\theta} z r dr dz \quad (3.29)$$

In the absence of sources or sinks of energy, the sum E_K plus E_p is a constant.

Other integral properties of interest are the mean potential temperature $\bar{\theta}$, the mean-square potential temperature $\bar{\theta}^2$ and the enstrophy $\bar{\eta}^2$ which are defined as

$$\begin{aligned} \bar{\theta} &= \int_0^1 \int_0^1 \hat{\theta} r dr dz \\ \bar{\theta}^2 &= \int_0^1 \int_0^1 \hat{\theta}^2 r dr dz \\ \bar{\eta}^2 &= \int_0^1 \int_0^1 \hat{\eta}^2 r dr dz \end{aligned} \quad (3.30)$$

These quantities have the following time dependence

$$\frac{d\bar{\theta}}{dt} = 0 \quad (3.31)$$

$$\frac{d\bar{\theta}^2}{dt} = -2N \int_0^1 \int_0^1 \hat{w} \hat{\theta} r dr dz \quad (3.32)$$

$$\frac{d\bar{\eta}^2}{dt} = -2 \int_0^1 \int_0^1 \hat{\eta}^2 \frac{\partial \hat{\psi}}{\partial z} \frac{dr dz}{r} + 2 \int_0^1 \int_0^1 \hat{\eta} \frac{\partial \hat{\theta}}{\partial r} r dr dz \quad (3.33)$$

The first term on the right hand side is a geometrical term which acts as a source or sink of enstrophy.

Since the rate of change of potential energy may be shown to be

$$\frac{dE_p}{dt} = - \int_0^1 \int_0^1 \hat{w} \hat{\theta} r dr dz \quad (3.34)$$

then we find that if $N \neq 0$ we may define a quantity which very much resembles available potential energy whose variations are equal to the variations in potential energy, namely $\bar{\theta}^2/2N$. However, in this study

we consider only the case $N = 0$, in which case $\bar{\theta}^2$ is a constant (in the absence of sources or sinks).

If we substitute the spectral expansions into the above expressions we obtain the following relations

$$E_K = \frac{1}{2} \sum_A \lambda_A^2 \hat{\psi}_A^2 \quad (3.35)$$

$$E_P = \frac{\alpha}{\pi} \sum_{n_A} (-)^{n_A} \hat{\theta}_{n_A}^0 / n_A \quad (3.36)$$

$$\bar{\theta} = \frac{2\alpha}{\pi} \sum_{n_A} \hat{\theta}_{n_A}^0 / n_A \quad n_A = \text{odd} \quad (3.37)$$

$$\bar{\theta}^2 = \sum_A \hat{\theta}_A^2 \quad (3.38)$$

$$\bar{\eta}^2 = \sum_A \lambda_A^4 \hat{\psi}_A^2 \quad (3.39)$$

In order for the spectral equations to preserve the total energy as well as the other invariants, the Jacobian terms represented by (3.26) and (3.27) must not contribute to the tendency of E_K or $\bar{\theta}^2$. It is straightforward to show this is the case, independent of the truncation simply through the symmetries of the interaction coefficients. Thus $\bar{\theta}^2$ is a constant and the rate of change of kinetic energy is given by

$$\frac{dE_K}{dt} = \sum_A \hat{k}_A \hat{\psi}_A \hat{\theta}_A \quad (3.40)$$

and both results are independent of truncation.

If, on the other hand, we consider the rate of change of potential energy we find that

$$\begin{aligned} \frac{dE_P}{dt} &= \frac{\alpha}{\pi} \sum_{n_A} (-)^{n_A} \frac{1}{n_A} \frac{d\hat{\theta}_{n_A}^0}{dt} \\ &= \frac{\alpha}{\pi} \sum_{n_A} (-)^{n_A} \frac{1}{n_A} \sum_{B,C} R_{ABC} \hat{\psi}_B \hat{\theta}_C \hat{k}_C n_A S_{BC}^A T_{BC}^0 \end{aligned} \quad (3.41)$$

Now it may be verified that

$$T_{bc}^{\circ} = \frac{\lambda^2}{2} [J_0(\hat{k}_b \lambda)]^2 \delta(i_b, i_c); \quad i_b \neq 0$$

$$T_{00}^{\circ} = 0$$

so that using the definition of R_{abc} , then (3.41) may be reduced to,

$$\frac{dE_p}{dt} = 4 \sum_{n_A} \sum_{n_B} \sum_{n_C} \sum_{i_B} (-)^{n_A} \hat{k}_B \hat{\psi}_{n_B}^{i_B} \hat{\theta}_{n_C}^{i_B} S_{n_B n_C}^{n_A} \quad (3.42)$$

and so to preserve total energy, we must show that the right hand side of (3.42) is the negative of the right hand side of (3.40). This is equivalent to demonstrating that

$$\sum_{n_A} (-)^{n_A} S_{n_B n_C}^{n_A} = -\frac{1}{4} \delta(n_B, n_C)$$

Now, in view of the property (3.23) and given $n_B \neq n_C$ then there are only two values of n_A which contribute to the summation, namely $n_A = n_B + n_C$, and $n_A = |n_B - n_C|$. Therefore, the summation becomes

$$\sum_{n_A} (-)^{n_A} S_{n_B n_C}^{n_A} = \left(\frac{1}{4}\right)(-)^{n_B + n_C} - \left(\frac{1}{4}\right)(-)^{|n_B - n_C|} = 0$$

On the other hand, if $n_C = n_B$, then since $n_A > 0$ there is only one value of n_A which contributes to the summation, i. e. $n_A = n_B + n_C$, in which case the desired result follows. The above result applies for an infinite number of components. In order to have a similar property hold for a truncated system we simply must permit values of n_A large enough so that when n_B and n_C are at their maximum values, that $n_A = n_B + n_C$. That is, if the maximum value of n_C is N_C and the maximum value of n_B is N_B , then the maximum value of n_A should be $N_B + N_C$. Note that this extension of the series applies only to $\hat{\theta}_{n_A}^{\circ}$. In a similar way we may show that this truncation also satisfies the condition that the mean potential temperature be a constant. For convenience we then define the following sets of components to be the standard truncation:

$$\hat{\psi}_{n_A}^{i_A} ; i_A = 1 \text{ to } i_{\max}, n_A = 1 \text{ to } n_{\max}$$

$$\hat{\theta}_{n_A}^{i_A} ; i_A = 1 \text{ to } i_{\max}, n_A = 1 \text{ to } n_{\max}$$

$$\hat{\theta}_{n_A}^0 ; i_A = 0 \quad n_A = 1 \text{ to } n_{\max}$$

and this will be termed a truncation of $n_{\max}$ by $i_{\max}$. In this way the total energy, the mean potential temperature and the mean-square potential temperature will be conserved in a non-dissipative system.

Aliasing has been eliminated. If two interacting components affect a third component, which is inside the truncation, the interaction is allowed. If, however, they affect a component outside the truncation, the interaction is not allowed. Since all components outside the truncation are orthogonal to those inside, the error caused by neglecting these interactions is also orthogonal to the components inside. Thus the error is not spuriously distributed amongst the components inside the truncation, as generally happens in grid point models.

4. Initial Conditions and Numerical Procedure

In all experiments, the fluid was assumed to be confined to a cylinder $3k_m \times 3k_m$ ($k=1$). The basic state potential temperature was set at 294°K and initially there was no motion. The initial two-dimensional temperature perturbation was assumed to be of the form

$$\hat{\theta} = \hat{\theta}(0) \exp(-57.5 r^2) \sin^2(2.5\pi z) \quad 0 \leq z \leq .4$$

$$\hat{\theta} = 0 \quad .4 \leq z \leq 1$$

$$\hat{\theta}(0) = .8^\circ\text{C}$$

(3.43)

which can be expressed in terms of the spectral functions G_A by integration over the domain.

The interaction coefficients (3.24) were evaluated by numerical quadrature to an accuracy of 7 significant decimals using Simpson's Rule with correction over 400 points.

For the time integration, a Hamming Predictor-Corrector with repeated interval having was used, with a basic timestep of 20 seconds. Cases were run with a basic timestep of 5 seconds with no appreciable difference in result.

A check on the accuracy of the numerical procedures was provided by the conservation of $\bar{\theta}^2$, $\bar{E}_T$ and $\bar{\theta}$ in the non-dissipative system.

It was also possible to construct from the non-dissipative spectral equations, a so-called maximum-simplification set, in the sense of Lorenz (1960). These equations, which are developed in Appendix I, have a simple solution in terms of Elliptic Functions which can provide a further check on the Hamming Predictor-Corrector.

The truncation actually used is given in Table I, the first corresponding to maximum simplification. An 8 x 8 truncation has 144 components altogether, and space resolution roughly equal to 330 metres in both directions.

Table 1.

Possible Standard Truncations and the Corresponding Number of Components and Equivalent Gridlengths

n_{max} z wavenumber	i_{max} r wavenumber	Number of Components	Resolution
1	1	maximum simplification	
6	6	84	430 metres
8	8	144	330 metres
16	16	544	180 metres

The interaction coefficient method is extremely time consuming for high-resolution experiments. Thus for the 16×16 model, a new method, based on the transform technique of Orszag (1970) was used. The application of the transform technique to the spectral convection equations is described in Appendix III.

IV THE TWO-DIMENSIONAL RESULTS

The numerical model based on equations (3.21) and (3.22) was integrated with and without diffusion, for several different resolutions.

The case of a rising bubble with no diffusion is considered first, because it is possible to examine the very powerful non-linear processes in isolation. The model produces, as a by-product, energy spectra, which can be used to throw light on the rapid formation of an intense thermal gradient immediately above the bubble.

With the addition of various forms of diffusion, results from the model can be compared with previous numerical investigations and the similarity theory of Scorer (1957).

1. Non-Dissipative Case

Figure 3 depicts the numerical simulation of a buoyant bubble with no dissipation. This particular integration uses a 8×8 truncation as defined above. In the convergence zone immediately above the warm bubble, a thermal cap or region of steep potential temperature gradient is formed. This gradient steepens as time goes on, until eventually the resolution

Figure 3 Evolution of the potential temperature field (solid line, at every $.1^{\circ}\text{C}$) and stream-function (dashed lines at every $5 \times 10^{10} \text{ cm}^3/\text{sec}$) for the low-resolution non-diffusive case. The time in minutes is given at the upper right hand corner of each chart.

Figure 4a Energy box diagram for the transfer of potential temperature variance for the low-resolution non-diffusive case. Numbers represent percentages of the total potential temperature variance.

Figure 4b Energy box diagram for the generation (vertical arrows) and transfer (horizontal arrows) of kinetic energy for the same case. Numbers represent percentages of the total kinetic energy at 6 minutes.

Figure 5 Time evolution of the one-dimensional spectra in the z direction of potential temperature variance (a), kinetic energy (b) and enstrophy (c). (d) is the one-dimensional spectrum of potential energy variance in the r direction. Amplitudes are normalized with respect to the respective integrated quantities $\overline{\Theta^2}$, $\overline{\epsilon_K}$ or $\overline{\eta^2}$ (Figure 6). Time is indicated by (-) at 1/2 minute, (- -) at 3 minutes, (.....) at 6 minutes and (- . -) at 9 minutes.

of the model is insufficient to represent it properly. Thus, after six minutes, the bubble appears to break into two and the maximum potential temperature of the bubble increases in a manifestation of Gibb's phenomenon. As the integration continues, the bubble runs into the top of the box, spreads sideways, then starts to descend along the sides. This will of course decelerate the vortex motion, and ultimately reverse the circulation. This stage of development has no atmospheric realism but indicates that the model continues to display physical realism well after most grid point models without dissipation would have "blown up".

The energy transfers of this integration are interesting and illustrate the nature of the equations we deal with as well as the difficulty associated with computational instability. Figures 4a and 4b shown an energy box diagram of the transfers among Z wave numbers of $\bar{\theta}^2$ and E_K respectively. The spectral set we have used contains only standing waves, with which we model a moving bubble. This means, for instance, that as the bubble passes through a node of one of the Z waves, the amplitude of that particular wave will change sign. Thus the variance in any particular wave is somewhat dependent on the position of the bubble. For this reason, we have grouped the waves in three groups with the arbitrary designations large, small, and very small. Waves 1 to 4 were chosen to be large, since, were the initial bubble simply displaced to the top of the box without change in shape or size, at least 96% of the potential temperature variance would remain in this group.

The rapid cascade of potential temperature variance shows up quite clearly in Figure 4a. This is an accelerating process as virtually no transfer takes place before 3 minutes. Between 6 and 9 minutes, the transfers begin to reverse direction, indicating that the smallest scales present are effectively "saturated". The kinetic energy transfers are more complicated, due to the presence of generation terms. The effect of the blocking theorem is quite striking, with comparatively little energy transfer to high wavenumbers and practically all the generation taking place in the large-scale. It is apparent that this blocking of transfer explains why the vortex

Figure 6 Variation with time of kinetic energy E_k (solid line - B) and enstrophy $\eta^2/2$ (dashed line - A) for the non-diffusive low resolution case.

flows in bubble convection are so "smooth".

In Figure 5 we present the one-dimensional spectra of potential temperature variance $\overline{\theta^2}$, kinetic energy E_k and enstrophy $\overline{\eta^2}$ in the Z direction. The spectra in the r direction are all very similar to each other, so we have included only the spectrum of $\overline{\theta^2}$. Because of the use of standing waves and the consequent wide variability in amplitude, we have used one-minute time averages, rather than instantaneous values. The spectra are drawn for 1/2 minute, 3 minutes, 6 minutes and 9 minutes. We have also plotted the time variation of E_k and $\overline{\eta^2}$ in Figure 6, which together with $\overline{\theta^2}$ have been used to normalize the wavenumber spectra.

The tremendous transfer of temperature variance in the Z direction is apparent. In the r direction, it will be noticed that the initial temperature variance does not drop off very rapidly with increasing wavenumber. This is partially due to the smaller scale in the r direction and also to the fact that Bessel functions provide a somewhat less efficient

representation than trigonometric functions. It would be tempting to conclude that the transfers in the r direction are weaker than in the Z direction. However, results from a high resolution experiment, discussed in the next section, indicate, that this is not true.

The time variation of E_K and $\overline{\eta^2}/2$ have been plotted in arbitrary units in Figure 6. The ratio of these two quantities defines a scale which apparently increases slightly at first, and then decreases more rapidly. It appears that, despite the blocking process, the conversion from potential to kinetic energy is causing the effective scale to become smaller and smaller.

It would seem that the transport of kinetic energy is comparatively unimportant, but this is not entirely so. An integration performed without the advection of vorticity, shows that it is very important in the formation of the slightly lop-sided vortex of Ogura (1962) with the maximum vertical velocity lying somewhat ahead of the vortex centre.

2. High Resolution

It was felt that the previous results were somewhat colored by the low resolution of the model. The computation time for an integration using the method of interaction coefficients increases with a high power of the resolution. Consequently, all high resolution experiments were performed using the more efficient transform method (Appendix C). This technique still yields energy spectra, but it is rather difficult to obtain energy transfers.

A 16×16 non-dissipative model was run with exactly the same initial conditions as the 8×8 model. This implied that all of the components of the potential temperature field with l_A or $n_A > 8$ were initially zero.

In Figure 7 is depicted the time evolution of the potential temperature field and the stream-function for the high-resolution experiment. The results do not differ appreciably from the low-resolution experiment

Figure 7 Evolution of the potential temperature field (solid lines at every $.1^{\circ}\text{C}$) and stream-function (dashed lines at every $5 \times 10^{10} \text{ cm}^3/\text{sec}$) for the high resolution non-diffusive case. The time in minutes is given at the upper right hand corner of each chart.

Figure 8 Time evolution of $\hat{\theta}(z)$ at the origin ($r = 0$) for the high resolution non-diffusive case. Time is indicated by (-) at 0 minutes, (- -) at 2 minutes, (. . .) at 4 minutes, and (- . . .) at 6 minutes.

Figure 9 Time evolution of the one-dimensional spectra in the Z direction of potential temperature variance (a) and kinetic energy (b) for the high resolution non-diffusive case. Spectra in the r direction of potential temperature variance and kinetic energy are plotted in (c) and (d) respectively. All spectra are normalized with respect to Θ^2 or E_k . Time is indicated by (-) at 1/2 minute, (- - -) at 3 minutes and (····) at 6 minutes.

Figure 10 Time evolution of the characteristic scale for the low-resolution (A) and high-resolution (B) non-diffusive cases. The scale is proportional to the square root of the ratio of kinetic energy E_K to enstrophy η^2 .

until after 4 minutes, and then the bubble splits into 4 minor bubbles instead of 2 as in the low-resolution case.

To more graphically illustrate the formation of the "cap" and the bubble "splitting" we have plotted in Figure 8 $\hat{\theta}(z)$ at the origin ($r=0$) for 0, 2, 4, and 6 minutes. It might be noted that the maximum potential temperature has only increased to $.814^\circ\text{C}$ instead of $.920^\circ\text{C}$ as in the 8×8 model.

In Figure 9a and 9b are presented the one-dimensional spectra of potential temperature variance $\overline{\theta^2}$ and kinetic energy E_K in the Z direction. Similar spectra in the r direction are presented in Figures 9c and 9d. These spectra are normalized by the appropriate values of E_K or $\overline{\theta^2}$ and are plotted at 1/2 minute, 3 minutes and 6 minutes.

As in the low-resolution experiment, it is apparent that the transfer of potential temperature variance is greater than the transfer of kinetic energy. It is also clear, however, that the transfers in the r direction are just as great as in the Z direction, which is not suggested in the low resolution results. This implies that the formation of a cap around

the side of the bubble is taking place just as rapidly as the cap ahead of the bubble.

In Figure 10 is plotted the time evolution of characteristic scale for the high and low-resolution experiments. This scale is proportional to the square root of the ratio of kinetic energy E_k to the enstrophy $\overline{\eta^2}$. The two scales are indistinguishable until after 5 minutes, when there is cascade to very high wavenumbers in the high resolution model.

As a final observation, we note that the high resolution model became "saturated" at almost the same time as the low-resolution model. In other words, higher resolution will not slow down non-linear instability very much because of the larger number of transfer routes available at higher resolution.

3. Inclusion of Dissipation

In a numerical model, there are two reasons for including dissipation terms. The first is to model the physical cascade of energy to smaller scales than the model can resolve. The second is to prevent numerical instabilities. In the spectral model there is no computational instability as such, but the results still become non-physical when the bubble splits up. The dissipation necessary to prevent instabilities or bubble splitting is not necessarily equal to the dissipation which is physically correct.

Various experiments with different diffusion formulations were performed to find out how much dissipation is necessary to prevent bubble splitting. The results of these experiments and comparison with a simple theory are shown in Appendix B. A rather unsuccessful attempt to prevent splitting using non-linear exchange with potential temperature components in the ϕ direction is described briefly in Chapter 6.

4. The Approach to Similarity

Before discussing the results of the dissipative two-dimensional model we will briefly summarize the similarity theory of bubble convection,

which was developed by Scorer (1953, 1957) and Batchelor (1954), and has also been discussed by Woodward (1959) and Ogura (1962).

When a finite quantity of heat is suddenly released at a point in the atmosphere it will rise and set up a turbulent motion, much like a ring vortex. If we assume that the distributions of velocity and potential temperature relative to the centre of the bubble, remain the same as the bubble rises, we can employ dimensional arguments to obtain a number of useful results.

Thus if Z and R represent the vertical and horizontal displacements of any defineable feature, then dimensional analysis leads to the following results;

$$Z = nR$$

$$W^2 = C^2 \bar{\theta} B R$$

where Z is measured from the virtual origin, i. e. from the apex of the cone swept out by the largest horizontal section of the thermal; n is a numerical coefficient to be determined by experiment; $\bar{\theta} B$ is the mean value of the local buoyancy term on a unit mass of fluid, ($\bar{\theta} B$ represents the average value of $\hat{\theta}$ in our non-dimensional system); C^2 is a constant of the same nature as n ; and W represents the vertical velocity.

The volume V of the thermal may be related to R by the relation

$$V = m R^3$$

where m is another number to be determined by experiment. The total buoyancy of the thermal is constant, so that if suffix 0 denotes the value at the moment of release, we must have

$$\bar{\theta} B m R^3 = \bar{\theta} B_0 V_0$$

writing $w = \frac{dz}{dt}$ and integrating, we obtain

$$Kz^2 = t \quad (4.1)$$

where $K = m^{1/2} / [2nc(\xi B_o V_o)^{1/2}]$

On the basis of (4.1) we may derive the following proportionality relationships,

$$\begin{aligned} w &\propto \tilde{\theta}_{max} \propto t^{-1/2} \\ \theta_{max} &\propto t^{-3/2} \\ G_E &\propto D_E \propto t^{-1/2} \\ \bar{Q} &\propto \text{constant} \\ Z_{cap} &\propto t^{+1/2} \end{aligned} \quad (4.2)$$

where θ_{max} and $\tilde{\theta}_{max}$ are the maximum values of potential temperature and the horizontally averaged potential temperature respectively; G_E represents the generation of kinetic energy; D_E represents the dissipation of kinetic energy; $\bar{Q}$ represents the mean vorticity; and Z_{cap} represents the height of the bubble cap measured from the virtual origin.

All of the relationships (4.2) indicate a bubble which is decelerating and thus the dissipative terms are expected to be fairly strong.

The spectral model was run with different diffusion formulations in order to approach similarity conditions. Various techniques were tried including the specification of the shape of the energy spectrum for high wavenumbers and the removal of any energy transferred into this region in excess of the assumed distribution. The diffusion formulation of Lilly (1962), which is based on an eddy coefficient of viscosity proportional to

Figure 11 Evolution of the potential temperature field (solid lines at every $.1^{\circ}\text{C}$) and stream-function (dashed lines at every $5 \times 10^{10} \text{ cm}^3/\text{sec}$) for the low-resolution diffusive case. The time in minutes is given at the upper right hand corner of each chart.

the absolute value of the deformation field, is quite complicated in spectral form because of the absolute value operator. We have therefore used a somewhat different technique which has the advantage of linearity in space, but maintains a time variation in the coefficient of eddy viscosity. In this formulation the coefficient of eddy viscosity is assumed to be proportional to the absolute value of the space-averaged vorticity ($\bar{Q}$).

$$\nu_{\hat{e}} = 2.95 \times 10^{-2} \bar{Q}$$

$$\nu_{\hat{Q}} = .25 \nu_{\hat{e}}$$

This coefficient of eddy viscosity is zero at $t=0$ and should become constant when similarity conditions are established. We have chosen the smaller value of $\nu_{\hat{Q}}$ because the blocking effect inhibits kinetic energy cascade and a larger viscosity would quickly remove kinetic energy from the smallest scales.

The use of this viscosity formulation results in a fairly large effective diffusion coefficient near the end of the experiment which means that the Reynolds number of this run is somewhat smaller than in previous numerical simulations.

The results of an 8×8 low resolution integration using this diffusion formulation are shown in figure 11. The solution is seen to be reasonably shape-preserving. However, there are at least two failings. Firstly, the vortex appears reluctant to detach itself from the bottom of the box. This results in a uniformly low horizontal velocity below the vortex centre. Secondly there is no tendency to form the characteristic mushroom shape.

It was suspected that the low horizontal velocity below the vortex centre was simply due to lack of resolution. A run using the 16×16 model with a similar diffusion formulation confirmed this suspicion. The failure to form a mushroom shape appears to be more fundamental and is probably due to the rather simple diffusion formulation.

Figure 12 Values of various quantities as functions of time are plotted and a log-log graph for the low-resolution diffusive case. The straight lines indicate the predictions from similarity theory.

A - W_{max} ; B - Θ_{max} ; C - height of the cap;
 D - eddy viscosity parameter and mean vorticity;
 E - dissipation of kinetic energy; F - Θ_{max}
 (maximum horizontally averaged potential temperature deviation); G - generation of kinetic energy

In order to investigate quantitatively the relationship between the height of significant features and time, it is necessary to determine the virtual origin of the bubble. We obtain this virtual origin by determining the trajectory of the vortex centre and extrapolating the ultimate straight line back to $r=0$. In this experiment the result was -750 metres. In the following discussion heights are measured from this virtual origin rather than from the true origin. We will consider the thermal outline of the bubble to be defined by the $.02^{\circ}\text{K}$ isentrope and in keeping with Ogura's usage we define Z_{cap} to be the foremost part of the thermal outline.

In Fig. 12 we present on a log-log graph the time variations of various quantities and their comparisons with similarity theory (given by the light straight lines). As both Lilly and Ogura observed, not all quantities enter similarity simultaneously, however the similarity relationships are quite well followed. In Table 2 we present a comparison of the similarity constants obtained in this experiment with those of Ogura and the tank experiments. Further we have also set $m=3$ in order to evaluate the constant C . It might be noted that whereas Ogura's thermal was taller and narrower than the tank experiment, our thermal is shorter and stouter. This result is to be expected because of the relatively low Reynold's number of our experiment.

Table 2. Comparison of observed and calculated similarity parameters.

	Tank Experiments	Numerical	
		Ogura	Spectral
$\mathcal{L}$	15°	12°	19°
h	4	5	3
$W_{\text{max}}/\dot{Z}$	2.2	2.3	$2.7(Z_{\text{cap}})$
$K^{-1}(B_0 V_0)^{-1/2}$	180	236	$140(Z_{\text{cap}})$
C	1.2	1.3	1.3

5. Energy Considerations

The concept of a similarity stage in buoyant convection presupposes the existence of a turbulent mechanism which degrades the energy of the large scale flow. This dissipation is quite rapid; the typical ratio of energy dissipation to generation (termed the energy dissipation fraction) has been estimated by Lilly (1966) to be about 0.60. For this reason, it is difficult to understand why Ogura (1962) and Arnason et al (1968) using highly diffusive numerical procedures, were concerned that their integrations should preserve the adiabatic invariants. For, if a model should reach a similarity stage, it is necessary for diffusion to take place explicitly or implicitly. Thus, if diffusion is implicit, as is the case with Arnason and Ogura, then there seems to be little point in determining the relative constancy of the adiabatic invariants.

In any case we have calculated the energy dissipation fraction for our model and compared the values with both Ogura and Arnason et al. These figures are given in Table 3. We note that for both Ogura's and Arnason's models this fraction is negative, indicating that there is some source of kinetic energy other than potential energy. On the other hand, the energy dissipation fraction in the spectral model increases to a value of 0.38 which is considerably less than the estimates given by Lilly (1966); but approximately the same value that was obtained by Lilly in 1962.

Table 3. Comparison of the energy dissipation fraction as calculated for three models of dry buoyant convection.

Time (min)	Energy Dissipation Fraction		
	Ogura	Arnason et al	Spectral
5	-.07	.00	.10
10	-.06	-.06	.28
15	-.04	-.14	.38

V THE THREE DIMENSIONAL SPECTRAL CONVECTION EQUATIONS

" The thermals were released from a thin hemispherical copper cup pivoted about a horizontal axis, the level of the fluid inside being made the same as outside so as to avoid initial impulse. The cup was turned over quickly by hand, and a negligible amount of motion was produced thereby. A dense white precipitate was used to make the buoyant fluid visible. Immediately after release the front surface of the thermal became covered with protruberances and the volume began to increase. "

R. S. Scorer - Journal of Fluid Mechanics (1957)

In this description of the first few moments in the evolution of a laboratory produced buoyant thermal, lies the justification for a three-dimensional simulation of bubble convection. All of Scorer's thermals, which were originally axisymmetric, quickly developed asymmetries. If these asymmetries were too gross he rejected them as being caused by inhomogeneities in the ambient fluid. However, in all the cases that he accepted and measured there was some measure of asymmetry.

It is quite apparent why these asymmetries occur. As the axisymmetric bubble rises it develops a front or cap where the density gradient is intense. This density gradient is gravitationally unstable to any

perturbation introduced there. Such a perturbation will manifest itself as a small thermal which will grow very quickly at the expense of the parent thermal because it is situated in a region of unstable lapse rate. Ultimately, of course, this small thermal or protruberance will decay due to dissipation to still smaller scales, advection by the axisymmetric flow to a less favorable region for growth, or because it has temporarily modified its parent thermal in such a way as to cut off its source of energy.

In the atmosphere, where buoyancy is derived from thermal differences, the process we have been describing is a rather complicated form of thermal dissipation. Thus each thermal scale transfers thermal energy to smaller scales by creating miniature carbon-copies of itself. Ultimately, the thermals will be so small as to be seriously affected by molecular conductivity.

Up to this point, there have been no laboratory or numerical investigations of this rather intriguing phenomenon. Indeed, there are a number of questions which have never been asked, let alone answered.

In this study we shall raise two questions and attempt to answer them through a numerical experiment. The first question is: What is the growth rate for unstable non-axisymmetric perturbations and how dependent is this growth rate on the potential temperature gradient in the "cap" of the parent thermal? The second question concerns the non-linear cascade of potential temperature variance. In a real buoyant thermal, the gradient of potential temperature in the "cap" presumably has some finite limit. Is this limit primarily maintained by transfer to smaller-scale axisymmetric modes or by transfer to non-axisymmetric modes?

The numerical model we shall use is based on the "convection system" derived in Chapter 2. Before discussing the development of the model we list again the "convection system" for easy reference.

$$\frac{\partial \hat{u}}{\partial t} + \hat{V} \cdot \nabla \hat{u} + \widehat{V' \cdot \nabla u'} = -\frac{\partial \hat{\Pi}}{\partial r} + \frac{\hat{V}^2}{r} + \frac{\hat{V}'^2}{r} + F_{\hat{u}} \quad (5.1)$$

$$\frac{\partial \hat{v}}{\partial t} + \hat{V} \cdot \nabla \hat{v} + \widehat{V' \cdot \nabla v'} = -\frac{\hat{u} \hat{v}}{r} - \frac{\hat{u}' \hat{v}'}{r} + F_{\hat{v}} \quad (5.2)$$

$$\frac{\partial \hat{w}}{\partial t} + \hat{V} \cdot \nabla \hat{w} + \widehat{V' \cdot \nabla w'} = -\frac{\partial \hat{\Pi}}{\partial z} + \hat{\theta} + F_{\hat{w}} \quad (5.3)$$

$$\frac{\partial \hat{\theta}}{\partial t} + \hat{V} \cdot \nabla \hat{\theta} + \widehat{V' \cdot \nabla \theta'} = D_{\hat{\theta}} \quad (5.4)$$

$$\frac{1}{r} \frac{\partial}{\partial r}(r \hat{u}) + \frac{\partial \hat{w}}{\partial z} = 0 \quad (5.5)$$

$$\frac{\partial u'}{\partial t} + \hat{V} \cdot \nabla u' + V' \cdot \nabla \hat{u} = -\frac{\partial \Pi'}{\partial r} + 2 \frac{V' \hat{V}}{r} + F_{u'} \quad (5.6)$$

$$\frac{\partial v'}{\partial t} + \hat{V} \cdot \nabla v' + V' \cdot \nabla \hat{v} = -\frac{1}{r} \frac{\partial \Pi'}{\partial \phi} - \frac{\hat{u} v'}{r} - \frac{\hat{v} u'}{r} + F_{v'} \quad (5.7)$$

$$\frac{\partial w'}{\partial t} + \hat{V} \cdot \nabla w' + V' \cdot \nabla \hat{w} = -\frac{\partial \Pi'}{\partial z} + \theta' + F_{w'} \quad (5.8)$$

$$\frac{\partial \theta'}{\partial t} + \hat{V} \cdot \nabla \theta' + V' \cdot \nabla \hat{\theta} = D_{\theta'} \quad (5.9)$$

$$\frac{1}{r} \frac{\partial}{\partial r}(r u') + \frac{1}{r} \frac{\partial v'}{\partial \phi} + \frac{\partial w'}{\partial z} = 0 \quad (5.10)$$

where $F_{\hat{u}}$, $F_{\hat{v}}$, $F_{\hat{w}}$, $D_{\hat{\theta}}$, etc are Fickian diffusion terms and N has been set equal to zero.

In essence the method is the following. Starting with the "convection system" we shall modify the tangentially averaged equations (5.1 to 5.5) so as to eliminate all interactions involving the "perturbation" variables. Equations (5.1 to 5.5) will then be essentially the axisymmetric set. Any interactions in the "perturbation" equations (5.6 to 5.10) involving the tangentially-averaged variables will use the solutions to the axisymmetric set. This procedure amounts to a "linearization" about the tangentially-averaged variables and is only valid if the amplitudes of the ϕ -dependent

"perturbations" are very much smaller than the amplitudes of the axisymmetric variables.

1. Boundary Conditions and Energetics

The "convection system" will be solved subject to the following boundary conditions

$$\begin{aligned}
 \hat{u} &= 0 \quad \text{at} \quad r = 0, \alpha & u' &= 0 \quad \text{at} \quad r = \alpha \\
 \frac{\partial \hat{u}}{\partial z} &= 0 \quad \text{at} \quad z = 0, 1 & \frac{\partial u'}{\partial z} &= 0 \quad \text{at} \quad z = 0, 1 \\
 \hat{w} &= 0 \quad \text{at} \quad z = 0, 1 & w' &= 0 \quad \text{at} \quad z = 0, 1 \quad \text{and} \quad r = 0 \\
 \frac{\partial \hat{w}}{\partial r} &= 0 \quad \text{at} \quad r = 0, \alpha & \frac{\partial w'}{\partial r} &= 0 \quad \text{at} \quad r = \alpha & (5.11) \\
 \hat{v} &= 0 \quad \text{at} \quad r = 0, \alpha & \frac{\partial v'}{\partial z} &= 0 \quad \text{at} \quad z = 0, 1 \\
 \frac{\partial \hat{v}}{\partial z} &= 0 \quad \text{at} \quad z = 0, \alpha & \theta' &= 0 \quad \text{at} \quad z = 0, 1 \quad \text{and} \quad r = 0 \\
 \hat{\theta} &= 0 \quad \text{at} \quad z = 0, \alpha & \frac{\partial \theta'}{\partial r} &= 0 \quad \text{at} \quad r = \alpha \\
 \frac{\partial \hat{\theta}}{\partial r} &= 0 \quad \text{at} \quad r = 0, \alpha
 \end{aligned}$$

These boundary conditions assume no normal flow and at least on the top and the bottom boundaries the normal derivatives of the tangential velocity components are zero. The boundary at $r = \alpha$ is not a real physical boundary and all that is required is that $\hat{u}$, $\hat{\theta}$, $\hat{w}$, u' , v' , w' and θ' are all small there.

The "convection system", like its axisymmetric subset has certain integral properties when dissipation is neglected. As mentioned above, the "convection system" will eventually be "linearized" and

energy conservation will no longer be possible. Nonetheless, it is desirable to show that it is energetically consistent.

We define four integral quantities, E_K - the total kinetic energy, E_P - the total potential energy, $\bar{\theta}$ - the mean potential temperature, and $\bar{\theta}^2$ - the total potential temperature variance, as follows

$$E_K = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \int_0^{\alpha} (\hat{u}^2 + \hat{v}^2 + \hat{w}^2 + \hat{u}'^2 + \hat{v}'^2 + \hat{w}'^2) r dr dz d\phi \quad (5.12)$$

$$E_P = -\frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} \int_0^{\alpha} (\hat{\theta} + \theta') z r dr dz d\phi = - \int_0^{\alpha} \int_0^{2\pi} \hat{\theta} z r dr dz \quad (5.13)$$

$$\bar{\theta} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} \int_0^{\alpha} (\hat{\theta} + \theta') r dr dz d\phi = \int_0^{\alpha} \int_0^{2\pi} \hat{\theta} r dr dz \quad (5.14)$$

$$\bar{\theta}^2 = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} \int_0^{\alpha} (\hat{\theta}^2 + \theta'^2) r dr dz d\phi \quad (5.15)$$

It can be shown that equations (5.1 to 5.10) without dissipation conserve the total energy ($E_K + E_P$), the mean potential temperature $\bar{\theta}$ and the potential temperature variance $\bar{\theta}^2$. The proofs are identical so only the proof for the conservation of $\bar{\theta}^2$ will be given.

If we multiply equation (5.4) through by $\hat{\theta}$ and equation (5.9) by θ' and integrate over the domain the following equations result

$$\begin{aligned} \frac{1}{2} \overline{\frac{\partial \hat{\theta}^2}{\partial t}} + \overline{\frac{\hat{y}}{2} \cdot \nabla \hat{\theta}^2} + \overline{\hat{\theta} \widehat{v' \cdot \nabla \theta'}} &= 0 \\ \frac{1}{2} \overline{\frac{\partial \theta'^2}{\partial t}} + \overline{\frac{\hat{y}}{2} \cdot \nabla \theta'^2} + \overline{\theta' \hat{v' \cdot \nabla \hat{\theta}}} &= 0 \end{aligned}$$

where we have neglected diffusion and the "bar" operator represents integration over r , z , and ϕ .

Since $\text{div } \hat{V} = 0$ and the normal component of $\hat{V}$ equals zero at the boundaries the second term in each equation is zero. From the fact that $\text{div } V' = 0$ and the normal component of V' equals zero at the boundaries we have

$$\overline{\hat{\theta} \nabla' \cdot \nabla \theta'} = - \overline{\theta' \nabla' \cdot \nabla \hat{\theta}}$$

$$\text{Thus } \frac{\partial}{\partial t} (\overline{\hat{\theta}^2 + \theta'^2}) = \frac{\partial}{\partial t} \overline{\theta^2} = 0$$

Similar proofs follow for $E_p + E_K$ and $\bar{\theta}$. The second term in all the prognostic equations of the "convection system" represent advections by the axisymmetric flow while the third terms represent transfers between the axisymmetric state and the ϕ dependent "perturbation" state.

2. The "Linearization" Assumption

The "convection system" of equations is still too complex to integrate. We will therefore assume all perturbation quantities are small with respect to tangentially averaged quantities. This implies that all interactions in the tangentially-averaged equations involving "perturbation" quantities will be dropped. As in Chapter 3 this assumption decouples the $\hat{V}$ equation (5.2) from the other equations. Thus if $\hat{V}$ is initially everywhere zero, then it will remain so. Consequently, equations (5.1 to 5.5) reduce down to the axisymmetric set of Chapter 3 and all $\hat{V}$ terms in the "perturbation" equations will be dropped.

The following set of equations result

$$\frac{\partial \hat{u}}{\partial t} + \hat{u} \frac{\partial \hat{u}}{\partial r} + \hat{w} \frac{\partial \hat{u}}{\partial z} = - \frac{\partial \hat{\pi}}{\partial r} + F\hat{u} \quad (5.16)$$

$$\frac{\partial \hat{w}}{\partial t} + \hat{u} \frac{\partial \hat{w}}{\partial r} + \hat{w} \frac{\partial \hat{w}}{\partial z} = - \frac{\partial \hat{\pi}}{\partial z} + \hat{\theta} + F\hat{w} \quad (5.17)$$

$$\frac{\partial \hat{\theta}}{\partial t} + \hat{u} \frac{\partial \hat{\theta}}{\partial r} + \hat{w} \frac{\partial \hat{\theta}}{\partial z} = D\hat{\theta} \quad (5.18)$$

$$\frac{1}{r} \frac{\partial (r\hat{u})}{\partial r} + \frac{\partial \hat{w}}{\partial z} = 0 \quad (5.19)$$

$$\frac{\partial u'}{\partial t} + \hat{u} \frac{\partial u'}{\partial r} + \hat{w} \frac{\partial u'}{\partial z} + u' \frac{\partial \hat{u}}{\partial r} + w' \frac{\partial \hat{u}}{\partial z} = - \frac{\partial \pi'}{\partial r} + F_{u'} \quad (5.20)$$

$$\frac{\partial v'}{\partial t} + \hat{u} \frac{\partial v'}{\partial r} + \hat{w} \frac{\partial v'}{\partial z} = - \frac{1}{r} \frac{\partial \pi'}{\partial \phi} - \frac{\hat{u} v'}{r} + F_{v'} \quad (5.21)$$

$$\frac{\partial w'}{\partial t} + \hat{u} \frac{\partial w'}{\partial r} + \hat{w} \frac{\partial w'}{\partial z} + u' \frac{\partial \hat{w}}{\partial r} + w' \frac{\partial \hat{w}}{\partial z} = - \frac{\partial \pi'}{\partial z} + \theta' + F_{w'} \quad (5.22)$$

$$\frac{\partial \theta'}{\partial t} + \hat{u} \frac{\partial \theta'}{\partial r} + \hat{w} \frac{\partial \theta'}{\partial z} + u' \frac{\partial \hat{\theta}}{\partial r} + w' \frac{\partial \hat{\theta}}{\partial z} = D\theta' \quad (5.23)$$

$$\frac{1}{r} \frac{\partial (ru')}{\partial r} + \frac{1}{r} \frac{\partial v'}{\partial \phi} + \frac{\partial w'}{\partial z} = 0 \quad (5.24)$$

This set of equations is not energy conserving, but energy transfers and conversions are consistent. The form of these equations is not suitable for spectral integration, so that further modifications will be made. However, these modifications do not involve any more physical assumptions.

3. Application of Helmholtz's Theorem

In the axisymmetric system (5.16 to 5.19) it is possible to eliminate the pressure $\hat{\pi}$ by cross-differentiating the $\hat{u}$ and $\hat{w}$ equations to obtain a vorticity equation. The introduction of stream function $\hat{\psi}$ reduces the system to two prognostic equations in $\hat{\theta}$ and $\hat{\psi}$. Because the vorticity is related to the stream function $\hat{\psi}$ by a second order Sturm-Liouville operator, it is possible to obtain a set of orthogonal functions to use in a spectral expansion of the equations.

In the case of the "perturbation" equations (5.20 to 5.24) this

straightforward approach cannot be used and it is necessary to resort to a different technique.

There is a three-dimensional analogue of Helmholtz's Theorem (see Lamb p. 208) which allows the three velocity components u' , v' , and w' to be written in terms of the components of a vector stream function. In addition, it is possible to cross-differentiate equations (5.20 to 5.22) to obtain prognostic equations for the three components of vorticity

$$\xi_r' = \frac{1}{r} \frac{\partial w'}{\partial \phi} - \frac{\partial v'}{\partial z} \quad \text{the radial component of vorticity} \quad (5.25)$$

$$\xi_\phi' = \frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial r} \quad \text{the meridional component of vorticity}$$

$$\xi_z' = \frac{1}{r} \frac{\partial (rv')}{\partial r} - \frac{1}{r} \frac{\partial u'}{\partial \phi} \quad \text{the vertical component of vorticity}$$

These prognostic equations do not contain π' . However, in cylindrical coordinates, the second order differential equations relating ξ_r' , ξ_ϕ' , and ξ_z' to the components of the stream function vector do not lead to eigenvalue problems and thus are useless for determining a set of spectral functions.

It was mentioned earlier that it is not absolutely necessary to expand the equations in terms of orthogonal functions. Robert (1966) for example, used non-orthogonal but linearly independent functions in his spectral model of the primitive equations. It would be possible to apply this technique to equations (5.20-5.24) but there are two objections. Firstly, it is necessary to apply a re-orthogonalization procedure after every timestep. Secondly, linear terms in the equation become in spectral form infinite series, rather than single terms.

The method eventually adopted was closely related to one used in the integration of filtered baroclinic models. It has a drawback when applied to the convection equations which will be discussed later. However, the method is sufficient for the purpose.

We expand u' and v' in terms of an ordinary stream function H' and a potential function S' which are defined with respect to the horizontal (r, ϕ) plane. Thus

$$u' = \frac{1}{r} \frac{\partial H'}{\partial \phi} + \frac{\partial S'}{\partial r} \quad v' = -\frac{\partial H'}{\partial r} + \frac{1}{r} \frac{\partial S'}{\partial \phi} \quad (5.26)$$

In terms of H' and S' , the continuity equation (5.24) can be written as

$$\nabla \cdot V = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial S'}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 S'}{\partial \phi^2} + \frac{\partial w'}{\partial z} = 0$$

or
$$\frac{\partial w'}{\partial z} = -\nabla_a^2 S' \quad (5.27)$$

where ∇_a^2 is the two-dimensional Laplacian operator in the (r, ϕ) plane

$$\nabla_a^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2}$$

The vertical component of vorticity ξ'_z can be written as

$$\xi'_z = \frac{1}{r} \frac{\partial}{\partial r} (r v') - \frac{1}{r} \frac{\partial u'}{\partial \phi} = -\nabla_a^2 H' \quad (5.28)$$

It is apparent that u' , v' and w' can be written in terms of the new variables H' and S' . Consequently the three prognostic equations (5.20 to 5.22) can be converted into two prognostic equations for H' and S' . By differentiating equation (5.21) by $\frac{1}{r} \frac{\partial}{\partial r} (r)$ and equation (5.20) by $\frac{1}{r} \frac{\partial}{\partial \phi}$ and subtracting we obtain a prognostic equation for H' .

$$\frac{\partial}{\partial t} \nabla_a^2 H' = -(\hat{u} \frac{\partial}{\partial r} + \hat{w} \frac{\partial}{\partial z}) \nabla_a^2 H' + \hat{Q} \frac{\partial w'}{\partial \phi} - \xi'_r \frac{\partial \hat{w}}{\partial r} + \nabla_a^2 H' \frac{\partial \hat{w}}{\partial z} \quad (5.29)$$

where $\hat{Q} = \frac{1}{r} \left(\frac{\partial \hat{w}}{\partial r} - \frac{\partial \hat{u}}{\partial z} \right)$ as in Chapter 3

$$\xi'_r = \text{the radial component of vorticity}$$

and we have dropped the dissipation terms. The first term represents advection of "perturbation" vorticity by axisymmetric flow and the other terms are "twisting-tilting" conversions into the other components of vorticity. This equation contains only non-linear terms on the right-hand side.

A prognostic equation for S' is a little more difficult to obtain. Operating with $\frac{1}{r} \frac{\partial^2}{\partial z \partial r}$ on equation (5.20) and with $\frac{1}{r} \frac{\partial^2}{\partial z \partial \phi}$ on equation (5.21) and adding we obtain

$$\frac{\partial}{\partial t} \nabla_a^2 \frac{\partial S'}{\partial z} = - \frac{\partial}{\partial z} \left[\frac{1}{r} \frac{\partial}{\partial r} (r A_u') + \frac{1}{r} \frac{\partial}{\partial \phi} A_v' \right] - \frac{\partial}{\partial z} \nabla_a^2 \pi' \quad (5.30)$$

where

$$A_u' = \hat{u} \frac{\partial u'}{\partial r} + \hat{w} \frac{\partial u'}{\partial z} + u' \frac{\partial \hat{u}}{\partial r} + w' \frac{\partial u'}{\partial z}$$

$$A_v' = \hat{u} \frac{\partial v'}{\partial r} + \hat{w} \frac{\partial v'}{\partial z} + \frac{\hat{u} v'}{r}$$

and we have again dropped dissipation terms. Now it is possible to obtain a diagnostic equation for the pressure π' by operating with $\frac{1}{r} \frac{\partial}{\partial r}$ on the u' equation, $\frac{1}{r} \frac{\partial}{\partial \phi}$ on the v' equation and $\frac{\partial}{\partial z}$ on the w' equation and adding. Instead we will eliminate the pressure from the system by using the operator ∇_a^2 on the w' equation (5.22). Thus

$$\frac{\partial}{\partial t} \nabla_a^2 w' = - \nabla_a^2 A_w' - \nabla_a^2 \frac{\partial \pi'}{\partial z} + \nabla_a^2 \theta' \quad (5.31)$$

where

$$A_w' = \hat{u} \frac{\partial w'}{\partial r} + \hat{w} \frac{\partial w'}{\partial z} + u' \frac{\partial \hat{w}}{\partial r} + w' \frac{\partial \hat{w}}{\partial z}$$

and we have dropped the dissipation terms. Now subtracting equation (5.31) from (5.30) to eliminate the pressure term we obtain

$$\frac{\partial}{\partial t} [\nabla_a^2 (\frac{\partial S'}{\partial z} - \nabla_a^2 w')] = \nabla_a^2 A_{w'} - \frac{\partial}{\partial z} \left[\frac{1}{r} \frac{\partial}{\partial r} (r A_u') + \frac{1}{r} \frac{\partial A_v'}{\partial \phi} \right] - \nabla_a^2 \theta' \quad (5.32)$$

Recalling that $\frac{\partial w'}{\partial z} = -\nabla_a^2 S'$ we see that this is a prognostic equation for S' in terms of the variables S' , H' and θ' . It might be noted that though the pressure Π' has been explicitly eliminated from (5.32) it is being implicitly calculated in the non-linear terms. From a finite-difference point of view such a highly differentiated equation as (5.32) would increase truncation error to such an extent as to be unusable. However, in spectral form all differentiations are done exactly, so there is no problem.

We have thus obtained a set of equations which can be conveniently integrated using the spectral method. We may summarize these equations as follows;

$$\frac{\partial}{\partial t} \nabla_a^2 H' = A_H' \quad (5.33)$$

$$\frac{\partial}{\partial t} [\nabla_a^2 (\frac{\partial S'}{\partial z} - w')] = A_S' - \nabla_a^2 \theta' \quad (5.34)$$

$$\frac{\partial \theta'}{\partial t} = A_{\theta'} \quad (5.35)$$

$$\frac{\partial w'}{\partial z} = -\nabla_a^2 S' \quad (5.36)$$

where $A_H' = -(\hat{u} \frac{\partial}{\partial r} + \hat{w} \frac{\partial}{\partial z}) \nabla_a^2 H' + \hat{Q} \frac{\partial w'}{\partial \phi} - \int_r' \frac{\partial \hat{w}}{\partial r} + \nabla_a^2 H' \frac{\partial \hat{w}}{\partial z}$

$$A_S' = +\nabla_a^2 A_{w'} - \frac{\partial}{\partial z} \left(\frac{1}{r} \frac{\partial}{\partial r} (r A_u') + \frac{1}{r} \frac{\partial A_v'}{\partial \phi} \right)$$

$$A_{\theta'} = -\hat{u} \frac{\partial \theta'}{\partial r} - \hat{w} \frac{\partial \theta'}{\partial z} - u' \frac{\partial \hat{\theta}}{\partial r} - w' \frac{\partial \hat{\theta}}{\partial z}$$

and $D_{\theta'}$ has been set equal to zero.

As mentioned earlier the method of expanding the components of velocity in terms of S' and H' has a drawback. It turns out that a satisfactory spectral system cannot be defined which satisfies the general boundary condition on u' at $r = \mathcal{L}$.

We can write u' as follows

$$u' = u'_{ROT} + u'_{DIV}$$

where $u'_{ROT} = \frac{1}{r} \frac{\partial H'}{\partial \phi}$ and $u'_{DIV} = \frac{\partial S'}{\partial r}$

The suffixes "ROT" and "DIV" stand for the rotational and divergent part of the wind on the (r, ϕ) plane. Now at $r = \mathcal{L}$ the most general boundary condition is

$$u' = u'_{ROT} + u'_{DIV} = 0$$

$$\text{or } u'_{ROT} = -u'_{DIV} \quad (5.37)$$

This general boundary condition is very difficult to apply in spectral form. If, for instance, u' is expanded in terms of orthogonal functions which satisfy the general boundary condition then the continuity equation (5.24) will turn into an infinite series in spectral form. This is clearly undesirable so a more restrictive boundary condition was chosen. Specifically, we require that

$$u'_{ROT} = u'_{DIV} = 0 \quad \text{at } r = \mathcal{L} \quad (5.38)$$

which implies that $H' = 0$ and $\frac{\partial S'}{\partial r} = 0$ at $r = \mathcal{L}$.

Spectral functions will be chosen which will individually satisfy this more restrictive boundary condition. However, this does not imply that a set of spectral functions so-defined cannot be used to represent a flow-field satisfying the more general boundary condition $u'_{ROT} = -u'_{DIV}$ at $r = \mathcal{L}$. Such a representation will be exact (in a least-squares sense)

as the number of spectral components goes to infinity. Thus the problem of the exactness of the representation of a general horizontal wind field U' , V' by the use of spectral functions which do not satisfy the general boundary condition merely boils down to a question of resolution.

At the moment, the statements in the proceeding paragraph are merely hypotheses. A more convincing demonstration of this property of the spectral representation will be provided after the spectral functions have been defined in the next section.

4. The Spectral Equations

Equations (5.33, 5.34, 5.35 and 5.36) together with the axisymmetric system comprise a complete set which can be solved numerically by the spectral method. The technique used to obtain a suitable set of orthogonal functions is the same as in Chapter 3; namely, by the solution of an eigenvalue problem. In this case, we consider two separate eigenvalue problems. Firstly,

$$\nabla_a^2 H' = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \right) H' = -\lambda^2 H' \quad (5.39)$$

subject to the boundary conditions (5.11 and 5.38) that $H' = 0$ at $r = a$, $\frac{\partial H'}{\partial z} = 0$ at $z = 0$. The eigensolutions of this problem are of the form $H' = J_M(kr) \exp(im\phi) f(z)$ where J_M is Bessel's function of the first kind of order M , k is a root of the equation $J_M(ka) = 0$, and $\frac{\partial f}{\partial z} = 0$ at $z = 0$. The eigenvalues are given by

$$\lambda^2 = k^2$$

The second eigenvalue problem we will consider is

$$\nabla_a^2 S' = -\lambda^2 S' \quad (5.40)$$

subject to the boundary conditions (5.11 and 5.38) that $\frac{\partial S'}{\partial r} = 0$ at $r = a$,

$\partial S'/\partial Z = 0$ at $Z = 0, 1$. The eigensolutions of this problem are of the form $S' = J_M(cr) \exp(im\phi) \delta(z)$ where J_M is Bessel's function of the first kind of order M , C is a root of the equation $J'_M(c\alpha) = 0$ and $\partial \delta/\partial z = 0$ at $Z = 0, 1$. The eigenvalues are given by

$$\lambda^2 = C^2$$

It is convenient to expand $f(z)$ and $\delta(z)$ in terms of the orthogonal functions $\cos n\pi z$, where n is a positive integer. This spectral expansion satisfies the boundary conditions $\partial H'/\partial z = \partial S'/\partial z = 0$ at $Z = 0, 1$; and is consistent with the spectral expansion used for the axisymmetric variables.

From the continuity equations (5.36) and the S' equation (5.34) it is apparent that W' and Θ' should be expanded in terms of the following functions

$$W' = J_M(cr) \sin n\pi z e^{im\phi}; \quad \Theta' = J_M(cr) \sin n\pi z e^{im\phi}$$

We now define the following set of spectral functions,

$$\begin{pmatrix} F_A \\ F_A^* \end{pmatrix} \equiv \begin{pmatrix} F(j_A, n_A) \\ F^*(j_A, n_A) \end{pmatrix} = R_A J_M(k_A r) \cos n_A \pi z \begin{pmatrix} \sin m\phi \\ \cos m\phi \end{pmatrix} \quad (5.41)$$

$$\begin{pmatrix} G_A \\ G_A^* \end{pmatrix} \equiv \begin{pmatrix} G(i_A, n_A) \\ G^*(i_A, n_A) \end{pmatrix} = P_A J_M(c_A r) \cos n_A \pi z \begin{pmatrix} \cos m\phi \\ \sin m\phi \end{pmatrix} \quad (5.42)$$

$$\begin{pmatrix} I_A \\ I_A^* \end{pmatrix} \equiv \begin{pmatrix} I(i_A, n_A) \\ I^*(i_A, n_A) \end{pmatrix} = P_A J_M(c_A r) \sin n_A \pi z \begin{pmatrix} \cos m\phi \\ \sin m\phi \end{pmatrix} \quad (5.43)$$

where C_A is the i_A th root of $J'_M(c\alpha) = 0$ and k_A is the j_A th root of $J_M(k\alpha) = 0$. P_A and R_A are normalization factors given by

$$P_A = \frac{C_A}{J_M(C_A \alpha)} \sqrt{\frac{8}{C_A^2 \alpha^2 - M^2}} \quad R_A = \frac{\sqrt{8}}{\alpha J_{M-1}(K_A \alpha)}$$

It should be pointed out that the positions of $\sin M\phi$ and $\cos M\phi$ in F_A and F_A^* are reversed from their positions in G_A , G_A^* , I_A and I_A^* .

We will define the volume integral of a product of two functions as follows.

$$\langle Z_A, Z_B \rangle = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \int_0^{\alpha} Z_A Z_B r dr dz d\phi$$

Then

$$\begin{aligned} \langle F_A, F_B \rangle &= \langle F_A^*, F_B^* \rangle = \delta(j_A, j_B) \delta(n_A, n_B) \\ \langle G_A, G_B \rangle &= \langle G_A^*, G_B^* \rangle = \delta(i_A, i_B) \delta(n_A, n_B) \end{aligned} \quad (5.44)$$

$$\langle I_A, I_B \rangle = \langle I_A^*, I_B^* \rangle = \delta(i_A, i_B) \delta(n_A, n_B)$$

$$\langle F_A, F_B^* \rangle = \langle G_A, G_B^* \rangle = \langle I_A, I_B^* \rangle = 0 \quad (5.45)$$

where $\delta(l, h) = 1$ if $l = h$ and equals zero otherwise. All integrals involving products of "starred" and "unstarred" functions are zero.

It should be noted that there is only one value of M allowed in the system so the Kronecker delta is only needed for the r and z wave-numbers. This occurs because the wave-wave interactions in the ϕ direction were eliminated in Chapter 2 and thus all components with different values of M will grow independently of one another. Consequently, it was considered sufficient to consider components of a "perturbation"

with only a single wavenumber in the ϕ - direction.

We now expand the variables H' , S' , Θ' and W' in terms of the functions F_A , G_A and I_A etc. Thus

$$H' = \sum_A (H_A F_A + H_A^* F_A^*) \quad (5.46)$$

$$S' = \sum_A (S_A G_A + S_A^* G_A^*) \quad (5.47)$$

$$W' = \sum_A \frac{C_A^2}{n_A \pi} (S_A I_A + S_A^* I_A^*) \quad (5.48)$$

$$\Theta' = \sum_A (\Theta_A I_A + \Theta_A^* I_A^*) \quad (5.49)$$

where the summation takes place over all possible values of n_A and all possible values of j_A or l_A .

Before substituting these expressions into equations (5.33 to 5.36) let us obtain the spectral expansions for U' and V' . Thus

$$U' = \sum_A \frac{R_A}{r} M J_M(K_A r) \cos n_A \pi z (H_A \cos M\phi - H_A^* \sin M\phi) \\ + \sum_A P_A J_M'(C_A r) \cos n_A \pi z (S_A \cos M\phi + S_A^* \sin M\phi) \quad (5.50)$$

$$V' = - \sum_A R_A J_M'(K_A r) \cos n_A \pi z (H_A^* \cos M\phi + H_A \sin M\phi) \\ + \sum_A \frac{P_A}{r} M J_M(C_A r) \cos n_A \pi z (S_A^* \cos M\phi - S_A \sin M\phi) \quad (5.51)$$

It will be observed that since $J_M(K_A \alpha) = 0$ and $J_M'(C_A \alpha) = 0$ that the boundary condition $U' = 0$ at $r = \alpha$ is satisfied identically. Similarly, since $\sin n_A \pi z = 0$ at $z = 0, 1$ the conditions on W' at the top and bottom boundaries are also satisfied. In like manner all the boundary conditions (5.11) are satisfied.

If we substitute the spectral expansions (5.49) into the thermodynamic

equation (5.35), multiply by $P_A J_M(C_A r) \sin n_A \pi z \cos M \phi$ and integrate over r , z and ϕ we obtain a prognostic equation for Θ_A . Thus

$$\dot{\Theta}_A = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \int_0^1 A_{\theta}' P_A J_M(C_A r) \sin(n_A \pi z) \cos(M \phi) r dr dz d\phi \quad (5.52)$$

$$= \langle A_{\theta}', I_A \rangle \quad (5.53)$$

The integration over ϕ is trivial because only a single value of M is allowed. Because of the orthogonality relation (5.45) any term in A_{θ}' containing H_B^* , S_B^* , or Θ_B^* (for any B) will integrate out to zero.

An equation for Θ_A^* can be obtained from (5.35) by multiplying through by $P_A J_M(C_A r) \sin(n_A \pi z) \sin(M \phi)$ and integrating. Thus

$$\begin{aligned} \dot{\Theta}_A^* &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \int_0^1 A_{\theta}' P_A J_M(C_A r) \sin(n_A \pi z) \sin(M \phi) r dr dz d\phi \\ &= \langle A_{\theta}', I_A^* \rangle \end{aligned} \quad (5.53 (*))$$

Again the orthogonality relations cause any term in A_{θ}' containing H_B , S_B or Θ_B (for any B) to integrate out to zero in (5.53(*)).

In a similar fashion equations can be obtained for each of H_A , H_A^* , S_A and S_A^* .

$$-K_A^2 \dot{H}_A = \langle A_H', F_A \rangle \quad (5.54)$$

$$-K_A^2 \dot{H}_A^* = \langle A_H', F_A^* \rangle \quad (5.54 (*))$$

$$\frac{C_A^2}{n_A \pi} (C_A^2 + n_A^2 \pi^2) \dot{S}_A = C_A^2 \Theta_A + \langle A_S', I_A \rangle \quad (5.55)$$

$$\frac{C_A^2}{n_A \pi} (C_A^2 + n_A^2 \pi^2) \dot{S}_A^* = C_A^2 \Theta_A^* + \langle A_S', I_A^* \rangle \quad (5.55 (*))$$

The integrals $\langle A_H', F_A \rangle$ and $\langle A_S', I_A \rangle$ also have the property that any term in A_H' or A_S' containing H_B^* , S_B^* or Θ_B^* (for any B) will integrate out to zero. Correspondingly, any term in $\langle A_H', F_A^* \rangle$ or $\langle A_S', I_A^* \rangle$ containing H_B , S_B or Θ_B (for any B) will also integrate out to zero.

This orthogonal property implies that the set of equations (5.53, 5.54 and 5.55) is completely decoupled from the set of (*) equations (5.53 (*), 5.54 (*) and 5.55 (*)). We will choose initial conditions such that there is a disturbance in potential temperature field (Θ), but no motion. Consequently $H'(r, z, \phi, 0) = S'(r, z, \phi, 0) = 0$. The decoupling of the "starred" and "unstarred" equations implies that if Θ_A^* (for all A) is set initially to zero, then Θ_A^* , H_A^* and S_A^* (for all A) will be zero for all time. It is possible to make all Θ_A^* initially zero simply by rotation of the coordinate system about the axis ($r = 0$). Thus we will drop the equations for Θ_A^* , H_A^* and S_A^* from our set.

The remaining equations (5.53, 5.54 and 5.55) along, with predictive equations for $\hat{\psi}$ and $\hat{\theta}$ are then completely self-contained.

In summary: The "convection system" equations (5.1 to 5.10) has been "linearized" by assuming that the amplitudes of all perturbation variables are small with respect to the amplitudes of axisymmetric quantities. The horizontal windfield (u', v') is then expanded in terms of a potential function S' and a stream function H' . Spectral functions suitable for expansion of the non-axisymmetric variables are obtained by considering eigenvalue problems for H' and S' . The non-axisymmetric variables are then expanded in terms of these spectral functions resulting in a set of prognostic equations for the amplitude coefficients of the spectral functions.

5. The Validity of the Spectral Expansion for a General Windfield

In section 3 of this chapter it was stated that a horizontal windfield which satisfied the general outer boundary condition (5.37) could be represented to any desired accuracy by the spectral functions since defined in

(5.41 to 5.43). We will not attempt to prove this hypothesis rigorously, but instead offer a demonstration of its validity.

Suppose there exists a horizontal wind field u^+ , v^+ which is represented in terms of a stream function H^+ and a potential function S^+ . We will choose H^+ and S^+ such that at $r = \mathcal{L}$; the sum $u_{ROT}^+ + u_{DIV}^+ = 0$, but u_{ROT}^+ and u_{DIV}^+ are not individually zero. A horizontal wind field which satisfies these conditions is defined by

$$S^+(r, \phi, z) = S_B^+ R_B J_M(k_B r) \cos n_B \pi z \cos M \phi$$

$$H^+(r, \phi, z) = H_B^+ P_B J_M(c_B r) \cos n_B \pi z \sin M \phi$$

where R_B , P_B are the normalization factors defined in (5.41 to 5.43), k_B is the j_B th root of $J_M(k\mathcal{L}) = 0$, c_B is the i_B th root of $J_M'(c\mathcal{L}) = 0$, n_B is a positive integer, and S_B^+ and H_B^+ are constants related by the following boundary condition

$$u_{ROT}^+ + u_{DIV}^+ = 0 \quad \text{at} \quad r = \mathcal{L}$$

$$\text{or} \quad S_B^+ R_B J_M'(k_B \mathcal{L}) + H_B^+ M P_B J_M(c_B \mathcal{L}) = 0$$

Thus S^+ is defined in terms of the spectral functions used to expand H' ; and H^+ is defined in terms of the spectral functions for S' . Consequently $u_{ROT}^+ \neq 0$ and $u_{DIV}^+ \neq 0$ at $r = \mathcal{L}$. This choice of S^+ and H^+ , defined in terms of functions which are the exact opposite of those used for S' and H' , will provide a good test of the efficiency of representation of the spectral functions (5.41 to 5.43)

The spectral representation for S^+ will be obtained by consideration of the continuity equation (5.27)

$$\nabla_2^2 S^\dagger = \frac{1}{r} \frac{\partial}{\partial r} (r u^\dagger) + \frac{1}{r} \frac{\partial^2 v^\dagger}{\partial \phi^2} = \nabla_2^2 S'$$

where $\nabla_2^2 S'$ is the spectral representation for $\nabla_2^2 S^\dagger$

Thus

$$\nabla_2^2 S^\dagger = -k_B^2 S^\dagger = \nabla_2^2 S' = - \sum_A P_A c_A^2 S_A J_M(c_A r) \cos n_A \pi z \cos M \phi$$

Multiplication of both sides of this equation by $P_A J_M(c_A r) \cos n_A \pi z \cos M \phi$ and integration over r , ϕ and z results in the following expression for S_A

$$S_A = S_{n_A}^{l_A} = \frac{1}{4} \frac{k_B^2}{c_A^2} S_B^\dagger \int_0^{\alpha} J_M(k_B r) J_M(c_A r) R_B P_A r dr$$

for $n_A = n_B$

or

$$S_A = \frac{2 k_B^3}{c_A} \frac{S_B^\dagger}{\sqrt{c_A^2 - M^2} (c_A^2 - k_B^2)}$$

We will test the convergence of this representation for $S^\dagger$ by calculating the mean-square error of the horizontal divergence field for a finite representation

$$\begin{aligned} E_{div}^B &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^\alpha (\nabla_2^2 S^\dagger - \nabla_2^2 S')^2 r dr dz d\phi \\ &= k_B^4 (S_B^\dagger)^2 - \sum_A c_A^4 S_A^2 \\ &= k_B^4 (S_B^\dagger)^2 \left[1 - 4 k_B^2 \sum_A \frac{c_A^2}{(c_A^2 - M^2)(c_A^2 - k_B^2)^2} \right] \end{aligned}$$

The error E_{DIV}^B converges fairly rapidly to zero, depending on which root of $J_M(k\alpha) = 0$ is chosen. Thus for k_1 , the smallest root of $J_M(k\alpha) = 0$, the expression in square brackets equals .0091 for $A=3$ and .0014 for $A=6$. If larger k_B are chosen, the series will converge less rapidly. In any case, for $C_A \gg k_B$ each term in the series is proportional to C_A^{-4} and will thus decrease rapidly.

If we follow the same procedure for the stream function $H^\dagger$ and define a mean-square error of the vertical component of vorticity we find

$$\begin{aligned} E_{ROT}^B &= \frac{1}{2\pi_0} \int_0^{2\pi} \int_0^1 \int_0^\alpha (\nabla_\alpha^2 H^\dagger - \nabla_\alpha^2 H')^2 r dr dz d\phi \\ &= C_B^4 (H_B^\dagger)^2 \left[1 - \frac{4 C_B^2}{(C_B^2 - M^2)} \sum_A \frac{k_A^2}{(k_A^2 - C_B^2)^2} \right] \end{aligned}$$

This expression also converges rapidly to zero, though less rapidly than the expression for E_{DIV}^B .

This example demonstrates that the spectral functions chosen can represent to any desired degree of accuracy this particular horizontal windfield which satisfies the general boundary condition $U'_{DIV} = -U'_{ROT}$ at $r = \alpha$.

6. Energy Definitions

The non-dissipative axisymmetric equations conserve total energy ($E_P + E_K$) and total potential temperature variance $\overline{\Theta^2}$. Similar integral quantities defined for the "linearized" non-dissipative perturbation equations (5.53 to 5.55) are not conserved. This does not imply, however, that these integral quantities are without value. For instance, the time evolution of certain of these integral quantities can be used as a measure of the growth or decay of the perturbation. In addition, there exist subsets of the perturbation equations which do have conservative properties that can

be used to test the numerics of the model.

Two integrated perturbation quantities are of interest: the total perturbation potential temperature variance $(\theta')^2$ and the total perturbation kinetic energy E'_K . In the absence of diffusion, the neglect of the terms $u' \frac{\partial \hat{\theta}}{\partial r}$ and $w' \frac{\partial \hat{\theta}}{\partial z}$ in equation (5.23) should result in the conservation of $(\theta')^2$. The reduced equation obtained by the neglect of these terms can be used to test the efficiency of the numerical formulations. However, if these terms are included the time evolution of $(\theta')^2$ can be used as a measure of the growth or decay of the perturbation.

The total perturbation potential temperature variance in spectral form becomes

$$\overline{(\theta')^2} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \int_0^\infty (\theta')^2 r dr dz d\phi = \sum_A \theta_A^2 \quad (5.56)$$

In addition to the conversion of potential energy into perturbation kinetic energy through the θ' term in equation (5.23), certain of the non-linear terms in equations (5.20 to 5.22) are also sources or sinks of total perturbation kinetic energy. The neglect of these terms: notably $u' \frac{\partial \hat{u}}{\partial r}$ and $w' \frac{\partial \hat{u}}{\partial z}$ in (5.20); $\frac{\hat{u} v'}{r}$ in (5.21); and $u' \frac{\partial \hat{w}}{\partial r}$, $w' \frac{\partial \hat{w}}{\partial z}$ in (5.22) results in a system of equations where the changes in perturbation kinetic energy (in the absence of dissipation) are equal to the conversions of potential energy. This reduced system of equations can again be used as a test of the numerics of the model.

The total perturbation kinetic energy in spectral form is as follows

$$\begin{aligned} E'_K &= \frac{1}{4\pi} \int_0^{2\pi} \int_0^1 \int_0^\infty \left[(u')^2 + (v')^2 + (w')^2 \right] r dr dz d\phi \\ &= \frac{1}{4\pi} \int_0^{2\pi} \int_0^1 \int_0^\infty \left[-H' \nabla_a^2 H' - S' \nabla_a^2 S' + (w')^2 \right] r dr dz d\phi \end{aligned}$$

$$\begin{aligned}
 E'_K &= \frac{1}{2} \sum_A K_A^2 H_A^2 + \frac{1}{2} \sum_A C_A^2 S_A^2 + \frac{1}{2} \sum_A \frac{C_A^4}{n_A^2 \pi^2} S_A^2 \\
 &= E_H + E_S + E_W
 \end{aligned} \tag{5.57}$$

where E_H is the total kinetic energy of the rotational part of the perturbation horizontal wind, E_S is the total kinetic energy of the divergent part of the perturbation horizontal windfield and E_W is the total kinetic energy of the perturbation vertical wind.

7. Initial Conditions

As in Chapter 3 the fluid was confined to a cylinder 3 km x 3 km ($\alpha = 1$). The basic state potential temperature Θ was set at 294°K and initially there was no motion. The initial tangentially averaged potential temperature distribution was given by (3.43).

It was the purpose of this experiment to investigate the growth of a small ϕ dependent "perturbation" on this time dependent axisymmetric state. Consequently, the initial potential temperature distribution of this "perturbation" is very important in its growth or decay. In order to examine qualitatively the effect of different initial distribution of the potential temperature "perturbation" it is necessary to examine equation (5.23)

$$\frac{\partial \Theta'}{\partial t} + \hat{u} \frac{\partial \Theta'}{\partial r} + \hat{w} \frac{\partial \Theta'}{\partial z} + u' \frac{\partial \hat{\Theta}}{\partial r} + w' \frac{\partial \hat{\Theta}}{\partial z} = 0$$

where we have neglected dissipation. Multiplying through by Θ' and integrating over ϕ we obtain a prognostic equation for $\widehat{(\Theta')^2}$. Thus

$$\frac{1}{2} \frac{\partial \widehat{(\Theta')^2}}{\partial t} = - \nabla \cdot \nabla \frac{\widehat{(\Theta')^2}}{2} - \frac{\partial \hat{\Theta}}{\partial r} \widehat{u' \Theta'} - \frac{\partial \hat{\Theta}}{\partial z} \widehat{w' \Theta'} \tag{5.58}$$

The first term on the right hand side simply represents advection of $\widehat{(\Theta')^2}$ by the axisymmetric flow and will not cause the "perturbation"

to either grow or decay. The last two terms, on the other hand, represent sources or sinks of "perturbation" potential temperature variance.

Now w' and θ' are in general positively correlated because if θ' is positive it will cause upward motion and if θ' is negative then it will cause downward motion. Considering the third term on the right hand side of (5.58) it appears that the "perturbation" will grow if $\partial \hat{\theta} / \partial z$ is negative and decay if $\partial \hat{\theta} / \partial z$ is positive. In the bubble cap $\partial \hat{\theta} / \partial z$ is large and negative so we will hypothesize that the maximum growth of the "perturbation" occurs there.

We will concern ourselves with the fastest growing "perturbation" because presumably others would either decay or grow more slowly and are likely to be of less importance.

Thus the initial conditions for the "perturbation" were chosen as follows

$$\begin{aligned} H' &= S' = 0 \\ \theta'(0, r, z, \phi) &= \theta'(0) J_M(c_8 r) \sin^2(4\pi z - \pi) \cos M\phi & 0 \leq r \leq .2 \\ & & .25 \leq z \leq .5 \\ \theta'(0, r, z, \phi) &= 0 & \text{elsewhere} \\ \theta'(0) &= .008^\circ \text{C} = .01 \hat{\theta}(0) \end{aligned} \quad (5.59)$$

where c_8 is the 8th root of $J'_M(c_8) = 0$

This potential temperature distribution represents a "perturbation" placed just above the cap of the parent thermal (see Figure 13) and if the hypothesis is correct it should be the fastest growing protruberance. The value of the ϕ -wavenumber was always set equal to two, for the following reason. The initial potential temperature perturbation θ' must be placed in the vicinity of the parent thermal in order for interaction to occur. This

means there will be variance in the Θ' field near but not at the origin $r=0$. Now the Bessel functions $J_M(r)$ have the property that $\lim_{r \rightarrow 0} \frac{d^{(M-1)} J_M(r)}{dr^{(M-1)}} = 0$. Therefore as M increases, it requires an increasing number of functions $J_M(c_A r)$ to represent Θ' . The value $M=2$ was chosen over $M=1$, because ϕ -wavenumber one simply represents a horizontal translation of the vertical axis of the thermal.

8. Truncation and Numerics

The standard truncation of the non-axisymmetric variables H' , S' and Θ' is defined as follows

$$H_A = H_{n_A}^{j_A} ; j_A = 1 \rightarrow j_{\max} ; n_A = 1 \rightarrow n_{\max}$$

$$S_A = S_{n_A}^{i_A} ; i_A = 1 \rightarrow i_{\max} ; n_A = 1 \rightarrow n_{\max}$$

$$\Theta_A = \Theta_{n_A}^{i_A} ; i_A = 1 \rightarrow i_{\max} ; n_A = 1 \rightarrow n_{\max}$$

In all cases actually considered $n_{\max} = i_{\max} = j_{\max} = 16$

All spectral equations were integrated using the transform technique described in Appendix C. A check on the accuracy of the numerics was provided by considering non-dissipative subsets of the spectral equations (5.53 to 5.55) which should theoretically conserve $(\Theta')^2$ or in which the change of total perturbation kinetic energy E'_K was due purely to potential energy conversions (see section 6 of this chapter).

VI RESULTS FROM THE THREE-DIMENSIONAL MODEL

In this chapter we present the results of experiments whose purpose was to simulate numerically the early stages in the breakdown of a two-dimensional atmospheric thermal into fully three-dimensional convection. The approach adopted was the investigation of the growth rates of small-amplitude non-axisymmetric perturbations superimposed on an initially axisymmetric thermal, under various experimental conditions

The growth of small-amplitude asymmetries on an axisymmetric parent thermal, even if only a "linearized" approximation to a highly non-linear phenomenon, is very complex because both symmetric and asymmetric variables are fully time-dependent. The time evolution of the perturbation is therefore strongly dependent on the time evolution of the parent thermal. In addition, the growth rate of the perturbation would likely depend on its shape, scale, and relative position with respect to the parent thermal.

The quasi-three-dimensional model developed in Chapter V was used to study the effect of the configuration of the parent thermal and the relative position of the perturbation with respect to it. The other factors were not tested. The perturbation scale has been chosen as the smallest that could be comfortably represented with the resolution available. The

Figure 13. Initial conditions for the "fastest-growing" non-axisymmetric perturbations. An (r, z) cross-section at $\phi = 0^\circ$ of the initial potential temperature field of the parent thermal and the perturbation is portrayed in (a). The non-axisymmetric perturbation is located on the "cap" of the parent thermal. In (b) is plotted an (r, ϕ) cross-section at $Z = 1.25$ km of the initial perturbation potential temperature field. The isopleths for the parent thermal are at every $.1^\circ\text{C}$ and for the non-axisymmetric perturbation at every $.003^\circ\text{C}$.

shape of the perturbation has been chosen to make the scale in each of the r , ϕ and Z directions approximately equal (see Figure 13b). Since equations (5.53 to 5.55) are linear with respect to perturbation quantities, the relative growth rate of the perturbation will be independent of its initial amplitude.

Though the parent thermal always had the same initial conditions (3.43) its time evolution could be controlled to some extent by changing the dissipation parameters in equations (3.21 and 3.22). Several diffusion constants were used varying between $\nu_{\hat{q}} = \nu_{\hat{\theta}} = 0$ to $\nu_{\hat{q}} = \nu_{\hat{\theta}} = 100 \text{ m}^2/\text{sec}$. These parameters control the magnitudes of the gradients of potential temperature $\frac{\partial \hat{\theta}}{\partial Z}$, $\frac{\partial \hat{\theta}}{\partial r}$ that can develop, and, thus, according to the hypothesis of Chapter 5, influence the growth or decay rate of the non-axisymmetric perturbation.

It was suggested in Chapter 5 that the "fastest-growing" perturbations would be those located at the "cap" of the parent thermal. To test this hypothesis several cases were run with various initial potential temperature perturbations having the same shape, scale, and amplitude, but different spatial positions with respect to the parent thermal. As expected, the "fastest-growing" perturbation was indeed located in the "cap" of the parent thermal. The initial conditions for this particular perturbation are described by equations (5.59) and depicted in Figure 13. The (r, z) variation of the initial perturbation and parent thermal on the plane $r = 0$ is shown in Figure 13a. A plan view (r, ϕ) of the initial perturbation is shown in Figure 13b. The assumed ϕ wavenumber of two implies that for any variable X , $X(r, z, \phi, t) = X(r, z, \phi + 180^\circ, t) = -X(r, z, \phi + 90^\circ, t)$. The potential temperature perturbation in Fig. 13 therefore represents two hot and two cold bubbles.

In Figures 14, 15, 16 and 17 are plotted (r, z) cross-sections of the time evolution of θ' , v' , u' and w' respectively, for this case. All are plotted on the plane $\phi = 0^\circ$ except for v' which is 45° out of phase with θ' , u' and w' . The time evolution of the parent thermal, as evidenced by the $\hat{\theta}$ and $\hat{\psi}$ fields in Figure 14 and 15, is very similar

Figure 14. Time evolution of the potential temperature field of the parent thermal and the "fastest-growing" perturbation. Isopleths for the parent thermal are at every $.1^{\circ}\text{C}$ and for the perturbation at every $.003^{\circ}\text{C}$. The kinematic viscosity and thermal diffusivity are equal to $100 \text{ m}^2/\text{sec}$ for the parent thermal, but the perturbation is non-diffusive. The time in minutes is given at the upper right hand corner of each chart.

Figure 15. Time evolution of the tangential velocity field (solid lines at every cm/sec) and axis-symmetric stream function (dotted lines at every 5×10^{10} cm³/sec) for the same case. The sign convention for the tangential velocity is negative for anti-clockwise motion around the axis $r = 0$ and positive for clockwise motion. This is an (r, z) cross-section at $\phi = 45^\circ$.

Figure 16. Time evolution of the radial velocity field (solid lines at every cm/sec) for the same case. The arrows indicate direction of motion. The dotted isopleths indicate .25 cm/sec.

Figure 17. Time evolution of the vertical velocity field (solid lines at every 3 cm/sec) for the same case. The arrows indicate direction of motion. The dotted isopleths indicate 1 cm/sec.

to the dissipative axisymmetric thermal of Figure 11.

The growing temperature perturbation remains on the "cap" of the parent thermal during the course of the integration, but is gradually advected to the outside. This advection by the velocity field of the parent thermal results in a "flattening" of the perturbation. The tendency to "flatten" in turn causes large gradients in the perturbation potential temperature field and cascade of variance to the highest vertical and radial wavenumbers in the model. This is manifested in the formation of an area of negative potential temperature at 8 minutes. Since there was no diffusion on the perturbation, the experiment was terminated at that point.

The perturbation velocity fields develop through the potential energy conversion term in the w' equation. Generally speaking, the radial velocities are much weaker than the tangential velocities and both are considerably weaker than the vertical velocities. The "flattening" of the temperature perturbation is reflected in the change in orientation of the zero axis in the u' and v' fields as time progresses.

Consideration of the continuity equation (2.8) and the boundary condition $\frac{\partial w}{\partial z} = 0$ at $z = 0, 1$, shows that the horizontal average of the vertical velocity $\overline{w}$ must vanish. For an axisymmetric thermal the upward vertical motion at the bubble centre must therefore be compensated by descending motion on the outside. However, for non-axisymmetric modes there is an extra degree of freedom and upward motion does not have to be compensated by downward motion in the same (r, z) plane. Hence, the difference in vertical motion fields between the axisymmetric thermal and the non-axisymmetric perturbations. In Figure 17 there are two areas of downward motion. The outer one is believed to be realistic while the inside one is probably due to the cascade of kinetic energy and consequent "saturation" of the highest horizontal and vertical wavenumbers.

In Figure 18 is plotted the total perturbation kinetic energy for the vertical motion, the divergent part of the horizontal wind and the rotational part of the horizontal wind, for the same case. The rotational part of

Figure 18. Time evolution of the total perturbation kinetic energy for the vertical motion (A), the divergent part of the horizontal wind (B) and the rotational part of the horizontal wind (C).

Figure 19. Time evolution of the maximum potential temperature gradient in the "cap" of the parent bubble (upper) and of the total perturbation potential temperature variance $(\theta')^2$ (lower). Letters A and B refer to cases with $\nu_{\hat{q}} = \nu_{\hat{\theta}} = 100 \text{ m}^2/\text{sec}$ and $\nu_{\hat{q}} = \nu_{\hat{\theta}} = 0$ respectively.

the horizontal wind grows quite slowly as might be expected from equation (5.29) for H' which is entirely non-linear on the right hand side.

Various cases were run with different diffusion parameters in the axisymmetric equations in order to investigate the effect of the potential temperature gradient in the "cap" of the parent bubble on the growth of the non-axisymmetric perturbation. The time evolutions of these perturbations are quite similar to those portrayed in Figures 14 to 17, so we have not plotted them.

We have plotted in Figure 19 the time evolution of $\left| \frac{\partial \hat{\theta}}{\partial z} \right|_{\max}$, the maximum gradient of potential temperature in the "cap" of the parent thermal, and $\overline{(\theta')^2}$, the total perturbation potential temperature variance, for two cases. Though the values of $\left| \frac{\partial \hat{\theta}}{\partial z} \right|_{\max}$ are plotted on the axis $r = 0$, they can be taken as a measure of the potential temperature gradient in the "cap" of the parent thermal. The two cases which have been chosen represent experiments performed with the maximum and the minimum values of the dissipation parameters for the axisymmetric equations.

For case A, which has previously been considered, $\nu_{\hat{\theta}} = \nu_{\hat{\theta}} = 100 \text{ m}^2/\text{Sec}$, while in case B, which in all other respects is identical to case A, $\nu_{\hat{\theta}} = \nu_{\hat{\theta}} = 0$. The growth rates in the two cases correspond to an e-folding time of about 4 minutes for case (A) and 3 minutes for case (B). The "cap" of the non-dissipative parent thermal (B) has a much larger temperature gradient than (A) and yet the growth rate of the superimposed perturbation is scarcely larger. A possible reason for this apparently weak dependence of perturbation growth rate on the gradient of potential temperature in the "cap", is the following. In all cases the axisymmetric flow field advected the perturbation in such a way that $\theta_{\max}$ remained spatially coincident with the "cap" of the parent thermal. However, the "cap" in the non-diffusive parent thermal shrinks greatly in width as its potential temperature gradient increases. After a time, only part of the potential temperature perturbation is still located in the region of the cap, the rest being in an area where $\frac{\partial \hat{\theta}}{\partial z}$ and $\frac{\partial \hat{\theta}}{\partial r}$ are small

Such sharp potential temperature gradients in the "cap" would tend to favor the growth of perturbations with smaller radial and vertical scales than those used in these experiments.

1. Large-Amplitude Non-Axisymmetric Perturbations

In Scorer's tank experiments the "cap" of the bubble eventually attained some small, but finite width (not necessarily constant with time). The finite limit for the "cap" width is set by the cascade of energy to smaller scales and ultimately by molecular dissipation. This cascade process is manifested as smaller-scale non-axisymmetric protruberances growing on the "cap" of the parent bubble.

An axisymmetric numerical model attempts to simulate this cascade process by some form of numerical dissipation. In essence, the dissipation terms are simulating transfers to both axisymmetric modes of smaller scale than the minimum resolvable scale and to all non-axisymmetric modes. In fact, however, the effect of the dissipation terms is to remove excess potential temperature variance that has cascaded to the smallest resolvable radial and vertical scales.

With the present model, it is possible to include some of these transfer processes explicitly, namely the transfers between ϕ -wavenumber λ modes and the axisymmetric modes. If the "linearization" assumption for the non-axisymmetric modes is dropped then there will be complete exchange of potential temperature variance between the parent bubble and the perturbation. In particular, the "fastest-growing" perturbation will remove potential temperature variance from the parent bubble. Now, if this variance happens to be removed primarily from the smallest-scale axisymmetric modes then the "cap" of the parent bubble will reach some equilibrium width and the "splitting" observed in Chapter 4 will not occur.

To determine how the growth of non-axisymmetric perturbations affected the parent thermal, the three-dimensional model developed in

Chapter 5 was modified in the following way. The "linearized" thermodynamic equations (5.18 and 5.23) were replaced by the more general set (5.4 and 5.9). The remaining equations for $\hat{\psi}$, u' , v' and w' remained unchanged. The initial amplitude of the perturbation was increased so that $\theta'_{\max}^{t=0} / \hat{\theta}_{\max}^{t=0} = .2$. There was dissipation on the perturbation, but none on the parent bubble.

As expected the potential temperature perturbation grew very rapidly at the expense of the parent thermal. In the "cap" of the parent thermal, away from the axis $r = 0$, the potential temperature gradient did not grow beyond a certain limit. However, on the axis $r = 0$, the perturbation had zero amplitude and could not interact with the parent thermal. Thus the axisymmetric potential temperature gradient on the axis grew rapidly and "splitting" eventually occurred.

The result of this experiment implies that the cascade and dissipative processes for an axisymmetric thermal cannot be simulated simply by including transfers to non-axisymmetric modes; the cascade to smaller-scale axisymmetric modes must also be modelled in some fashion.

VII SUMMARY AND CONCLUSIONS

The motion of a buoyant bubble in neutral surroundings has been investigated with two numerical models developed from the anelastic equations for shallow convection. The first, an axisymmetric non-dissipative spectral model, was used to investigate the cascade of potential temperature variance in the formation of the bubble "cap". With the addition of dissipation, the results from this model were compared with the laboratory tank experiments of Scorer and with other numerical experiments. The second model, a "linearized" three-dimensional spectral model was used to investigate the growth of asymmetries on an initially axisymmetric buoyant thermal.

The effect of Fjortoft blocking in an axisymmetric model was demonstrated by comparison of the resulting kinetic energy and potential temperature variance spectra. Though conversions from potential to kinetic energy counteracted the blocking, the net effect was a much milder cascade of kinetic energy than of potential temperature variance.

Other results from the non-dissipative axisymmetric model were that the "cap" of the bubble formed as rapidly at the side of the bubble as at the top. The "splitting" of the bubble that occurred when too much potential temperature variance was cascaded into the smallest resolvable

scales, seemed to always occur at the same time, independent of resolution.

The dissipative form of the axisymmetric model yielded results which could be favorably compared with the similarity theory of Scorer. It appeared, though, that the Fickian dissipation formulation was inadequate to model convection with Reynold's number as high as is observed in tank experiments. For a given resolution, a certain value of kinematic viscosity and thermal diffusivity is necessary to prevent the bubble "splitting", and, in general, these dissipation parameters imply an unrealistically low experimental Reynold's number. It would also appear, that the (eddy) Prandtl number should be less than one because otherwise the blocking effect implies that too much kinetic energy will remain in low wavenumber modes and the vortex will expand too rapidly.

The three-dimensional experiments indicated that the "cap" of the axisymmetric parent thermal is indeed unstable to non-axisymmetric modes. The fastest growing of these modes, if non-dissipative, have e-folding times of the order of 3-4 minutes.

However, if a numerical simulation of bubble convection is to retain physical realism for any length of time it must include dissipation terms. The magnitude of the dissipation parameter, together with the minimum resolvable scale define a characteristic time-scale. For a model with insufficient resolution this diffusive time-scale could be shorter than the e-folding growth times of non-axisymmetric perturbations. In this case, a fully three-dimensional model of bubble convection would be unjustifiable.

The spectral approach to the numerical integration of a bubble convection model was largely vindicated during the course of this experiment. The results were, of course, unaliased and energetically consistent. The technique provided the clearest and most reliable picture yet of the energy conversions and cascades involved in two-dimensional bubble convection. For the three-dimensional model the spectral approach implied the velocity fields were always exactly non-divergent. An equivalent grid-point model requires elaborate smoothing procedures in order to satisfy this constraint.

The spectral technique had the disadvantage of excessive computing time though the introduction of the transform method alleviated that to some extent. The major difficulty in computation involved the expansion in Bessel functions in the radial direction. The possibility of using piecewise continuous quasi-orthogonal functions (Wang and Douglas (1971)) might alleviate this problem.

With the transform technique of spectral integration it is possible in principle to handle non-linear dissipation and perhaps even precipitation. This would imply the expansion of discontinuous fields into a finite series of spectral functions, but, after all, this experiment has just demonstrated that the creation of a near discontinuity in the thermal field of a rising bubble can be modelled successfully with the spectral technique.

REFERENCES

- Arnason, G., R.S. Greenfield and E.A. Newburg, 1968: A numerical experiment in dry and moist convection including the rain stage. J. Atmos. Sci., 25, 404-415.
- Baer, F., 1964: Integration with the spectral vorticity equation. J. Atmos. Sci., 21, 260-286.
- Batchelor, G.K., 1953: The condition for dynamical similarity of motions of a frictionless perfect-gas atmosphere. Quart. J. R. Meteor. Soc., 79, 224-235.
- Batchelor, G.K., 1954: Heat convection and buoyant effects in fluids. Quart. J. R. Meteor. Soc., 80, 339-358.
- Benard, H., 1900: Les tourbillons cellulaires dans une nappe liquide. Revue general des sciences pures et appliquees, 11, 1261-1271 and 1309-1328.
- Chandrasekhar, S., 1961: Hydrodynamic and hydromagnetic stability. Oxford University Press, p. 652.
- Cooley, J.W., and J.W. Tukey, 1965: An algorithm for the machine calculation of complex Fourier series. Math. Computation, 19, 297-301.
- Deardorff, J.W., 1965: A numerical study of pseudo three-dimensional parallel plate convection. J. Atmos. Sci., 22, 419-435.
- Deardorff, J.W., 1966: The counter-gradient heat flux in the lower atmosphere and in the laboratory. J. Atmos. Sci., 23, 503-506.
- Dutton, J.A. and G.H. Fichtl, 1969: Approximate equations of motion for gases and liquids. J. Atmos. Sci., 26, 241-254.
- Fjortoft, R., 1953: On the changes in the spectral distribution of kinetic energy for two-dimensional non-divergent flow. Tellus, 5, 225-230.
- Fox, D.G., 1970: Numerical simulation of three-dimensional convective elements. Conference on Cloud Physics. August 24-27, 1970. Fort Collins, Colorado.

- Gough, D.O., 1969: The anelastic approximation for thermal convection. J. Atmos. Sci., 26, 448-456.
- Hacque, S.M.A., 1964: Finite amplitude convection in a conditionally unstable stagnant air mass. Tellus, 16, 147-159.
- Herring, J.R., 1963: Investigation of problems in thermal convection. J. Atmos. Sci., 20, 325-338.
- Herring, J.R., 1964: Investigation of problems in thermal convection: rigid boundaries. J. Atmos. Sci., 21, 277-290.
- Jeffries, H., 1926: The stability of a layer of fluid heated below. Phil. Mag., 2, 833-844.
- Jeffries, H., 1928: Some cases of instability in fluid motions. Proc. Roy. Soc. (London) A, 118, 195-208.
- Kuo, H.L., 1961: Solution of the non-linear equations of cellular convection and heat transport. J. fluid Mech., 10, 611-634.
- Lamb, H., 1932: Hydrodynamics. Cambridge University Press. p. 738.
- Lilly, D.K., 1962: On the numerical simulation of buoyant convection. Tellus, 14, 148-172.
- Lilly, D.K., 1964: Numerical solutions for the shape-preserving two-dimensional thermal convection element. J. Atmos. Sci., 21, 83-98.
- Lilly, D.K., 1966: Theoretical models of convection elements and ensembles. 1965-1966 Seminar Series sponsored by Travellers Research Centre, Inc., Advances in Numerical Weather Prediction.
- Lilly, D.K., 1970: Lectures in sub-synoptic scales of motion and two-dimensional turbulence. Summer School of Space Physics of the Centre National d'Etudes Spatiales, Lanion, France.
- Lorenz, E.N., 1960: Maximum simplification of the dynamic equations. Tellus, 12, 243-254.
- Lorenz, E.N., 1963: Deterministic non-periodic flow. J. Atmos. Sci., 20, 130-141.

- Malkus, W. V. R., 1954: Discrete transitions in turbulent convection. Proc. Roy. Soc. (London) A, 225, 185-195.
- Malkus, J. S., and G. Witt, 1959: The evolution of a convective element: a numerical calculation. *The Atmosphere and the Sea in Motion*, New York, Rockefeller Institute Press, 509 pp.
- Mason, B. J. and R. Emig, 1961: Calculations of the ascent of a saturated buoyant parcel with mixing. Quart. J. Roy. Meteor. Soc., 87, 212-223.
- Morton, B. R., 1959: Forced plumes. J. Fluid Mech., 5, 151-163.
- Ogura, Y., 1962: Convection of isolated masses of buoyant fluid: a numerical calculation. J. Atmos. Sci., 19, 492-502.
- Ogura, Y., 1963: The evolution of a moist convective element in a shallow, conditionally unstable atmosphere: a numerical calculation. J. Atmos. Sci., 20, 407-423.
- Ogura, Y., and N. A. Phillips, 1962: Scale analysis of deep and shallow convection in the atmosphere. J. Atmos. Sci., 19, 173-179.
- Orszag, S. A., 1970: Transform method for the calculation of vector-coupled sums: Application to the spectral form of the vorticity equation. J. Atmos. Sci., 27, 890-895.
- Pellew, A., and R. V. Southwell, 1940: On maintained convective motion in a fluid heated from below. Proc. Roy. Soc. (London) A, 176, 312-343.
- Phillips, N. A., 1959: An example of non-linear computational instability. *The Atmosphere and Sea in Motion*, Rockefeller Institute Press, New York, pp. 501-504.
- Priestly, C. H. B., 1954: Convection from a large horizontal surface. Australian J. Phys., 7, 176-201.
- Rayleigh, Lord, 1916: On convective currents in a horizontal layer of fluid when the higher temperature is on the under side. Phil. Mag., 32, 529-546.

- Richards, J.M., 1961: Experiments on the penetration of an interface by buoyant thermals. J. Fluid Mech., 11, 369-384.
- Richards, J.M., 1963: Experiments on the motion of isolated cylindrical thermals through unstratified surroundings. Int. J. Air Water Poll., 7, 17-34.
- Robert, A.J., 1966: The integration of a low-order spectral form of the primitive meteorological equations. J. Meteor. Soc. of Japan, 44, 237-245.
- Rouse, H., C.S. Yih and H.W. Humphreys, 1952: Gravitational convection from a boundary source. Tellus, 4, 201-210.
- Saltzman, B., 1961: Finite amplitude free convection as an initial value problem. J. Atmos. Sci., 19, 329-344.
- Sasaki, Y., 1967: Some dynamical aspects of atmospheric convection. Tellus, 19, 45-53.
- Scorer, R.S., 1957: Experiments on convection of isolated masses of buoyant fluid. J. Fluid Mech., 2, 583-594.
- Scorer, R.S., and F.H. Ludlam, 1953: Bubble theory of penetrative convection. Quart. J. Roy. Meteor. Soc., 79, 94-103.
- Segel, L.A., 1965: The structure of non-linear cellular solutions to the Boussinesq equations. J. Fluid Mech., 21, 345-358.
- Takeda, T., 1966: Effects of the prevailing wind with vertical shear on the convective cloud accompanied with heavy rainfall. J. Meteor. Soc. of Japan, 44, 129-143.
- Takeda, T., 1971: Numerical simulation of a precipitating convective cloud: the formation of a "long-lasting" cloud. J. Atmos. Sci., 28, 350-376.
- Telford, J.W., 1966: The convective mechanism in clear air. J. Atmos. Sci., 23, 652-666.
- Veronis, G., 1959: Cellular convection with a finite amplitude in a rotating fluid. J. Fluid Mech., 5, 401-435.

Wang, H.H., and J. Douglas, 1971: Numerical solutions of the one-dimensional primitive equations using Galerkin approximation: March 1971: Publications 320-3280, I.B.M., Palo Alto Research Centre, California, U.S.A.

Woodward, B., 1959: The motion in and around isolated thermals. Quart. J.R. Meteor. Soc., 85, 144-151.

APPENDIX A

Maximum Simplification of the Spectral Convection Equations

The very simplest form of the axisymmetric spectral equations (3.21 and 3.22) which retains non-linear interactions is the following set.

$$\hat{\theta}_{2n_A}^0 = -2\pi \hat{k}_A \mathcal{L}^{-1} n_A \hat{\psi}_{n_A}^{i_A} \hat{\theta}_{n_A}^{i_A} \quad (8.1)$$

$$\hat{\theta}_{n_A}^{i_A} = 2\pi \hat{k}_A \mathcal{L}^{-1} n_A \hat{\psi}_{n_A}^{i_A} \hat{\theta}_{2n_A}^0 - N \hat{k}_A \hat{\psi}_{n_A}^{i_A} \quad (8.2)$$

$$\lambda_A^2 \hat{\psi}_{n_A}^{i_A} = \hat{k}_A \hat{\theta}_{n_A}^{i_A} \quad (8.3)$$

where $N \equiv \frac{1}{\Theta} \frac{\partial \Theta}{\partial z}$ is the constant basic state lapse rate and i_A and n_A are the horizontal and vertical wavenumbers respectively.

This system of equations, with the addition of dissipation terms, is similar to the one discussed by Lorenz (1963).

The kinetic energy, potential energy and the mean-square potential temperature variance of the system are defined by equations (3.35), (3.36) and (3.38) respectively

$$E_K = \lambda_A^2 (\hat{\psi}_{n_A}^{i_A})^2 / 2$$

$$E_P = \mathcal{L} \hat{\theta}_{2n_A}^0 / 2\pi n_A$$

$$\overline{\theta^2} = (\hat{\theta}_{2n_A}^0)^2 + (\hat{\theta}_{n_A}^{i_A})^2$$

If we consider the case of a convectively neutral basic state ($N=0$), as in chapter 4, then $\overline{\theta^2}$ and the sum $E_P + E_K$ are both conserved.

However, it is possible to obtain analytic solutions for the non-dissipative equations (8.1 to 8.3) when $N \neq 0$. In this more general case the following constraints are satisfied.

$$2\pi n_A \mathcal{L}^{-1} (E_K + E_P) = E = \text{constant} \quad (8.4)$$

$$2N(E_K + \overline{\Theta^2}/2N) = T^2 = \text{constant} \quad (8.5)$$

The first constraint (8.4) expresses the conservation of kinetic plus potential energy as before. In the second constraint (8.5), the quantity $\overline{\Theta^2}/2N$ is rather reminiscent of the available potential energy, at least if $N > 0$ (stable stratification). If the non-linear terms were to be neglected in (8.1) and (8.2) then the system would degenerate to simple gravity wave oscillations for $N > 0$ and to exponential growth for $N < 0$; but the same constraint (8.5) would still be satisfied.

Defining $C_1 = \frac{\lambda_A^2 \pi n_A}{\mathcal{L}}$ $C_2 = \frac{N \mathcal{L}}{\pi n_A E}$ $C_3 = \frac{\hat{K}_A \pi n_A}{\mathcal{L}}$

and $\psi = \hat{\psi}_{n_A}^{i_A}$ $\theta_1 = \hat{\theta}_{n_A}^{i_A}$ $\theta_0 = \hat{\theta}_{2n_A}^0$

we can proceed to solve equations (8.1-3) by substituting from the constraints (8.4 and 8.5) into equation (8.3).

$$\begin{aligned} \text{Thus } \dot{\psi} &= \frac{\hat{K}_A}{\lambda_A^2} \theta_1 \\ &= C_3 \sqrt{[D_1 + D_2 - \psi^2][\psi^2 - (D_1 - D_2)]} \end{aligned} \quad (8.6)$$

where $D_1 = E(2 - C_2)/2C_1$

$$D_2 = \sqrt{(C_2 - 2)^2 E^2 + 4(T^2 - E^2)} / 2C_1$$

We shall consider the case where $\psi(t=0) = 0$, in which case $T^2 \geq E^2$. This implies that D_2 is real and furthermore that $D_2 \geq |D_1|$. Thus $D_1 + D_2 \geq 0$ and $D_1 - D_2 \leq 0$; in fact $D_1 + D_2 = |\psi_{\text{maximum}}|$.

Making the substitution $\psi = \sqrt{D_1 + D_2} \cos \phi$ the equation (8.6) reduces to the form

$$\frac{d\phi}{dt} = -C_3 \sqrt{2D_2 \left[1 - \left(\frac{D_1 + D_2}{2D_2} \right) \sin^2 \phi \right]} \quad (8.7)$$

Since $0 \leq (D_1 + D_2)/2D_2 \leq 1$ this equation has solutions of the following form

$$t(\phi) = \frac{1}{C_3 \sqrt{2D_2}} \left[F(\pi/2 | m) - F(\gamma | m) \right] \quad (8.8)$$

where $m = (D_1 + D_2)/2D_2$

$$\gamma = \arccos(\psi / \sqrt{D_1 + D_2})$$

$F(\gamma | m)$ is the elliptic function of the first kind.

If we consider the case of wavenumber one ($L_A = \lambda_A = 1$) then the initial bubble is a superposition of $\hat{\Theta}_2^0 \sin 2\pi Z$ and $\hat{\Theta}_1^1 \sin \pi Z J_0(\hat{k}_1 r)$. The constant E , equal to $\hat{\Theta}_2^0(t=0)$, is thus defined by the initial position of the bubble centre, Z_i .

$$Z_i < 1/2 \quad \text{implies} \quad E > 0$$

$$Z_i = 1/2 \quad \text{implies} \quad E = 0$$

$$Z_i > 1/2 \quad \text{implies} \quad E < 0$$

The case $E=0$ has a slightly different solution than (8.8) because of the definition of C_2 .

Solution (8.8) for ψ can be substituted into equations (8.1 to 8.3) to obtain solutions for θ_0 and θ_1 . In the case $N \gg 0$ these solutions take the form of modified gravity waves. In all other cases (neutral, weakly stable, or unstable) the solutions represent a bubble rising to the top of the box through its own buoyancy, and then being carried across the top and down the other side through the strength of the vortex it has

generated. If N is small and positive the upward motion is inhibited somewhat by the stable stratification. If $N < 0$ the bubble rises and grows very quickly in amplitude because of the energy drawn from the unstable basic state. However, as the bubble rises Θ_0 becomes large and negative (stably stratified) so that in equation (8.2) it tends to neutralize the effect of negative N thus preventing unbounded growth.

In the special case $N = 0$ which implies $C_2 = 0$, the time for the bubble to rise to the top of the box is given by

$$t(\pi/2) = \frac{1}{C_3} \sqrt{\frac{C_1}{2T}} F\left(\frac{\pi}{2} \middle| \frac{E+T}{2T}\right)$$

Thus the ascent time for a non-dissipative bubble rising through neutral surroundings is inversely proportional to the square root of the initial perturbation in the potential temperature field. This result seems to be approximately correct even for the high resolution non-dissipative models and can be used as a rule-of-thumb for estimating running times.

APPENDIX B

On Non-linear Computational Instability

In chapter 4 we mentioned non-linear computational instability in connection with the rapid transfer of potential temperature variance to the smallest scales. We make this connection because it has been shown (eg. Phillips (1959), Robert et al (1970)) that this type of instability occurs because of interactions of the smallest wavelengths in the model. This type of instability may also occur in the presence of damping, and it would be desirable to find some criterion which would indicate if this phenomenon is likely to occur. The condition which is important in this regard is believed to be the development of an infinite gradient in the potential temperature and/or vertical motion fields.

One such criterion may be obtained by consideration of the vertical gradients of potential temperature and vertical velocity at the cap of a bubble. Consider the point on the vertical axis of the bubble which moves along with the cap. This point is, in effect, on a sort of frontal surface and thus $\left| \frac{\partial w}{\partial z} \right|$ and $\left| \frac{\partial \theta}{\partial z} \right|$ will always be relative maxima. The height of this point we denote by z_f . The governing equations (in dimensional form) are

$$\frac{d\theta}{dt} = -k_1 \theta \quad (9.1)$$

$$\frac{dw}{dt} = C\theta - C_p \frac{\partial(\pi \theta)}{\partial z} - k_2 w \quad (9.2)$$

where $C = \frac{g}{H}$ and k_1, k_2 are damping coefficients. In order to make the problem tractable we will drop the pressure term in equation (9.2). The significance of this omission will be discussed later. If we differentiate equations (9.1) and (9.2) with respect to z and re-arrange the results we obtain

$$\frac{d}{dt}\left(\frac{\partial \theta}{\partial z}\right) = -\frac{\partial \theta}{\partial z} \frac{\partial w}{\partial z} - k_1 \frac{\partial \theta}{\partial z} \quad (9.3)$$

$$\frac{d}{dt}\left(\frac{\partial w}{\partial z}\right) = C \frac{\partial \theta}{\partial z} - \left(\frac{\partial w}{\partial z}\right)^2 - k_2 \frac{\partial w}{\partial z} \quad (9.4)$$

If we now set $X = \frac{\partial w}{\partial z}$ and $Y = \frac{\partial \theta}{\partial z}$ we may treat the total time derivatives as a simple time derivative moving with the flow and the system reduces to two ordinary differential equations

$$\frac{dY}{dt} = -XY - k_1 Y \quad (9.5)$$

$$\frac{dX}{dt} = CY - X^2 - k_2 X \quad (9.6)$$

which have the solutions

$$Y(t) = Y_0 k_1 k_2 (k_2 - k_1) / L(t) \quad (9.7)$$

$$X(t) = C Y_0 k_1 k_2 (1 - \exp[(k_1 - k_2)t]) / L(t)$$

where $L(t) = (k_2 - k_1) e^{k_1 t} (k_1 k_2 + C Y_0) + C Y_0 (k_1 \exp[(k_1 - k_2)t] - k_2)$

and $Y_0 = \left(\frac{\partial \theta}{\partial z}\right)_{t=0}$ at $Z = Z_f$

There are similar solutions for the special cases $k_1 = k_2$ and k_1 or k_2 equal zero.

Singularities will occur in $X(t)$ and $Y(t)$ unless $C Y_0 + k_1 k_2 \geq 0$. In the special case where $C Y_0 = -k_1 k_2$ then the steady state solutions of (9.5) and (9.6) will be unstable with respect to small perturbations. Thus the stability criterion for this system is

$$k_1 k_2 > \frac{d}{dt} \left| \frac{\partial \theta}{\partial z} \right|_{t=0} \quad \text{at} \quad Z = Z_f \quad (9.8)$$

Before discussing the interpretation of this result we should mention the effect of the omission of the pressure term. The calculated pressure fields of Arnason et al (1968) indicate that the bubble behaves initially as a blob of fluid moving slowly through a fluid of different viscosity and ultimately as a solid object moving through a viscous fluid (see Lamb, pp. 595-601). Thus the pressure force can be considered as a dissipative mechanism for kinetic energy, and can be incorporated roughly into K_2 .

The interpretation of the instability criterion is a little difficult due to the drastic simplifications we have made with respect to the dissipative terms. Subsequent discussion will be confined to Fickian diffusion, in which case K_1 and K_2 may be thought of as diffusion coefficients multiplied by some wavenumber squared. An obvious choice for the wavenumber, is that corresponding to the smallest vertical scale, that is twice the vertical gridlength. If we further assume that the eddy diffusion coefficients are the same, then the criterion becomes

$$\gamma_0 > \frac{l_z^2}{\pi^2} \left(\frac{g}{\Theta} \left| \frac{\partial \Theta}{\partial z} \right|_{t=0}^{z=z_f} \right)^{1/2} \quad (9.9)$$

where γ_0 is the eddy diffusion coefficient and l_z is the effective vertical grid size.

Several integrations were performed to test this criterion and the results are compared with (9.9) in table . The observed critical viscosities are obtained by finding the coefficient of viscosity which will prevent the saturation of the shortest wavescales (and the resulting reverse cascade of energy). These observed critical viscosities are very roughly determined and thus the criterion (9.9) would seem to be of some use.

TABLE 4 Comparison of observed and theoretical critical viscosities necessary for the maintenance of computational stability.

Resolution	$\left \frac{\partial \theta}{\partial z} \right _{t=0}^{z=z_f}$ $^{\circ}\text{C}/\text{met.}$	Critical viscosity	
		Theoretical	Observed
6x6	1.2×10^{-3}	160	115
6x6	2.4×10^{-4}	72	52
6x6	1.2×10^{-2}	510	370
8x8	1.2×10^{-3}	90	85

APPENDIX C

The Transform Method Applied to the Integration of the Spectral Convection Equations

The transform method which is about to be described, was used to integrate the three-dimensional equations and the 16×16 set of two-dimensional equations. For the 16×16 set (544 components) it was approximately 4 times as fast as the interaction coefficient technique.

The use of the transform technique will be demonstrated by applying it to Eq. (3.22), the axisymmetric thermodynamic equation.

$$\hat{\Theta}_A = \hat{\Theta}_{n_A}^{i_A} = - \iint_0^1 J(\sum_B \hat{\psi}_B F_B, \sum_c \hat{\Theta}_c G_c) G_A dr dz - \hat{K}_A \hat{\psi}_A N - \nu_{\hat{\Theta}} \lambda_A^2 \hat{\Theta}_A \quad (10.1)$$

The transform method, of course, applies only to the non-linear terms, so that the last two (linear) terms will be dropped in the discussion that follows. After expansion and re-grouping Eq. (10.1) becomes

$$\begin{aligned} \hat{\Theta}_A = - \iint_0^1 [\sum_{n_B} R_1(r, n_B) \sin n_B \pi z \sum_{n_c} R_2(r, n_c) \cos n_c \pi z \\ + \sum_{n_B} R_3(r, n_B) \cos n_B \pi z \sum_{n_c} R_4(r, n_c) \sin n_c \pi z] N_A J_0(K_A r) \sin n_A \pi z r dr dz \end{aligned} \quad (10.2)$$

where $R_1(r, n_B) = \sum_{i_B} \hat{K}_B N_B \hat{\psi}_{n_B}^{i_B} J_0(\hat{K}_B r)$

$$R_2(r, n_c) = \sum_{i_c} n_c \pi N_c \hat{\Theta}_{n_c}^{i_c} J_0(\hat{K}_c r)$$

$$R_3(r, n_B) = \sum_{i_B} n_B \pi N_B \hat{\psi}_{n_B}^{i_B} J_1(\hat{K}_B r)$$

$$R_4(r, n_c) = \sum_{i_c} \hat{K}_c N_c \hat{\Theta}_{n_c}^{i_c} J_1(\hat{K}_c r)$$

It should be noted that R_1 , R_2 , R_3 and R_4 are only functions of r

and the vertical wavenumber n . In the numerical procedure the first step is always the summation over r wavenumber (i_b or i_c) to produce functions R_1 , R_2 , R_3 and R_4 . The second step involves integrations over z .

Thus Eq. (10.2) can be re-written as

$$\hat{\Theta}_A = - \int_0^\alpha \sum_{n_B} \sum_{n_C} [R_1(r, n_B) R_2(r, n_C) S_{AB}^C + R_3(r, n_B) R_4(r, n_C) S_{AC}^B] N_A J_0(\hat{k}_A r) r dr$$

where
$$S_{AB}^C = \int_0^1 \cos n_C \pi z \sin n_A \pi z \sin n_B \pi z dz \quad (10.3)$$

which takes on the values $1/4$, $-1/4$ or 0 as before. The sum over n_B and n_C is trivial because most values of S_{AB}^C are equal to zero. Thus if

$$P(r, n_A) = \sum_{n_B} \sum_{n_C} [R_1(r, n_B) R_2(r, n_C) S_{AB}^C + R_3(r, n_B) R_4(r, n_C) S_{AC}^B]$$

then Eq. (10.3) becomes

$$\hat{\Theta}_A = - \int_0^\alpha P(r, n_A) N_A J_0(\hat{k}_A r) r dr \quad (10.4)$$

This integral must be performed by quadrature. To save computing time, it is desirable to reduce the number of points used in the quadrature to a minimum. Gaussian quadrature and other schemes were not successful, so a different technique was devised.

The integral of an arbitrary function $Q(r)$ can be approximated as follows

$$\int_0^\alpha Q(r) N_A J_0(\hat{k}_A r) r dr \approx \sum_{i=0}^{S-1} w(r_i, \hat{k}_A) Q(r_i)$$

where $w(r_i, \hat{k}_A)$ is a set of weights which will be different for each value of $\hat{k}_A$. We require that the quadrature scheme be exact for $Q(r) = J_0(k_j r)$

$j = 1, 2, 3, 4, \dots, s$. This will define a set of linear algebraic equations.

$$W(0, \hat{k}_i) J_0(0) + W(r_i, \hat{k}_i) J_0(\hat{k}_i r_i) + \dots + W(r_{s-1}, \hat{k}_i) J_0(\hat{k}_i r_{s-1}) = 0$$

$$W(0, \hat{k}_i) J_0(0) + W(r_i, \hat{k}_i) J_0(\hat{k}_i r_i) + \dots + W(r_{s-1}, \hat{k}_i) J_0(\hat{k}_i r_{s-1}) = \frac{N_i J_0^2(\hat{k}_i)}{2}$$

$$W(0, \hat{k}_i) J_0(0) + W(r_i, \hat{k}_i) J_0(\hat{k}_i r_i) + \dots + W(r_{s-1}, \hat{k}_i) J_0(\hat{k}_i r_{s-1}) = 0$$

The solution of this set of equations by matrix inversion gives the set of weights $W(r_i, \hat{k}_j)$. Thus

$$\hat{\theta}_A = - \int_0^{\infty} P(r, n_A) N_A J_0(\hat{k}_A r) r dr \approx - \sum_{i=0}^{s-1} P(r_i, n_A) W(r_i, \hat{k}_A) \quad (10.5)$$

The accuracy with which the quadrature is performed depends on the value of S chosen. In general the function $P(r, n_A)$ is composed of the summation of terms such as $J_0(\hat{k}_B r) J_0(\hat{k}_C r)$ and $J_1(\hat{k}_B r) J_1(\hat{k}_C r)$. These product terms can be expanded in a Fourier-Bessel series. Thus

$$J_0(\hat{k}_B r) J_0(\hat{k}_C r) = \sum_j T_j J_0(\hat{k}_j r)$$

$$J_1(\hat{k}_B r) J_1(\hat{k}_C r) = \sum_j V_j J_0(\hat{k}_j r)$$

If i_B and i_C are the rootnumbers corresponding to $\hat{k}_B$ and $\hat{k}_C$ it is found that the coefficients T_j and V_j diminish rapidly with increasing j if $j > i_C + i_B$. As a result of this, the value of S was set equal to $3.0 i_{max}$. For example, in an 8×8 system, the quadrature was performed over $3 \times 8 = 24$ points. This type of quadrature with $S = 24$ was only slightly less accurate than a 400 point Simpsons Rule quadrature.

The results of a time integration of an 8×8 system using the transform method were indistinguishable from the results using the traditional interaction coefficient technique.

It might be asked why Fast Fourier Transform methods (Cooley and Tookey (1965)) were not used on the Z integrations. There were two reasons for this. Firstly, all interactions were of the cos-sin-sin type which gives a very sparse interaction matrix. Secondly, the domain of integration was from 0 to π rather than 0 to 2π and the Fast Fourier Transform does not work so well over this domain.