

THE NUCLEAR MANY-BODY PROBLEM
USING THE BOUNDARY CONDITION MODEL

by

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ABSTRACT

In this thesis the Feshbach-Lomon Boundary Condition Model for nucleon-nucleon interactions is applied to the Independent Pair Model (or Brueckner theory) of infinite nuclear matter, and to the modified Brueckner theory which includes also the hole-hole interactions. This is accomplished by the construction of a pseudopotential which is equivalent to the Boundary Condition Model in the two-nucleon problem. The pseudopotential is then used in the many-body problem in place of the complicated phenomenological nucleon-nucleon potentials used in more standard treatments. Because of the very simple form of the pseudopotential, the integral equation for the nuclear matter K matrix can be handled relatively easily and without resorting to perturbative approaches. Further, the dependence of the K matrix on the centre of mass momentum of the interacting pair is treated more fully than in previous treatments. Numerical calculations are performed for the case when the nucleon-nucleon interaction vanishes in all but 1S_0 states, and in particular the effects arising from the hole-hole interactions and from the centre of mass momentum dependence of the K matrix are discussed. The singularity of the K matrix, the presence of which has been associated with a superconducting state of the system, is investigated for the case when the interaction potential vanishes in all but 1S_0 states. The question of whether a singularity exists when the interacting particles have a non-zero centre of mass momentum is also discussed. Finally a value for the size of the energy gap is found.

TABLE OF CONTENTS

	<u>Page</u>
ACKNOWLEDGMENTS.....	ii
ABSTRACT.....	iii
CHAPTER I	
INTRODUCTION AND SUMMARY.....	1
CHAPTER II	
THE FESHBACH-LOMON BOUNDARY CONDITION MODEL AND ITS REFORMULATION IN TERMS OF A PSEUDOPOTENTIAL	7
2.1 The Feshbach-Lomon Boundary Condition Model.....	7
2.2 The Pseudopotential.....	12
CHAPTER III	
THE TWO-BODY SCATTERING PROBLEM USING THE PSEUDOPOTENTIAL.....	21
3.1 Some Aspects of Scattering Theory.....	21
3.2 Use of the Pseudopotential.....	29
CHAPTER IV	
THE NUCLEAR MANY-BODY PROBLEM AND THE INDEPENDENT PAIR MODEL.....	37
4.1 Introduction.....	37
4.2 The Independent Pair Model of Nuclear Matter.....	40
CHAPTER V	
USE OF THE PSEUDOPOTENTIAL IN THE NUCLEAR MANY- BODY PROBLEM.....	67
5.1 The K Matrix Elements for Singlet Spin States.....	69

TABLE OF CONTENTS (Cont'd)

	<u>Page</u>
CHAPTER V (Cont'd)	
5.2 The K Matrix Elements for Triplet Spin States.....	88
5.3 Expressions for the Potential Energy per Particle and the Single Particle Potential using the Pseudopotential.....	97
CHAPTER VI	
INTERACTION IN 1S_0 STATES.....	107
CHAPTER VII	
SINGULARITIES IN THE K MATRIX AND THE ENERGY GAP.....	125
CHAPTER VIII	
CONCLUSIONS.....	142
APPENDIX I.....	145
APPENDIX II.....	150
APPENDIX III.....	154
FIGURE CAPTIONS.....	157
FIGURES I - VI.....	159
REFERENCES.....	165

CHAPTER I

INTRODUCTION AND SUMMARY

This thesis will be devoted to applying the Feshbach-Lomon Boundary Condition Model for nucleon-nucleon scattering to the Independent Pair Model (or Brueckner theory) of an infinite nucleus. In this Introduction we shall give a brief summary of some of the previous and more standard works and we shall also indicate the contribution of this thesis to the field.

The problem of explaining the properties of nuclei on the basis of the forces between the constituents is an old one in physics. The problem is in fact two-fold, since it involves, firstly, knowing the nature of the forces between nucleons, and secondly, once these forces are known, being able to handle the equations for a many-particle system. The first is indeed still an unsolved problem. Meson theory has been moderately successful in explaining the nuclear force and giving some of its qualitative features but most of the information about the forces between two nucleons has been obtained from nucleon-nucleon scattering experiments. To correlate the scattering data it has been customary to

represent the nuclear force by phenomenological local potentials. These potentials have become increasingly complex as more and more scattering data has been obtained.

Assuming that nuclear forces can be represented adequately by such phenomenological local potentials, the problem is to calculate properties of nuclei on the basis of these potentials. We have no knowledge of many-body forces so at most the theoretical nuclear physicist hopes to explain the properties of nuclei on the basis of the "known" two-body forces. In particular we wish to be able to calculate the energy per particle of a hypothetical configuration of nucleons called nuclear matter. The concept of nuclear matter results from an extrapolation of the semi-empirical mass formula of Weiszacker, which formula gives a good over-all fit to the binding energy of all known nuclei. It is believed on the basis of this formula that a very large number A of $A/2$ protons and $A/2$ neutrons would form a stable configuration with an energy per particle of approximately -15 Mev, provided that the Coulomb force between the protons did not exist. This hypothetical configuration is called nuclear matter.

It was not until the work of Brueckner and associates¹⁾ that a systematic method for calculating the energy per particle of nuclear matter was devised. Brueckner's idea was to replace the potential by a reactance matrix

(or K matrix), which is similar to the reactance matrix of scattering theory, and express the properties of nuclear matter (e.g., the energy per particle) in terms of this quantity. Bethe²⁾ has discussed the theory in detail pointing out the complications and developing its logical consistency. The most lucid description of the physics behind the theory however has been given by Gomes, Walecka, and Weisskopf.³⁾ Brueckner's theory can be looked upon as an "independent pair model" of nuclear matter in that it consists of taking into account the interaction of any pair of particles as exactly as possible, neglecting the interactions of all other particles among themselves and with the pair. The presence of the other particles is indirectly felt however through the exclusion principle. The energy of the system is then the sum of all the two body correlation energies. Extensive numerical calculations using the Brueckner theory of nuclear matter and phenomenological nucleon-nucleon potentials have been carried out, most notably by Brueckner and Gammel¹⁾ and by Moszkowski and Scott.^{4,5)} In both cases excellent results have been obtained. It has been conjectured that an improvement over Brueckner's theory can be obtained by including the so-called "hole-hole interactions".^{6,7)} (These are explained later.) At present no extensive numerical calculations have been performed to investigate this conjecture. It is felt however that the inclusion of

these terms will cause large corrections to the form of the K matrix but quite small corrections to the energy per particle.⁸⁾

In this thesis we shall also use the Brueckner theory (with the hole-hole interactions included) for our discussion of nuclear matter, but our starting point, namely the description of nuclear forces, will be different from that used by other workers. The description we use is the Feshbach-Lomon Boundary Condition Model⁹⁾: the interaction in each state is represented by an energy independent boundary condition on the logarithmic derivative of the wave function of the two nucleons at an energy independent core. In presenting their model Feshbach and Lomon were motivated by developments in the meson theory of nuclear forces which suggested that 1) use of simplified local potentials for describing nuclear forces is only possible if the nucleons are relatively far apart and, 2) there exists some region in which many virtual mesons are present, or alternatively in which the interaction energy is very large so that the behavior of the nucleons is relatively insensitive to their kinetic energy at infinity. This region has no sharp boundary but one can say that an energy independent description will hold for sufficiently small interparticle distances. These two developments suggest the following phenomenological model for the nucleon-nucleon interaction. The energy independent core is represented by means of a set of energy

independent boundary conditions at some energy independent core radius r_0 which may be state dependent. For r greater than r_0 the nuclear forces are represented by potentials indicated by meson theory. In their original analysis of the experimental data however, Feshbach and Lomon dropped the local potential external to the core. The resulting model is called the Boundary Condition Model (BCM) and forms the starting point of our work.

In order to apply the BCM to the Brueckner theory we must reformulate the BCM in terms of potential. This potential (or pseudopotential), which we give in the next Chapter, has an extremely simple form. It is much simpler than the usual phenomenological nucleon-nucleon potentials, and for this reason leads to a great deal of simplification of the many-body equations.

In the course of our discussion of the Brueckner theory we go beyond previous treatments by introducing methods to handle the centre of mass momentum dependence of the nuclear matter K matrix. We introduce certain expansions only the first terms of which have been given before.

In order to give some numerical results, we investigate in some detail the case when the interaction vanishes in all but 1S_0 states. In this case the exact expression for the nuclear matter K matrix can be written quite simply. We investigate in particular the centre of mass momentum

dependence of the K matrix and the effect of the hole-hole interactions.

As a final point, we discuss, for the case when the interaction vanishes in all but 1S_0 states, the singularity of the nuclear matter K matrix which has been associated with a superconducting state of nuclear matter.¹⁰⁾

CHAPTER II

THE FESHBACH-LOMON BOUNDARY CONDITION MODEL AND ITS REFORMULATION IN TERMS OF A PSEUDOPOTENTIAL

This Chapter will be devoted to a discussion of the Feshbach-Lomon Boundary Condition Model⁹⁾ of nucleon-nucleon scattering, and to a recasting of this model into a form making it more amenable for use in the nuclear many-body problem.

2.1 The Feshbach-Lomon Boundary Condition Model.

The Feshbach-Lomon approach to nucleon-nucleon scattering is radically different from the usual phenomenological models which have been suggested. Most of the models which have been suggested for the correlation of nucleon-nucleon scattering data assume a local potential $V(\vec{r}, \sigma, \tau)$ between the nucleons, where \vec{r} is the internucleon radius vector, σ represents the spin operators of each nucleon, and τ the isotopic spin. These models have become more and more complex in structure as more scattering data has become available.¹¹⁾

Progress in the meson theory of nuclear forces has indicated however that use of simplified local potentials for describing the nuclear forces is only possible if the

nucleons are relatively far apart. When the distance between the nucleons is less than one meson Compton wavelength, higher order effects, corresponding to the exchange of many mesons, must be taken into account. In this case, which is important in high-energy nucleon-nucleon scattering, a non-local potential must be used to describe the nuclear forces.

Motivated by these developments, Feshbach and Lomon introduce an extreme non-local interaction to handle these higher order effects. In particular, they consider the interaction in each state to be represented by an energy independent boundary condition on the logarithmic derivative of the wavefunction of the two nucleons, this logarithmic derivative taken at some energy independent core radius r_0 which may be state dependent. For r greater than r_0 , they would assume the usual local potentials indicated by meson theory, for example, the static one and two pion exchange potentials (OPEP and TPEP). It should be noted that Feshbach and Lomon represent the higher order effects by an energy independent boundary condition. It is felt that in the region where the higher order effects are important (and thus where the interaction energy is large) that the behavior of the nucleons will be relatively insensitive to their kinetic energy at infinity.

In their original analysis of the experimental data, however, Feshbach and Lomon dropped the local potential external to the core. The resulting approximation, which we shall call the Boundary Condition Model (BCM), will be

the starting point for the work of this thesis. The more sophisticated model, that is, boundary condition plus potential tail, is now being used by Feshbach and Lomon for an analysis of nucleon-nucleon scattering data. For the present, however, let us proceed with a quantitative discussion of the BCM.

The basic assumption of the BCM is expressed in terms of the boundary condition

$$\left. \frac{d\Psi(r)}{dr} \right|_{r=r_0} = \frac{F}{r_0} \Psi(r_0)$$

where Ψ is a state or group of states with the same total angular momentum, and where F and r_0 are energy-independent parameters. In particular, for two nucleons in a singlet spin state, $\Psi = \psi_\ell(r)$, the radial wave function of the two nucleons in the center of mass coordinate system (ℓ is the orbital angular momentum in units of \hbar), and the boundary condition becomes

$$(2.1) \quad \left. \frac{d\psi_\ell(r)}{dr} \right|_{r=r_{0\ell}} = \frac{f_\ell}{r_{0\ell}} \psi_\ell(r_{0\ell})$$

A similar simple structure holds for the wave function of two nucleons in a triplet spin state when J , the total angular momentum, equals ℓ . In this case, $\Psi = \psi_{JJ}(r)$

and the boundary condition is

$$(2.2) \quad \left. \frac{d\psi_{JJ}(r)}{dr} \right|_{r=r_{oJJ}} = \frac{f_{JJ}}{r_{oJJ}} \psi_{JJ}(r_{oJJ})$$

For the other two triplet states (that is, those where $\ell=J\pm 1$) the situation is not quite so simple since there can be coupling between these states. In this case, Ψ is a unicumular matrix

$$\Psi = \begin{bmatrix} \psi_{J\ell} \end{bmatrix} = \begin{pmatrix} \psi_{J,J-1}(r) \\ \psi_{J,J+1}(r) \end{pmatrix}$$

and F is a real 2x2 hermitian matrix

$$F = \begin{pmatrix} f_{J,J-1} & f_J^{(t)} \\ f_J^{(t)} & f_{J,J+1} \end{pmatrix}$$

and the boundary condition is

$$(2.3) \quad \begin{pmatrix} \frac{d\psi_{J,J-1}}{dr} \\ \frac{d\psi_{J,J+1}}{dr} \end{pmatrix}_{r=r_{oJ}} = \frac{1}{r_{oJ}} \begin{pmatrix} f_{J,J-1} & f_J^{(t)} \\ f_J^{(t)} & f_{J,J+1} \end{pmatrix} \begin{pmatrix} \psi_{J,J-1}(r_{oJ}) \\ \psi_{J,J+1}(r_{oJ}) \end{pmatrix}$$

Thus equations (2.1) to (2.3) form the BCM. The boundary conditions are clearly equivalent to the phase shifts and are therefore just another representation of the data. Indeed for singlet states of the neutron-proton system

$$\psi_\ell(r) = A_\ell \left[h_\ell^{(2)}(kr) + e^{2i\delta_\ell} h_\ell^{(1)}(kr) \right] \quad \text{for } r > r_{o\ell}$$

where A_ℓ is a normalization constant, δ_ℓ is the phase shift, k is the relative wave number, and the h 's are spherical Hankel functions:

$$h_\ell^{(1)}(x) = j_\ell(x) + in_\ell(x)$$

$$h_\ell^{(2)}(x) = j_\ell(x) - in_\ell(x)$$

where j_ℓ is a spherical Bessel function and n_ℓ is a spherical Neumann function. Thus we can write

$$\psi_\ell(r) = 2A_\ell e^{i\delta_\ell} \cos \delta_\ell \left[j_\ell(kr) - \tan \delta_\ell n_\ell(kr) \right]$$

direct substitution of which into equation (2.1) leads to

$$(2.4) \quad \tan \delta_\ell = \frac{f_\ell j_\ell(kr_{o_\ell}) - kr_{o_\ell} j'_\ell(kr_{o_\ell})}{f_\ell n_\ell(kr_{o_\ell}) - kr_{o_\ell} n'_\ell(kr_{o_\ell})}$$

Here and in what follows a prime on a function means differentiation with respect to its argument. This equation has been given by Feshbach and Lomon (see their equation 7). They also show the relationship between the triplet boundary condition parameters, and the phase shifts and mixing parameters for these states. We need not write down these equations here.

2.2 The Pseudopotential

We would like to apply the BCM to the nuclear many-body problem. We shall discuss the many-body problem in some detail in Chapter IV, but it is sufficient for our purposes now to state that it is formulated in terms of the nucleon-nucleon potential. Now the BCM contains no reference to a potential, so we shall in this Section reformulate the BCM in terms of a potential as the first step towards applying the model to the many-body problem. This potential, or pseudopotential, which we shall now construct will then be used in the many-body theory in place of the very complicated phenomenological potentials used in more standard treatments.

Thus, we are seeking a potential which vanishes for $r > r_0$ and such that the wave function of the Schrodinger equation describing the scattering of two particles via this potential satisfies the required boundary conditions, that is, equation (2.1), (2.2), or (2.3) depending upon the angular momentum of the state in question. We assert that such a potential is

$$\begin{aligned}
 (2.5a) \quad V(r) = & \frac{\hbar^2}{m} \sum_{\ell=0}^{\infty} \left[\frac{f_{\ell}}{r_{0\ell}} \delta(r-r_{0\ell}^+) - \delta(r-r_{0\ell}^-) \frac{\partial}{\partial r} \right] P_{\ell 0} \\
 & + \frac{\hbar^2}{m} \sum_{J=0}^{\infty} \left\{ \left[\frac{f_{JJ}}{r_{0JJ}} \delta(r-r_{0JJ}^+) - \delta(r-r_{0JJ}^-) \frac{\partial}{\partial r} \right] P_{JJ1} \right. \\
 & \left. + \left[\frac{S}{r_{0J}} \delta(r-r_{0J}^+) - \delta(r-r_{0J}^-) \frac{\partial}{\partial r} \right] (P_{J,J+1,1} + P_{J,J-1,1}) \right\}
 \end{aligned}$$

where m is the mass of one nucleon, r is the distance between the two nucleons, the f 's and r_0 's are the boundary parameters occurring in equations (2.1), (2.2), and (2.3), and the P 's are projection operators having the property that $P_{J\ell S}$ gives a non-zero result only when operating on a state having total angular momentum J , orbital angular momentum ℓ , and spin S . The operator S is defined by

$$(2.5b) \quad S_{J,J-1,1}^M = f_{J,J-1} \psi_{J,J-1,1}^M + f_J^{(t)} \psi_{J,J+1,1}^M$$

$$(2.5c) \quad S_{J,J+1,1}^M = f_J^{(t)} \psi_{J,J-1,1}^M + f_{J,J+1} \psi_{J,J+1,1}^M$$

$$(2.5d) \quad S_{J\ell S}^M = 0 \text{ for } \ell \neq J \pm 1, S \neq 1$$

where the $\psi_{J\ell S}^M$ are the orthonormal "spin-angle" wave functions.¹²⁾ (They are eigenfunctions of J^2 and J_z .)

Finally, the δ -functions occurring in equation (2.5a) are Dirac δ -functions. The arguments of the δ -functions contain, for example, $r_{0\ell}^+$ and $r_{0\ell}^-$ by which we mean

$$(2.5e) \quad r_{0\ell}^{\pm} = r_{0\ell} \pm \varepsilon$$

where ε is an infinitesimally small number. We interpret the potential in the sense of $\varepsilon \rightarrow 0$.

We prove our assertion as follows. The Schrodinger equation describing the scattering of two particles (each of mass m) by a potential V is, in the centre of mass system,

$$(2.6) \quad \nabla^2 \Psi + k^2 \Psi = \frac{m}{\hbar^2} V \Psi$$

where k is the relative wave number. Now we can write

$$(2.7) \quad \Psi(\vec{r}, \sigma) = \sum_{J=0}^{\infty} \sum_{M=-J}^{+J} \sum_{S=0}^1 \sum_{\ell=|J-S|}^{J+S} \psi_{J\ell S}(r) \psi_{J\ell S}^M(\hat{r}, \sigma)$$

where \hat{r} stands for the angles the radius vector \vec{r} makes with some coordinate axes, and σ stands for the spin operators of the particles. Substituting equation (2.7) into the LHS of equation (2.6) yields

LHS of (2.6) =

$$= \sum_{J=0}^{\infty} \sum_{M=-J}^{+J} \sum_{S=0}^1 \sum_{\ell=|J-S|}^{J+S} \psi_{J\ell S}^M \left\{ \psi_{J\ell S}'' + \frac{2}{r} \psi_{J\ell S}' - \frac{\ell(\ell+1)}{r^2} \psi_{J\ell S} + k^2 \psi_{J\ell S} \right\}$$

If V is given by equation (2.5), then substituting equation (2.7) into the RHS of equation (2.6) yields

RHS of (2.6) =

$$\begin{aligned} &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \left\{ \psi_{\ell\ell 0}^m \left[\frac{f_{\ell}}{r_{o\ell}} \delta(r-r_{o\ell}^+) - \delta(r-r_{o\ell}^-) \frac{\partial}{\partial r} \right] \psi_{\ell\ell 0} \right\} \\ &+ \sum_{J=0}^{\infty} \sum_{M=-J}^{+J} \left\{ \psi_{JJ1}^M \left[\frac{f_{JJ}}{r_{oJJ}} \delta(r-r_{oJJ}^+) - \delta(r-r_{oJJ}^-) \frac{\partial}{\partial r} \right] \psi_{JJ1} \right. \\ &+ \psi_{J,J-1,1}^M \left[\frac{f_J^{(t)}}{r_{oJ}} \delta(r-r_{oJ}^+) \psi_{J,J+1,1} + \left(\frac{f_{J,J-1}}{r_{oJ}} \delta(r-r_{oJ}^+) - \delta(r-r_{oJ}^-) \frac{\partial}{\partial r} \right) \psi_{J,J-1,1} \right] \\ &\left. + \psi_{J,J+1,1}^M \left[\frac{f_J^{(t)}}{r_{oJ}} \delta(r-r_{oJ}^+) \psi_{J,J-1,1} + \left(\frac{f_{J,J+1}}{r_{oJ}} \delta(r-r_{oJ}^+) - \delta(r-r_{oJ}^-) \frac{\partial}{\partial r} \right) \psi_{J,J+1,1} \right] \right\} \end{aligned}$$

Equating both sides of equation (2.6) and using the orthogonality of the ψ 's leads to the following equations

$$(2.8a) \quad \psi''_{\ell\ell 0} + \frac{2}{r} \psi'_{\ell\ell 0} - \frac{\ell(\ell+1)}{r^2} \psi_{\ell\ell 0} + k^2 \psi_{\ell\ell 0} =$$

$$= \left[\frac{f_{\ell}}{r_{o_{\ell}}} \delta(r-r_{o_{\ell}}^+) - \delta(r-r_{o_{\ell}}^-) \frac{\partial}{\partial r} \right] \psi_{\ell\ell 0}$$

$$(2.8b) \quad \psi''_{JJ1} + \frac{2}{r} \psi'_{JJ1} - \frac{J(J+1)}{r^2} \psi_{JJ1} + k^2 \psi_{JJ1} =$$

$$= \left[\frac{f_{JJ}}{r_{o_{JJ}}} \delta(r-r_{o_{JJ}}^+) - \delta(r-r_{o_{JJ}}^-) \frac{\partial}{\partial r} \right] \psi_{JJ1}$$

$$(2.8c) \quad \psi''_{J,J-1,1} + \frac{2}{r} \psi'_{J,J-1,1} - \frac{(J-1)J}{r^2} \psi_{J,J-1,1} + k^2 \psi_{J,J-1,1} =$$

$$= \frac{f_J^{(t)}}{r_{o_J}} \delta(r-r_{o_J}^+) \psi_{J,J+1,1} + \left(\frac{f_{J,J-1}}{r_{o_J}} \delta(r-r_{o_J}^+) - \delta(r-r_{o_J}^-) \frac{\partial}{\partial r} \right) \psi_{J,J-1,1}$$

$$(2.8d) \quad \psi''_{J,J+1,1} + \frac{2}{r} \psi'_{J,J+1,1} - \frac{(J+1)(J+2)}{r^2} \psi_{J,J+1,1} + k^2 \psi_{J,J+1,1} =$$

$$= \frac{f_J^{(t)}}{r_{o_J}} \delta(r-r_{o_J}^+) \psi_{J,J-1,1} + \left(\frac{f_{J,J+1}}{r_{o_J}} \delta(r-r_{o_J}^+) - \delta(r-r_{o_J}^-) \frac{\partial}{\partial r} \right) \psi_{J,J+1,1}$$

We will now show that $\psi_{\ell\ell 0}$, the radial wave function of two particles in a singlet spin state with orbital angular momentum ℓ , satisfies the boundary condition (2.1).

Integrating both sides of (2.8a) from α and β where $\alpha < r_{o_\ell}^-$ and $\beta > r_{o_\ell}^+$ yields

$$\begin{aligned} \psi'_{\ell\ell 0}(\beta) - \psi'_{\ell\ell 0}(\alpha) + 2 \left[\frac{\psi_{\ell\ell 0}(\beta)}{\beta} - \frac{\psi_{\ell\ell 0}(\alpha)}{\alpha} \right] + \int_{\alpha}^{\beta} dr \psi_{\ell\ell 0}(r) \left[\frac{1}{r^2} - \frac{\ell(\ell+1)}{r^2} + k^2 \right] \\ = \frac{f_\ell}{r_{o_\ell}} \psi_{\ell\ell 0}(r_{o_\ell}^+) - \psi_{\ell\ell 0}(r_{o_\ell}^-) \end{aligned}$$

Let us now take $\beta \rightarrow r_{o_\ell}^+$ and $\alpha \rightarrow r_{o_\ell}^-$. It is known that the wave function for a δ -function potential is continuous at the non-vanishing point of the δ -function but that its first derivative is discontinuous there. The same will be true of the wave function for our pseudopotential (i.e. $\psi_{\ell\ell 0}(r_{o_\ell}^+) = \psi_{\ell\ell 0}(r_{o_\ell}^-)$ but $\psi'_{\ell\ell 0}(r_{o_\ell}^+) \neq \psi'_{\ell\ell 0}(r_{o_\ell}^-)$). Thus when we take $\beta \rightarrow r_{o_\ell}^+$ and $\alpha \rightarrow r_{o_\ell}^-$, all but the first two terms on the LHS of the above equation vanish and we are left with

$$(2.9) \quad \psi'_{\ell\ell 0}(r_{o_\ell}^+) - \psi'_{\ell\ell 0}(r_{o_\ell}^-) = \frac{f_\ell}{r_{o_\ell}} \psi_{\ell\ell 0}(r_{o_\ell}^+) - \psi_{\ell\ell 0}(r_{o_\ell}^-)$$

or

$$\psi'_{\ell\ell 0}(r_{o_\ell}^+) = \frac{f_\ell}{r_{o_\ell}} \psi_{\ell\ell 0}(r_{o_\ell}^+)$$

which is the boundary condition (2.1). It should be clear from the methods we have used, that this result would still be obtained if we also included a potential inside the core, provided this potential is finite at r_{o_ℓ} . We shall not include such a potential however in the work of this thesis.

By performing identical operations on equations (2.8b) to (2.8d), we find that the function ψ_{JJ1} satisfies (2.2) and the functions $\psi_{J,J+1,1}$ and $\psi_{J,J-1,1}$ satisfy (2.3).

This proves our assertion and thus the potential, or pseudopotential, given by equation (2.5) is completely equivalent to the BCM. We have now reformulated the BCM in terms of a potential as was our aim.

The introduction of a pseudopotential to replace a boundary condition has also been considered by Huang and Yang¹³⁾ in another problem. They were interested in the hard sphere Bose gas and introduced a pseudopotential to replace the condition that the wave function must vanish on the surface of the spheres. As they point out, however, their pseudopotential, equation (12) in their paper, is quite general and could in fact have been used in the above nuclear problem. The Huang and Yang pseudopotential is really very complicated and for practical purposes must be replaced by some approximate form. No such approximation need be made with our pseudopotential. The difference in the form of our pseudopotential and that of Huang and Yang arises from the difference in the approaches taken for the derivation. Huang and Yang require the analytic form of the wave function inside the boundary radius to be identical to the form outside. They then construct a pseudopotential involving differential operators and delta functions at the origin so that the wave function is finite there. Thus, the solution

of the Schrodinger equation containing the Huang and Yang pseudopotential is a smooth function whereas the solution of the Schrodinger equation containing our pseudopotential has a discontinuous derivative at the boundary surface. Thus, even if the two wave functions were the same outside the boundary region, they would be different inside. We cannot attach any physical significance to the wave functions inside the boundary region so this difference should cause no concern. There may indeed be many pseudopotentials which we could construct to replace the BCM but we shall use (2.5) because of its simplicity.

We should point out that the pseudopotential (2.5) is not hermitian. This is also true of the Huang and Yang pseudopotential and as they have mentioned, this non-hermiticity is unimportant in the two-body problem since in the region having physical significance (i.e., outside the boundary surface) the wave function is identical to the wave function for the physical potential. We can easily construct a hermitian pseudopotential from (2.5).

We have

$$(2.10) \quad V(r) =$$

$$= \frac{\hbar^2}{m} \sum_{\ell=0}^{\infty} \left[\frac{f_{\ell}}{r_{o_{\ell}}} \delta(r-r_{o_{\ell}}^+) - \delta(r-r_{o_{\ell}}^-) \frac{\vec{\partial}}{\partial r} - \frac{\vec{\partial}}{\partial r} \delta(r-r_{o_{\ell}}^-) + \right. \\ \left. + \frac{2}{r_{o_{\ell}}} \delta(r-r_{o_{\ell}}^+) - \frac{2}{r_{o_{\ell}}} \delta(r-r_{o_{\ell}}^-) \right] P_{\ell 0 0} \\ + \frac{\hbar^2}{m} \sum_{J=0}^{\infty} \left\{ \left[\frac{f_{JJ}}{r_{o_{JJ}}} \delta(r-r_{o_{JJ}}^+) - \delta(r-r_{o_{JJ}}^-) \frac{\vec{\partial}}{\partial r} - \frac{\vec{\partial}}{\partial r} \delta(r-r_{o_{JJ}}^-) + \right. \right. \\ \left. \left. + \frac{2}{r_{o_{JJ}}} \delta(r-r_{o_{JJ}}^+) - \frac{2}{r_{o_{JJ}}} \delta(r-r_{o_{JJ}}^-) \right] P_{JJ1} \right. \\ \left. + \left[\frac{g}{r_{o_J}} \delta(r-r_{o_J}^+) - \delta(r-r_{o_J}^-) \frac{\vec{\partial}}{\partial r} - \frac{\vec{\partial}}{\partial r} \delta(r-r_{o_J}^-) + \right. \right. \\ \left. \left. + \frac{2}{r_{o_J}} \delta(r-r_{o_J}^+) - \frac{2}{r_{o_J}} \delta(r-r_{o_J}^-) \right] (P_{J,J+1,1} + P_{J,J-1,1}) \right\}$$

This hermitian pseudopotential is also completely equivalent to the BCM as can be shown by the methods used above. The pseudopotential (2.10) differs from (2.5) by containing terms proportional to $\frac{\vec{\partial}}{\partial r}$, which operates only on functions to the left, and additional terms to take care of the fact that the wave function for this pseudopotential is discontinuous at the

non-vanishing points of the δ -functions. Equation (2.10) is more complicated than (2.5) and we decided to use the simplest form possible hoping that the non-hermiticity would lead to no difficulties in the many-body problem. It is possible however that the non-hermiticity may lead to difficulties at high energies.

As an illustration of the use of our pseudopotential (2.5) we shall examine in the next Chapter some aspects of the two-body scattering problem. As we shall see, we can solve the Schrodinger equation exactly when the potential is given by (2.5).

CHAPTER III

THE TWO-BODY SCATTERING PROBLEM USING THE PSEUDOPOTENTIAL

In this Chapter we shall consider some aspects of scattering theory, and as an illustration of the use of our pseudopotential we shall apply the theory to the two-body scattering problem when the interaction potential is given by our singlet spin pseudopotential, that is, when the potential is given by the first line of (2.5). The first derivative of the wave function of the Schrodinger equation containing the pseudopotential is discontinuous and we shall point out explicitly here the way in which this discontinuity manifests itself.

3.1 Some Aspects of Scattering Theory

We consider the scattering of a particle of mass μ from a potential V which is independent of spin and isotopic spin. The Hamiltonian H of the system then is

$$H = H_0 + V$$

where H_0 describes the unperturbed motion of the particle. we shall take

$$H_0 = \frac{p^2}{2\mu}$$

where \vec{p} is the linear momentum of the particle. The eigenstates and eigenvalues of H_0 are given by

$$(3.1a) \quad H_0 |\vec{k}\rangle = E_k |\vec{k}\rangle$$

$$(3.1b) \quad E_k = \frac{\hbar^2 k^2}{2\mu}$$

The eigenvectors $|\vec{k}\rangle$ form an orthogonal set with normalization and completeness conditions as follows

$$(3.2a) \quad \left(\frac{1}{2\pi}\right)^3 \int_{\text{all momentum space}} d\vec{k} |\vec{k}\rangle \langle \vec{k}| = 1$$

$$(3.2b) \quad \langle \vec{k} | \vec{k}' \rangle = (2\pi)^3 \delta(\vec{k} - \vec{k}')$$

where $d\vec{k} = dk_x dk_y dk_z$ and $\delta(\vec{k} - \vec{k}')$ is a three dimensional Dirac δ -function. Further, the projection of $|\vec{k}\rangle$ on the eigenstate $|\vec{r}\rangle$ of the position operator of the particle is

$$(3.3) \quad \langle \vec{r} | \vec{k} \rangle = e^{i\vec{k} \cdot \vec{r}}$$

where the eigenvectors $|\vec{r}\rangle$ form an orthogonal set with normalization and completeness conditions

$$(3.4a) \quad \int_{\text{all space}} d\vec{r} |\vec{r}\rangle \langle \vec{r}| = 1$$

$$(3.4b) \quad \langle \vec{r} | \vec{r}' \rangle = \delta(\vec{r} - \vec{r}')$$

We assume that H has the same continuous spectrum as H_0 . Now we shall be interested in the "standing wave" eigenstates of H which we denote by $|\psi_k\rangle$. They satisfy the equation

$$(3.5a) \quad |\psi_k\rangle = |\vec{k}\rangle + \frac{\rho}{E_k - H_0} V |\psi_k\rangle$$

where ρ denotes principal value. That this is indeed an eigenstate of H can be seen by multiplying both sides on the left by $E_k - H_0$. We can write equation (3.5a) as

$$(3.5b) \quad |\psi_k\rangle = |\vec{k}\rangle + G_k V |\psi_k\rangle$$

where

$$(3.6) \quad G_k = \frac{\rho}{E_k - H_0}$$

G_k is called the Green's operator for standing waves; it is a solution of

$$(E - H_0) G = 1$$

Let us consider the coordinate representation of $|\psi_k\rangle$, that is, $(\vec{r}|\psi_k\rangle = \psi_k(\vec{r})$. From equations (3.5) and (3.3) we have

$$(\vec{r}|\psi_k\rangle) \equiv \psi_k(\vec{r}) = (\vec{r}|\vec{k}\rangle + (\vec{r}|G_k V|\psi_k\rangle) = e^{i\vec{k}\cdot\vec{r}} + (\vec{r}|G_k V|\psi_k\rangle$$

Now,

$$(\vec{r}|G_k V|\psi_k) = \int d\vec{r}' (\vec{r}|G_k|\vec{r}') (\vec{r}'|V|\psi_k)$$

using (3.4a). But

$$(\vec{r}|G_k|\vec{r}') = \left(\frac{1}{2\pi}\right)^3 \int d\vec{k}'' (\vec{r}|\vec{k}') (\vec{k}'|G_k|\vec{k}'') (\vec{k}''|\vec{r}')$$

using (3.2a), and using (3.6), (3.1) and (3.2b)

$$(3.7) \quad (\vec{k}'|G_k|\vec{k}'') = (\vec{k}'| \frac{\rho}{E_k - H_0} |\vec{k}'') = \frac{\rho}{E_k - E_{k'}} (2\pi)^3 \delta(\vec{k}' - \vec{k}'')$$

so that

$$(3.8) \quad (\vec{r}|G_k|\vec{r}') = \left(\frac{1}{2\pi}\right)^3 \frac{2\mu}{\hbar^2} \rho \int \frac{d\vec{k}' e^{i\vec{k}' \cdot (\vec{r} - \vec{r}')}}{k^2 - k'^2}$$

Further, $(\vec{r}'|V|\psi_k) = \int d\vec{r}'' (\vec{r}'|V|\vec{r}'') (\vec{r}''|\psi_k)$ which for a local potential, (that is, one where $(\vec{r}'|V|\vec{r}'') = V(\vec{r}') \delta(\vec{r}' - \vec{r}'')$) becomes

$$(\vec{r}'|V|\psi_k) = V(\vec{r}') \psi_k(\vec{r}')$$

so that finally we have

$$(3.9) \quad \psi_k(\vec{r}) = e^{i\vec{k} \cdot \vec{r}} + \int d\vec{r}' G_k(\vec{r}, \vec{r}') V(\vec{r}') \psi_k(\vec{r}')$$

where for convenience we now write $(\vec{r}|G_k|\vec{r}') = G_k(\vec{r}, \vec{r}')$.

We are also interested in the "reactance operator, K" which is defined by

$$(3.10) \quad K|\vec{k}) = V|\psi_k)$$

Multiplying by $(\vec{k}' |$ we obtain an equation for the reactance matrix elements

$$(3.11) \quad (\vec{k}' | K | \vec{k}) = (\vec{k}' | V | \psi_k)$$

which for a local potential becomes

$$(3.12) \quad (\vec{k}' | K | \vec{k}) = \int d\vec{r} e^{-i\vec{k}' \cdot \vec{r}} V(\vec{r}) \psi_k(\vec{r})$$

We can also derive an integral equation for $(\vec{k}' | K | \vec{k})$. Using equation (3.5) in (3.11) we have

$$\begin{aligned} (\vec{k}' | K | \vec{k}) &= (\vec{k}' | V | \psi_k) = (\vec{k}' | V | \vec{k}) + (\vec{k}' | V \frac{\rho}{E_k - H_0} V | \psi_k) \\ &= (\vec{k}' | V | \vec{k}) + (\vec{k}' | V \frac{\rho}{E_k - H_0} K | \vec{k}) \end{aligned}$$

Finally, using (3.2) and (3.1) we get

$$(3.13) \quad (\vec{k}' | K | \vec{k}) = (\vec{k}' | V | \vec{k}) + \left(\frac{1}{2\pi}\right)^3 \frac{2\mu}{\hbar^2} \rho \int \frac{d\vec{k}'' (\vec{k}' | V | \vec{k}'') (\vec{k}'' | K | \vec{k})}{k^2 - k''^2}$$

where, for a local potential,

$$(3.14) \quad (\vec{k}' | V | \vec{k}) = \int d\vec{r} e^{-i\vec{k}' \cdot \vec{r}} V(\vec{r}) e^{i\vec{k} \cdot \vec{r}}$$

We shall be concerned mainly with (3.9) to (3.14).

Let us now suppose that V is a central potential, that is $V = V(r)$. In this case the angle dependence of $\psi_k(\vec{r})$ is

particularly simple. We can write

$$(3.15) \quad \psi_{\mathbf{k}}(\vec{r}) = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} i^{\ell} \psi_{\mathbf{k}}^{\ell}(r) Y_{\ell m}^*(\hat{\mathbf{k}}) Y_{\ell m}(\hat{\mathbf{r}})$$

where the Y's are the usual spherical harmonics with orthonormality condition

$$(3.16) \quad \int d\hat{\mathbf{r}} Y_{\ell m}^*(\hat{\mathbf{r}}) Y_{\ell' m'}(\hat{\mathbf{r}}) = \delta_{\ell\ell'} \delta_{mm'}$$

The integration extends over the full solid angle, and the δ -functions on the RHS are Kronecker δ -functions.

Now the expansion of the plane wave in terms of spherical harmonics is given by¹⁴⁾

$$(3.17) \quad e^{i\vec{k} \cdot \vec{r}} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} i^{\ell} j_{\ell}(kr) Y_{\ell m}^*(\hat{\mathbf{k}}) Y_{\ell m}(\hat{\mathbf{r}})$$

Using equations (3.16) and (3.17) we can rewrite equation (3.8) in the form

$$(3.18a) \quad G_{\mathbf{k}}(\vec{r}, \vec{r}') = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} G_{\mathbf{k}}^{\ell}(r, r') Y_{\ell m}(\hat{\mathbf{r}}) Y_{\ell m}^*(\hat{\mathbf{r}}')$$

where

$$(3.18b) \quad G_{\mathbf{k}}^{\ell}(r, r') = \frac{2\mu}{\hbar^2} \frac{2}{\pi} \int_0^{\infty} dk' k'^2 \frac{j_{\ell}(k'r) j_{\ell}(k'r')}{k^2 - k'^2}$$

In Appendix I we show that

$$(3.19) \quad \frac{2}{\pi} \rho \int_0^\infty \frac{dk' k'^2 j_\ell(k'r) j_\ell(k'r')}{k^2 - k'^2} =$$

$$= \begin{cases} k j_\ell(kr) n_\ell(kr') & \text{if } r' > r \\ k j_\ell(kr') n_\ell(kr) & \text{if } r > r' \end{cases}$$

Using equations (3.15) to (3.18a) we obtain the following integral equation for $\psi_k^\ell(r)$ from (3.9)

$$(3.20) \quad \psi_k^\ell(r) = j_\ell(kr) + \int_0^\infty dr' r'^2 G_k^\ell(r, r') V(r') \psi_k(r')$$

Further using (3.15) to (3.17) we can write equation (3.12) as

$$(3.21a) \quad (\vec{k}' | K | \vec{k}) = (4\pi)^2 \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} (k' | K_\ell | k) Y_{\ell m}(\hat{k}') Y_{\ell m}^*(\hat{k})$$

where

$$(3.21b) \quad (k' | K_\ell | k) = \int_0^\infty dr r^2 j_\ell(k'r) V(r) \psi_k^\ell(r)$$

Using (3.16), (3.17), and (3.21) we obtain the following integral equation for $(k' | K_\ell | k)$ from (3.13)

$$(3.22) \quad (k' | K_\ell | k) = (k' | V_\ell | k) + \frac{2\mu}{\hbar^2} \frac{2}{\pi} \rho \int_0^\infty \frac{dk'' k''^2 (k' | V_\ell | k'') (k'' | K_\ell | k)}{k^2 - k''^2}$$

where

$$(3.23) \quad (k' | V_\ell | k) = \int_0^\infty dr r^2 j_\ell(k'r) V(r) j_\ell(kr)$$

Let us now consider equation (3.20) again. In the limit $r \rightarrow \infty$ we get

$$\psi_k(r) \rightarrow j_\ell(kr) + n_\ell(kr) \frac{2\mu k}{\hbar^2} \int_0^\infty dr r^2 j_\ell(kr) V(r) \psi_k(r)$$

where we have made use of (3.19). Now the phase shift is defined by the requirement that as $r \rightarrow \infty$

$$\psi_k^\ell(r) \rightarrow j_\ell(kr) - \tan \delta_\ell n_\ell(kr)$$

so that we have

$$\tan \delta_\ell = - \frac{2\mu k}{\hbar^2} \int_0^\infty dr r^2 j_\ell(kr) V(r) \psi_k^\ell(r)$$

which using (3.21b) gives the important relationship

$$(3.24) \quad (k | K_\ell | k) = - \frac{\hbar^2}{2\mu k} \tan \delta_\ell$$

We have now discussed enough of the formal theory of scattering for our purposes and can now turn to a particular application of the theory.

3.2 Use of the Pseudopotential

We shall be concerned in this Section with equations (3.20) to (3.24). These equations have been derived for the scattering of a particle of mass μ from a spin and isotopic spin independent central potential. These equations can be applied also to the scattering of two nucleons in a singlet spin state from an isotopic spin independent central potential when $2\mu=m$, the mass of one nucleon, r is the distance between the two nucleons, k is their relative wave number, and ℓ their orbital angular momentum. We shall then examine these equations when the potential is given by our pseudopotential for singlet spin and orbital angular momentum ℓ states, that is for

$$(3.25) \quad V(r) = \frac{\hbar^2}{m} \left[\frac{f_\ell}{a} \delta(r-a^+) - \delta(r-a^-) \frac{a}{\partial r} \right], \quad a=r_{0\ell}$$

We shall see that we can solve (3.20) exactly using this potential. Substituting (3.25) into (3.20) yields

$$(3.26a) \quad \psi_\ell(r) = j_\ell(kr) + \frac{f_\ell}{a} g^\ell(r, a^+) \psi_\ell(a^+) - g^\ell(r, a^-) \psi'_\ell(a^-)$$

For simplicity we now write $\psi_\ell(r) = \psi_k^\ell(r)$ (i.e., we drop the reference to k) and we also write

$$(3.26b) \quad g^\ell(r, r') = \frac{\hbar^2 a^2}{m} G_k^\ell(r, r') \equiv \frac{2a^2}{\pi} \rho \int \frac{\infty dk' k'^2 j_\ell(k'r) j_\ell(k'r')}{k^2 - k'^2} \equiv$$

$$\equiv ka^2 \begin{cases} j_\ell(kr) n_\ell(kr') & \text{if } r' > r \\ j_\ell(kr') n_\ell(kr) & \text{if } r > r' \end{cases}$$

We now need expressions for $\psi_\ell(a^+)$ and $\psi_\ell^i(a^-)$. We notice that $\psi_\ell(r)$ is a continuous function of r since $g^\ell(r, r')$ is. Thus, $\psi_\ell(a^+) = \psi_\ell(a)$ and therefore

$$(3.27) \quad \psi_\ell(a) = j_\ell(ka) + \frac{f_\ell}{a} g^\ell(a, a) \psi_\ell(a) - g^\ell(a, a) \psi_\ell^i(a^-)$$

using also $g^\ell(a, a^-) = g^\ell(a, a^+) = g^\ell(a, a)$.

Consider $\psi_\ell^i(r)$ however:

$$\psi_\ell^i(r) = k j_\ell^i(kr) + \frac{f_\ell}{a} g_r^\ell(r, a^+) \psi_\ell(a^+) - g_r^\ell(r, a^-) \psi_\ell^i(a^-)$$

where

$$g_r^\ell(r, r') = \frac{dg^\ell(r, r')}{dr}$$

It is discontinuous at $r=a^-$ since $g_r^\ell(r, r')$ is discontinuous when $r=r'$:

$$(3.28) \quad g_r^\ell(r, r') = \frac{2a^2}{\pi} \rho \int_0^\infty dk k'^3 \frac{j_\ell(k'r) j_\ell(k' r')}{k^2 - k'^2} = k^2 a^2 \begin{cases} j_\ell^i(kr) n_\ell(kr') & \text{if } r' > r \\ j_\ell(kr') n_\ell(kr) & \text{if } r > r' \end{cases}$$

from differentiating equation (3.26b).

Thus, what do we mean by $\psi_\ell^i(a^-)$? To answer this we look back at equation (2.9). It is seen there that we consider

$$(3.29) \quad \psi_\ell^i(a^-) = \lim_{a \rightarrow a^-} \psi_\ell^i(a) \quad \text{where } a < a^-$$

If the limit is not taken in this manner than the pseudo-potential will not lead to the required boundary condition. In equation (3.26a) we must interpret $\psi_\ell^{\dagger}(a^-)$ as given by (3.29) and we have then

$$(3.30) \quad \psi_\ell^{\dagger}(a^-) = k j_\ell^{\dagger}(ka) + \frac{f_\ell}{a} g_r^\ell(a^-, a) \psi_\ell(a) - g_r^\ell(a^-, a) \psi_\ell^{\dagger}(a^-)$$

using $g_r^\ell(a^-, a^+) = g_r^\ell(a^-, a)$. The important point is that the value of the first variable must be less than that of the second.

Now equations (3.27) and (3.30) can be solved for $\psi_\ell(a)$ and $\psi_\ell^{\dagger}(a)$ and we find

$$(3.31a) \quad \psi_\ell(a) = \frac{j_\ell(ka) [1 + g_r^\ell(a^-, a)] - k j_\ell^{\dagger}(ka) g_r^\ell(a, a)}{1 + g_r^\ell(a^-, a) - \frac{f_\ell}{a} g_r^\ell(a, a)}$$

$$(3.31b) \quad \psi_\ell^{\dagger}(a^-) = \frac{k j_\ell^{\dagger}(ka) [1 - \frac{f_\ell}{a} g_r^\ell(a, a)] + \frac{f_\ell}{a} j_\ell(ka) g_r^\ell(a^-, a)}{1 + g_r^\ell(a^-, a) - \frac{f_\ell}{a} g_r^\ell(a, a)}$$

so

$$\psi_\ell(r) = j_\ell(kr) + \frac{1}{a} g_r^\ell(r, a) \left\{ \frac{f_\ell j_\ell(ka) - k a j_\ell^{\dagger}(ka)}{1 + g_r^\ell(a^-, a) - \frac{f_\ell}{a} g_r^\ell(a, a)} \right\}$$

Now, from (3.26b) and (3.28)

$$g_r^\ell(a, a) = k a^2 j_\ell(ka) n_\ell(ka)$$

$$g_r^\ell(a^-, a) = k^2 a^2 j_\ell^{\dagger}(ka) n_\ell(ka)$$

Thus

$$(3.31c) \quad 1 + g_r^l(a^-, a) - \frac{f_l}{a} g^l(a, a) = -ka j_l(ka) \left[f_l n_l(ka) - ka n_l^*(ka) \right]$$

where we have made use of the relationship¹⁵⁾

$$j_l(x) n_l^*(x) - n_l(x) j_l^*(x) = \frac{1}{x^2}$$

Finally then

$$(3.32) \quad \psi_l(r) = j_l(kr) - \frac{g^l(r, a)}{ka^2 j_l(ka)} \left\{ \frac{f_l j_l(ka) - ka j_l^*(ka)}{f_l n_l(ka) - ka n_l^*(ka)} \right\}$$

We notice that for $r > a$, $\frac{g^l(r, a)}{ka^2 j_l(ka)} = n_l(kr)$ and from equation (2.4),

$$\left\{ \right\} = \tan \delta_l$$

so that

$$\psi_l(r) = j_l(kr) - \tan \delta_l n_l(kr), \quad r > a$$

as it should be.

Let us now consider $(k^* | K_l | k)$. Using (3.25) in (3.21b) gives

$$(k^* | K_l | k) = \frac{\hbar^2 a}{m} j_l(k^* a) \left[f_l \psi_l(a) - a \psi_l^*(a^-) \right]$$

which using (3.31) becomes

$$\begin{aligned}
 (3.33a) \quad (k^+ | K_\ell | k) &= \frac{\hbar^2 a}{m} j_\ell(k^+ a) \left\{ \frac{f_\ell j_\ell(ka) - ka j_\ell^i(ka)}{1 + g_\ell^i(a^-, a) - \frac{f_\ell}{a} g_\ell^i(a, a)} \right\} \\
 &= - \frac{\hbar^2}{mk} \frac{j_\ell(k^+ a)}{j_\ell(ka)} \left\{ \frac{f_\ell j_\ell(ka) - ka j_\ell^i(ka)}{f_\ell n_\ell(ka) - ka n_\ell^i(ka)} \right\}
 \end{aligned}$$

Again from equation (2.4), $\left\{ \right\} = \tan \delta_\ell$ so that

$$(k | K_\ell | k) = - \frac{\hbar^2}{mk} \tan \delta_\ell$$

Thus we see that equation (3.24) is indeed satisfied.

It is instructive for later work to rewrite $(k^+ | K_\ell | k)$ with the g 's written in their integral form. From (3.33a) we have, using (3.26b) and (3.28)

$$\begin{aligned}
 (3.33b) \quad (k^+ | K_\ell | k) &= \\
 &= \frac{\frac{\hbar^2 a}{m} j_\ell(k^+ a) [f_\ell j_\ell(ka) - ka j_\ell^i(ka)]}{1 - \frac{2a}{\pi} \rho \int_0^\infty \frac{dk'' k''^2 j_\ell(k'' a) [f_\ell j_\ell(k'' a) - k'' a j_\ell^i(k'' a^-)]}{k^2 - k''^2}}
 \end{aligned}$$

Let us now solve directly the integral equation (3.22) for $(k^+ | K_\ell | k)$.

We have

$$(k^{\dagger} | V_{\ell} | k) =$$

$$= \int_0^{\infty} dr r^2 j_{\ell}(k^{\dagger} r) V(r) j_{\ell}(kr) = \frac{\hbar^2 a}{m} \left[f_{\ell} j_{\ell}(k^{\dagger} a^{+}) j_{\ell}(ka^{+}) - ka j_{\ell}(k^{\dagger} a^{-}) j_{\ell}(ka^{-}) \right]$$

substitution of which into (3.22) gives

$$(3.34) \quad (k^{\dagger} | K_{\ell} | k) = f_{\ell} a j_{\ell}(k^{\dagger} a^{+}) K_1(k) - a^2 j_{\ell}(ka^{-}) K_2(k)$$

where

$$(3.35) \quad K_1(k) = \frac{\hbar^2}{m} j_{\ell}(ka^{+}) + \frac{2}{\pi} \rho \int_0^{\infty} \frac{dk'' k''^2 j_{\ell}(k'' a^{+}) (k'' | K_{\ell} | k)}{k^2 - k''^2}$$

$$(3.36) \quad K_2(k) = \frac{\hbar^2 k}{m} j_{\ell}^{\dagger}(ka^{-}) + \frac{2}{\pi} \rho \int_0^{\infty} \frac{dk'' k''^3 j_{\ell}^{\dagger}(k'' a^{-}) (k'' | K_{\ell} | k)}{k^2 - k''^2}$$

We now substitution equation (3.34) for $(k^{\dagger} | K_{\ell} | k)$ into (3.35) and (3.36) to get equations for $K_1(k)$ and $K_2(k)$. We find, using the definitions of g and g_r ,

$$K_1(k) = \frac{\hbar^2}{m} j_{\ell}(ka^{+}) + \frac{f_{\ell}}{a} K_1(k) g^{\ell}(a, a) - K_2(k) g^{\ell}(a, a)$$

$$K_2(k) = \frac{\hbar^2 k}{m} j_{\ell}^{\dagger}(ka^{-}) + \frac{f_{\ell}}{a} K_1(k) g_r^{\ell}(a^{-}, a) - K_2(k) g_r^{\ell}(a^{-}, a^{-})$$

the solutions of which are

$$K_1(k) = \frac{\frac{\hbar^2}{m} j_\ell(ka) \left[1 + g_r^\ell(a^-, a^-) \right] - \frac{\hbar^2 k}{m} j_\ell^\dagger(ka) g^\ell(a, a)}{1 + g_r^\ell(a^-, a^-) - \frac{f_\ell}{a} g^\ell(a, a) + \frac{f_\ell}{a} g^\ell(a, a) \left[g_r^\ell(a^-, a) - g_r^\ell(a^-, a^-) \right]}$$

$$K_2(k) = \frac{\frac{\hbar^2 k}{m} j_\ell^\dagger(ka) \left[1 - \frac{f_\ell}{a} g^\ell(a, a) \right] + \frac{\hbar^2}{m} j_\ell(ka) \frac{f_\ell}{a} g_r^\ell(a^-, a)}{1 + g_r^\ell(a^-, a^-) - \frac{f_\ell}{a} g^\ell(a, a) + \frac{f_\ell}{a} g^\ell(a, a) \left[g_r^\ell(a^-, a) - g_r^\ell(a^-, a^-) \right]}$$

Substituting these expressions back into (3.34) gives finally

$$(3.37) \quad (k^\dagger | K_\ell | k) =$$

$$= \frac{\hbar^2 a}{m} j_\ell(k^\dagger a) \left\{ \frac{f_\ell j_\ell(ka) - k a j_\ell^\dagger(ka) - f_\ell \left[g_r^\ell(a^-, a) - g_r^\ell(a^-, a^-) \right]}{1 + g_r^\ell(a^-, a^-) - \frac{f_\ell}{a} g^\ell(a, a) + \frac{f_\ell}{a} \left[g_r^\ell(a^-, a) - g_r^\ell(a^-, a^-) \right] g^\ell(a, a)} \right\}$$

Comparing the two equations for $(k^\dagger | K_\ell | k)$, that is, (3.37) and (3.33a), we see that they are different and thus $(k | K_\ell | k)$ as given by (3.37) will not be proportional to $\tan \delta_\ell$ as it should be. We have used no illegal procedures in obtaining (3.37) from the integral equation (3.22) for $(k^\dagger | K_\ell | k)$. However, by solving the integral equation for $(k^\dagger | K_\ell | k)$ we cannot take into account the fact that $\psi_\ell^\dagger(r)$ is discontinuous

at $r = a^-$ and it is for this reason that the two methods of obtaining $(k' | K_\rho | k)$ give different results. The integral equation (3.22) for $(k' | K_\rho | k)$ is ambiguous since $(k' | K_\rho | k)$ given by (3.33a) and by (3.37) are both solutions. This does not mean that we cannot deal with this equation however. We could deal with this equation by dropping the distinction between a^+ and a^- in the pseudopotential, that is, by using simply a . In the final result however the distinction must be made, and the correct choice will be governed by the result obtained by constructing $(k' | K_\rho | k)$ from the solution of the Schrodinger equation, that is, by (3.33). We emphasize this point here since in the next Chapter on the many-body problem we shall meet a similar integral equation.

This completes our discussion of the two-body problem. We have given an exact solution of the Schrodinger equation for the case when the potential is given by our singlet spin pseudopotential and we have illustrated how the discontinuity in the first derivative of the wave function manifests itself.

We shall now examine the much more complex nuclear many-body problem and the application of our pseudopotential to this case. We shall see that in the many-body problem as in the above two-body problem, use of the pseudopotential instead of a more standard phenomenological nucleon-nucleon potential leads to equations which are much simpler to handle.

CHAPTER IV

THE NUCLEAR MANY-BODY PROBLEM AND THE INDEPENDENT PAIR MODEL

4.1 Introduction

In the next Chapter we shall apply the BCM, via our nucleon-nucleon pseudopotential (2.5), to the nuclear many-body problem. Firstly, however, we must make some introductory remarks about the problem.

In this discussion we shall be concerned with a hypothetical configuration of nucleons called "nuclear matter". The concept of nuclear matter results from an extrapolation of the semi-empirical mass formula of Weiszacker, which formula gives a good over-all fit to the binding energy of all known nuclei. It is believed on the basis of this formula that a large number A of $A/2$ protons and $A/2$ neutrons would in their lowest energy state form a stable configuration with an energy per particle of approximately -15 Mev, provided of course that the Coulomb force between the protons did not exist. It is this hypothetical configuration that is called nuclear matter.

A second property attributed to nuclear matter is constancy of density of nucleons, $\rho = \frac{A}{\Omega}$, where Ω is the

volume of the configuration. It is known experimentally that the density of nucleons near the centre of all but the very lightest nuclei is roughly equal to .18 nucleons per (fermi)³ and it is this value which nuclear matter is assumed to possess.

The problem then is to explain these properties of nuclear matter on the basis of the forces between the nucleons. We have no knowledge of many-body forces and only an incomplete knowledge of two-body forces as obtained by scattering experiments. At most then, one would hope to calculate the properties of nuclear matter assuming the forces between the constituents to be given by those indicated by two-body scattering experiments.

Thus, we shall consider the following to be the Hamiltonian for nuclear matter:

$$H = \sum_{i=1}^A T_i + \frac{1}{2} \sum_{i=1}^A \sum_{j=1}^A V_{ij}$$

where T_i is the kinetic energy operator for nucleon i and V_{ij} is the interaction potential between nucleons i and j which shall be taken to be given from scattering experiments.

If $|\Psi\rangle$ is the normalized wave function of the ground state of nuclear matter then the energy E of the ground state is

$$E = (\Psi | \sum_{i=1}^A T_i + \frac{1}{2} \sum_{i=1}^A \sum_{j=1}^A V_{ij} | \Psi)$$

The earliest calculations to obtain E were perturbation calculations based on the independent particle model of nuclear matter, and using the total interaction energy as the perturbation. Neither of the known perturbation methods yielded satisfactory results: the Wigner-Brillouin series converges too slowly and the Raleigh-Schrodinger series, when stopped at any given order, contains terms of order A^2 (instead of A), which is clearly unphysical. These are the unlinked cluster terms. However, Goldstone,¹⁶⁾ in one of the most important papers in this field, considered the many-body problem using field theoretic techniques and obtained a new perturbation series in which no unlinked cluster terms appear.

Goldstone's elegant approach in terms of diagrams gives new series for $|\Psi\rangle$ and E and the problem then is to find some suitable approximation for them. The approximation series introduced first by Brueckner¹⁾ gives a good first approximation to the Goldstone series and consists in picking out all of the "ladder" diagrams in the Goldstone series for the energy. The Brueckner approximation can be looked upon as an independent pair model of nuclear matter in that it considers the interactions between any pair of particles, neglecting the interaction of all the other particles among themselves and with the pair. The most lucid description of the physics behind the

Brueckner approximation has been given by Gomes, Walecka, and Weisskopf³⁾ and this paper will be the starting point for our more quantitative discussion in the next Section.

4.2 The Independent Pair Model of Nuclear Matter.

Let us consider first the simplest of all models of nuclear matter, the degenerate Fermi gas model, wherein one completely ignores the interactions between the nucleons. Nuclear matter is considered to be a Fermi gas enclosed in a volume Ω and composed of four different kinds of particles: $A/4$ protons of each spin and $A/4$ neutrons of each spin. The particles will occupy single particle levels which are eigenstates of the momentum operator. We denote a single particle state by $|\alpha\rangle$ where the letter α stands for the quantum numbers \vec{k}_α , σ_α , τ_α where \vec{k}_α is the momentum of the particle and σ_α and τ_α are the spin and isospin, respectively. If $|\vec{r}\rangle$ is the eigenvector of the position operator for a particle, then

$$(4.1) \quad \langle \vec{r} | \alpha \rangle = \frac{1}{\sqrt{\Omega}} e^{i\vec{k}_\alpha \cdot \vec{r}} \sigma_\alpha \tau_\alpha$$

Thus, we consider the momentum and position eigenfunctions of a particle to have the following normalizations:

$$(4.2) \quad \begin{aligned} \langle \vec{k} | \vec{k}' \rangle &= \delta_{\vec{k} \vec{k}'} \\ \langle \vec{r} | \vec{r}' \rangle &= \delta(\vec{r} - \vec{r}') \\ \langle \vec{r} | \vec{k} \rangle &= \frac{1}{\sqrt{\Omega}} e^{i\vec{k} \cdot \vec{r}} \end{aligned}$$

The nuclear matter wave function is then a product of these single particle wave functions

$$(4.3) \quad |\Psi\rangle = \sqrt{\frac{1}{A!}} \mathcal{A} |a\rangle_1 |\beta\rangle_2 |\gamma\rangle_3 \dots |\omega\rangle_A$$

where \mathcal{A} is the antisymmetrizing operator. In the above, the subscripts 1,2,...,A indicate which particle occupies which single particle level. There are A single particle levels occupied for a system of A particles.

The total energy of the ground state of the system is the one where the particles occupy all levels up to the Fermi level and its energy is³⁾

$$(4.4) \quad E_0 = \frac{\hbar^2}{2m} \sum_0^{k_F} k_a^2 = \frac{3}{5} \frac{\hbar^2}{2m} k_F^2$$

where k_F is the Fermi momentum, and is related to the density $\rho = \frac{A}{V}$ by

$$(4.5) \quad k_F = \left(\frac{3}{2} \pi^2 \rho \right)^{1/3}$$

In the independent pair model of nuclear matter (IPAM) one goes beyond the above simple picture by including two particle correlations. That is, one treats the correlation between a given pair exactly and takes into account in an average fashion the other interactions of the particles among themselves and with the pair. One treats the scattering of two particles in nuclear matter in a manner similar to the

scattering of two particles in free space, there being one very important difference in these two cases however: the effect of the Pauli Principle. In the zeroth order approximation to the ground state of nuclear matter (i.e. equation (4.3)) all levels up to the Fermi level are occupied, and thus two particles in this Fermi sea cannot perform a real scattering process into different levels since these levels are already occupied by other particles. Although one assumes that the other particles do not interact directly with the pair, they still have an important effect on the interaction of the pair through the Pauli Principle. The mathematical formulation of this model was first given by Bethe and Goldstone¹⁷⁾ and the physical ideas underlying the model were illuminated by Gomes, Walecka, and Weisskopf.³⁾ The equation for the wave function of the two particles is not the usual Schrodinger equation but the Bethe-Goldstone (BG) equation, which we shall now derive using the approach of Gomes, Walecka, and Weisskopf.

In the IPAM the energy of the system is written as

$$E = E_0 + \frac{1}{2} \sum_{\alpha, \beta} \Delta \epsilon_{\alpha\beta}$$

where the summation is over all occupied levels, and where $\Delta \epsilon_{\alpha\beta}$ represents the energy shift coming from the interaction of a pair of particles in levels α and β . Each $\Delta \epsilon_{\alpha\beta}$ is

calculated by assuming that there is interaction only between the pair of particles in levels α and β . Since the interaction vanishes in levels $\gamma \neq \alpha$ or β the wave function of a particle in any of these levels is unchanged, that is, is still given by the zeroth order approximation. However, as a result of the interaction, the wave function $|\psi_{\alpha\beta}\rangle$ of particles in levels α and β is changed from a product of zeroth order wave functions and it is this new wave function which we shall now obtain.

In order to obtain an equation for $|\psi_{\alpha\beta}\rangle$ we consider, in the spirit of the above discussion, the following equation

$$(4.6) \quad H = \sum_{i=1}^A H_i + U_{\alpha\beta}$$

$$H_i = T_i + U_i$$

$$U_{\alpha\beta} = \sum_{i=1}^A \sum_{j=1}^A V_{ij} q_{\alpha}^i q_{\beta}^j$$

where T_i is the kinetic energy operator for particle i ; U_i is a self-consistent single particle potential felt by particle i (Because of the translational invariance of nuclear matter, it cannot depend on the position coordinate of particle i . It can depend on the momentum of the particle, of course.); V_{ij} is the interaction between particles i and j as determined from two-body scattering experiments; q_{α}^i is

the following operator

$$(4.7) \quad q_a^i = \sum_{\gamma > F} |\gamma\rangle_i \langle \gamma| + |a\rangle_i \langle a|$$

where $\sum_{\gamma > F}$ means summation over all single particle states

$|\gamma\rangle$ where $k_\gamma > k_F$.

In (4.7) the single particle states are eigenfunctions of $T+U$. That is,

$$(4.8) \quad (T+U) |\gamma\rangle = \epsilon_\gamma |\gamma\rangle$$

where $\epsilon_\gamma = \epsilon(k_\gamma)$ is the energy eigenvalue. Since U is independent of the position coordinate, the single particle eigenfunctions are identical to those discussed in connection with the Fermi gas model at the beginning of this Section.

The ground state of a system with the Hamiltonian (4.6) is the one in which only particles in levels α and β interact, that is, the levels $|\alpha\rangle$ and $|\beta\rangle$ are influenced by the interaction and their wave function is changed. We write the ground state wave function of the system as

$$(4.9) \quad |\Psi\rangle = \sqrt{\frac{1}{A!}} G |\psi_{\alpha\beta}\rangle_{12} |\gamma\rangle_3 |\delta\rangle_4 \dots |\omega\rangle_A$$

That is, we suppose that $k_\alpha, k_\beta, k_\gamma, \dots, k_\omega$ are all less than k_F . We consider the pair wave function $|\psi_{\alpha\beta}\rangle$ to be normalized, i.e., $(\psi_{\alpha\beta}|\psi_{\alpha\beta})=1$. The wave function (4.9) differs from (4.3) by having $|\psi_{\alpha\beta}\rangle$ in place of $|\alpha\rangle|\beta\rangle$, which from now on we shall write as $|\alpha\beta\rangle$.

Now if we consider $|\psi_{\alpha\beta}\rangle$ expanded in terms of the complete set of single particle eigenfunctions it is evident that because of the antisymmetrization $|\psi_{\alpha\beta}\rangle$ cannot contain any components with wave numbers less than k_F except for $|\alpha\rangle$ and $|\beta\rangle$. Thus we can write

$$(4.10) \quad (\alpha\beta|\psi_{\alpha\beta}) = 1$$

$$q_\alpha^1|\psi_{\alpha\beta}\rangle_{12} = q_\beta^2|\psi_{\alpha\beta}\rangle_{12} = |\psi_{\alpha\beta}\rangle_{12}$$

The interaction $U_{\alpha\beta}$ then is equivalent to the ordinary interaction in the levels α and β and vanishes for all other levels as far as the ground state is concerned. Thus the Hamiltonian describes a system of particles moving in an average field but where a direct interaction takes place only between particles in levels α and β , giving justification to the appellation, independent pair model.

In Appendix II we show that if the wave function (4.9) is to be an eigenfunction of (4.6) the following equation

must be satisfied.

$$(4.11) \quad (H_1 + H_2 - \epsilon_{\alpha\beta}) |\psi_{\alpha\beta}^A\rangle = - Q_{\alpha\beta} V |\psi_{\alpha\beta}^A\rangle$$

This is the Bethe-Goldstone equation. The new symbols are as follows: $|\psi_{\alpha\beta}^A\rangle$ is the antisymmetric wave function of particles 1 and 2 ($|\psi_{\alpha\beta}^A\rangle = \frac{1}{\sqrt{2}} [|\psi_{\alpha\beta}\rangle - |\psi_{\beta\alpha}\rangle]$); $\epsilon_{\alpha\beta}$ is the energy eigenvalue; and $Q_{\alpha\beta}$ the following operator:

$$(4.12) \quad Q_{\alpha\beta} = Q + |\alpha\beta^A\rangle \langle\alpha\beta^A|, \quad Q = \sum_{\gamma>F} \sum_{\delta>F} |\gamma\delta\rangle \langle\gamma\delta|$$

where $|\alpha\beta^A\rangle = \frac{1}{\sqrt{2}} [|\alpha\beta\rangle - |\beta\alpha\rangle]$. In equations (4.11) and (4.12) we have, for simplicity, dropped the labels 1 and 2 in some places. In the two-particle bras and kets (e.g. $\langle\gamma\delta|$ and $|\gamma\delta\rangle$) the first letter refers to the state of particle 1 and the second to that of particle 2.

It is worthwhile to point out that there can be no terms of the following form in $Q_{\alpha\beta}$

$$\sum_{\gamma>F} |\alpha\gamma\rangle \langle\alpha\gamma|, \quad (k_\alpha < k_F)$$

since such terms would give rise to single particle virtual excitations and must be ruled out for nuclear matter by conservation of centre of mass momentum.

We shall now obtain an expression for the energy shift $\Delta \epsilon_{\alpha\beta} = \epsilon_{\alpha\beta} - \epsilon_{\alpha} - \epsilon_{\beta}$ due to the pair interaction. Substituting (4.12) into (4.11) gives

$$(H_1 + H_2 - \epsilon_{\alpha\beta}) |\psi_{\alpha\beta}^A\rangle = - \sum_{\gamma > F} \sum_{\delta > F} |\gamma\delta\rangle (\gamma\delta | V | \psi_{\alpha\beta}^A) - |\alpha\beta^A\rangle (\alpha\beta^A | V | \psi_{\alpha\beta}^A)$$

Multiplying by $\langle \alpha\beta^A |$ on the left yields

$$\langle \alpha\beta^A | H_1 + H_2 - \epsilon_{\alpha\beta} | \psi_{\alpha\beta}^A \rangle = - \langle \alpha\beta^A | V | \psi_{\alpha\beta}^A \rangle$$

But $(H_1 + H_2) |\alpha\beta^A\rangle = (\epsilon_{\alpha} + \epsilon_{\beta}) |\alpha\beta^A\rangle$ and $\langle \alpha\beta^A | \psi_{\alpha\beta}^A \rangle = 1$ so we get

$$\begin{aligned} \epsilon_{\alpha} + \epsilon_{\beta} - \epsilon_{\alpha\beta} &= - \langle \alpha\beta^A | V | \psi_{\alpha\beta}^A \rangle \\ (4.13) \quad \Delta \epsilon_{\alpha\beta} &= \langle \alpha\beta^A | V | \psi_{\alpha\beta}^A \rangle = \langle \alpha\beta | V | \psi_{\alpha\beta} \rangle - \langle \alpha\beta | V | \psi_{\beta\alpha} \rangle \end{aligned}$$

Now since $\Delta \epsilon_{\alpha\beta}$ is the energy shift of a pair, the total energy of the system is then

$$(4.14) \quad E = E_0 + \frac{1}{2} \sum_{\alpha < F} \sum_{\beta < F} \Delta \epsilon_{\alpha\beta}$$

where $\sum_{\alpha < F}$ means summation over all states $|\alpha\rangle$ where $k_{\alpha} < k_F$.

Using (4.13) we can obtain an important fact about the form of the spacial representation of $|\psi_{\alpha\beta}^A\rangle$, i.e. of $(\vec{r}_1 \vec{r}_2 | \psi_{\alpha\beta}^A) = \psi_{\alpha\beta}^A(\vec{r}_1 \vec{r}_2)$. Writing Q as $1 - \bar{Q}$ where

$$\bar{Q} = \sum_{\gamma < F} \sum_{\delta < F} |\gamma\delta\rangle (\gamma\delta |$$

we obtain from substituting into (4.11)

$$(H_1 + H_2 - \epsilon_{\alpha\beta}) |\psi_{\alpha\beta}^A\rangle = -V |\psi_{\alpha\beta}^A\rangle + \sum_{\gamma < F} \sum_{\delta < F} |\gamma\delta\rangle (\gamma\delta | V | \psi_{\alpha\beta}^A) - |\alpha\beta^A\rangle (\alpha\beta^A | V | \psi_{\alpha\beta}^A)$$

Let us consider the $(\vec{r}_1 \vec{r}_2 |$ projection of this equation. We have

$$\text{LHS} = (H_1 + H_2 - \epsilon_{\alpha\beta}) (\vec{r}_1 \vec{r}_2 | \psi_{\alpha\beta}^A)$$

$$\text{RHS} = -(\vec{r}_1 \vec{r}_2 | V | \psi_{\alpha\beta}^A) + \sum_{\gamma < F} \sum_{\delta < F} (\vec{r}_1 \vec{r}_2 | \gamma\delta) (\gamma\delta | V | \psi_{\alpha\beta}^A) - (\vec{r}_1 \vec{r}_2 | \alpha\beta^A) (\alpha\beta^A | V | \psi_{\alpha\beta}^A)$$

Now, since V is diagonal in \vec{r} -space,

$$(\vec{r}_1 \vec{r}_2 | V | \psi_{\alpha\beta}^A) = (\vec{r}_1 \vec{r}_2 | V | \vec{r}_1 \vec{r}_2) (\vec{r}_1 \vec{r}_2 | \psi_{\alpha\beta}^A)$$

$$= V(\vec{r}_1 \vec{r}_2) \psi_{\alpha\beta}^A(\vec{r}_1 \vec{r}_2) \quad \text{writing } (\vec{r}_1 \vec{r}_2 | V | \vec{r}_1 \vec{r}_2) = V(\vec{r}_1 \vec{r}_2).$$

$$\sum_{\gamma < F} \sum_{\delta < F} (\vec{r}_1 \vec{r}_2 | \gamma\delta) (\gamma\delta | V | \psi_{\alpha\beta}^A) = \int d\vec{r}_1' d\vec{r}_2' G(\vec{r}_1 - \vec{r}_1') G(\vec{r}_2 - \vec{r}_2') V(\vec{r}_1' \vec{r}_2') \psi_{\alpha\beta}^A(\vec{r}_1' \vec{r}_2')$$

where

$$G(\vec{r} - \vec{r}') = \sum_{\gamma < F} (\vec{r} | \gamma) (\gamma | \vec{r}') = \frac{1}{\Omega} \sum_{\gamma < F} e^{i\vec{k}_\gamma \cdot (\vec{r} - \vec{r}')} \sigma_{\gamma\sigma_\gamma}^* \tau_\gamma \tau_\gamma^*$$

If $V(\vec{r}_1 \vec{r}_2) \rightarrow 0$ when $|\vec{r}_1 - \vec{r}_2| \rightarrow \infty$ we have then

$$\left. \begin{aligned} (\vec{r}_1 \vec{r}_2 | V | \psi_{\alpha\beta}^A) &\rightarrow 0 \\ \sum_{\gamma < F} \sum_{\delta < F} (\vec{r}_1 \vec{r}_2 | \gamma\delta) (\gamma\delta | V | \psi_{\alpha\beta}^A) &\rightarrow 0 \end{aligned} \right\} \quad \text{For } |\vec{r}_1 - \vec{r}_2| \rightarrow \infty$$

where we have used also the fact that $G(\vec{r}-\vec{r}')$ is a delta function of finite width and vanishes as $|\vec{r}-\vec{r}'| \rightarrow \infty$.

Thus for $|\vec{r}_1-\vec{r}_2| \rightarrow \infty$ the BG equation becomes

$$(H_1+H_2-\epsilon_{\alpha\beta})(\vec{r}_1\vec{r}_2|\psi_{\alpha\beta}^A) = -(\alpha\beta|V|\psi_{\alpha\beta}^A)(\vec{r}_1\vec{r}_2|\alpha\beta^A).$$

But using (4.13) the RHS can be rewritten:

$$(\epsilon_\alpha+\epsilon_\beta-\epsilon_{\alpha\beta})(\vec{r}_1\vec{r}_2|\alpha\beta^A) \equiv (H_1+H_2-\epsilon_{\alpha\beta})(\vec{r}_1\vec{r}_2|\alpha\beta^A).$$

so that

$$(\vec{r}_1\vec{r}_2|\psi_{\alpha\beta}^A) \rightarrow (\vec{r}_1\vec{r}_2|\alpha\beta^A) \text{ when } |\vec{r}_1-\vec{r}_2| \rightarrow \infty$$

Thus, at large distances the BG wave function "heals" to the unperturbed wave function of a pair. This important fact leads to the concept of the "healing distance" of the BG wave function and is a very important result from the model.

Using (4.13) we can also obtain a simpler equation for the BG wave function. Substituting (4.13) into (4.11) yields

$$(H_1+H_2-\epsilon_\alpha-\epsilon_\beta)|\psi_{\alpha\beta}^A) = -QV|\psi_{\alpha\beta}^A) +(\alpha\beta^A|V|\psi_{\alpha\beta}^A) \left[|\psi_{\alpha\beta}^A) - |\alpha\beta^A) \right]$$

Now $(\alpha\beta^A|V|\psi_{\alpha\beta}^A)$ is infinitesimally small for a large volume of nuclear matter (for a square well potential of depth V_0 and range b this term is of the order of $\frac{V_0 b^3}{\Omega}$) and thus we can drop the second term on the RHS. Finally then, we have

the following BG equation for infinite nuclear matter

$$(4.15) \quad (H_1 + H_2 - \epsilon_a - \epsilon_\beta) |\psi_{a\beta}\rangle = - QV |\psi_{a\beta}\rangle$$

This is the form of the equation originally given by Bethe and Goldstone and we shall deal with this equation rather than the more complicated equation (4.11).

The solution of (4.15) can be written

$$|\psi_{a\beta}\rangle = |\alpha\beta\rangle + \frac{1}{\epsilon_a + \epsilon_\beta - H_1 - H_2} QV |\psi_{a\beta}\rangle$$

or

$$(4.16) \quad |\psi_{a\beta}\rangle = |\alpha\beta\rangle + \sum_{\gamma > F} \sum_{\delta > F} \frac{|\gamma\delta\rangle \langle \gamma\delta | V | \psi_{a\beta} \rangle}{\epsilon_a + \epsilon_\beta - \epsilon_\gamma - \epsilon_\delta}$$

The concept of self-consistency arises in the following way. Since $\Delta\epsilon_{a\beta}$ is the energy shift of a pair, if a particle is in the state β then the interaction with all other particles must be $\sum_{a < F} \Delta\epsilon_{a\beta}$ hence we get a new single particle potential U^\dagger

$$U^\dagger = \sum_{a < F} \Delta\epsilon_{a\beta}$$

To assure self-consistency we must choose U (equation (4.6)) so that the resulting U^\dagger is as close to it as possible.

Thus we have the following equations for the independent pair model of nuclear matter

$$(4.17a) \quad |\psi_{\alpha\beta}\rangle = |\alpha\beta\rangle + \sum_{\gamma>F} \sum_{\delta>F} \frac{|\gamma\delta\rangle \langle\gamma\delta|V|\psi_{\alpha\beta}\rangle}{\epsilon_{\alpha} + \epsilon_{\beta} - \epsilon_{\gamma} - \epsilon_{\delta}}$$

$$\epsilon_{\alpha} = \epsilon(k_{\alpha}) = \frac{\hbar^2 k_{\alpha}^2}{2m} + U(k_{\alpha})$$

$$(4.17b) \quad \Delta\epsilon_{\alpha\beta} = (\alpha\beta|V|\psi_{\alpha\beta}) - (\alpha\beta|V|\psi_{\beta\alpha})$$

$$(4.17c) \quad U(k_{\beta}) = \sum_{\alpha<F} \Delta\epsilon_{\alpha\beta}$$

$$(4.17d) \quad E = E_0 + \frac{1}{2} \sum_{\alpha<F} \sum_{\beta<F} \Delta\epsilon_{\alpha\beta}$$

It should be noted that because of the self-consistency requirement the value of the BG wave function $|\psi_{\alpha\beta}\rangle$ is required for $k_{\alpha}, k_{\beta} > k_F$. The wave function in this case is taken to be (4.17a), using the principal value of the integral. We shall no longer use \mathcal{P} to denote the principal value of an integral but will simply understand from this point on that all singular integrals arising will be taken to be principal value integrals.

It should be noted further that in this discussion we have completely neglected "propagation off the energy shell" effects. This phrase was coined by Brueckner and the concept explained in some detail by Bethe²⁾, by Brueckner and Gammel¹⁾, and by

Brueckner and Goldman¹⁸⁾. The whole point of the matter is that the energy denominator occurring in (4.17a) cannot strictly speaking be written $\epsilon_\alpha + \epsilon_\beta - \epsilon_\gamma - \epsilon_\delta$ since the energies of the two particles in the virtual excited states (i.e. in levels γ and δ) depend not only on these states but also on the states of the two holes remaining below the Fermi sea. The wave function of the two particles in these states thus depends on four quantum numbers (γ, δ and two for the holes) and satisfies an equation which is similar to (but more complicated than) equation (4.17a). In the energy denominators of this more complicated equation, the energy of the particles in the virtual excited states will depend on additional quantum numbers and the wave function describing these states will satisfy another more complicated equation with still more complicated energy denominators occurring. This situation is repeated without end. Fortunately however the dependence on the additional quantum numbers is not great and what can be done is approximate the effect by introducing in the first step some average excitation energy in the energy denominator, thus terminating the infinite set of equations. Brueckner and Gammel¹⁾ have done this (they call the average excitation energy Δ) and their results are indeed not strongly dependent on Δ . As mentioned above, we shall neglect this complication entirely in this thesis.

Let us now introduce an operator K defined as follows

$$(4.18) \quad K|a\beta) = V|\psi_{a\beta})$$

so that from (4.17) we have

$$(4.19a) \quad (a'\beta'|K|a\beta) = (a'\beta'|V|a\beta) + \sum_{\gamma>F} \sum_{\delta>F} \frac{(a'\beta'|V|\gamma\delta)(\gamma\delta|K|a\beta)}{\epsilon_{a'} + \epsilon_{\beta} - \epsilon_{\gamma} - \epsilon_{\delta}}$$

$$(4.19b) \quad \Delta\epsilon_{a\beta} = (a\beta|K|a\beta) - (a\beta|K|\beta a)$$

Equations (4.19) were first introduced by Brueckner in his investigations of the nuclear many-body problem. Equations (4.18) and (4.19) will form the basis of the remainder of the work of this thesis.

We shall now put (4.18) and (4.19) in a more useable form by noticing some properties of the two-body potential. This potential is diagonal in isotopic spin space (since we neglect charge) and further can connect only states of the same total spin of the two particles. Thus in equations (4.18a) and (4.19a) we can suppress the isotopic spin part of the wave function, and in addition we need only consider pure total spin states of the two particle system. Thus, we shall take

$$|K_1 K_2; \chi_{S m_S}) \text{ in place of } |a\beta)$$

and

$$|\psi_{k_1 k_2}; \chi_{S m_S}) \text{ in place of } |\psi_{\alpha\beta})$$

where we now write the wave vectors of particles 1 and 2 as \vec{k}_1 and \vec{k}_2 rather than \vec{k}_α and \vec{k}_β , and where $\chi_{S m_S}$ is the normalized spin eigenfunction. (If \vec{S} is the total spin vector of the two particle system and S_z the z-projection of \vec{S} , then $\chi_{S m_S}$ is an eigenfunction of S^2 and S_z with eigenvalues $S(S+1)$ and m_S respectively.)

From (4.18a) then, the equation for $|\psi_{k_1 k_2}; \chi_{S m_S})$ is

$$|\psi_{k_1 k_2}; \chi_{S m_S}) = |k_1 k_2; \chi_{S m_S}) + \sum_{m_S'=-S}^{+S} \sum_{k_1' > k_F} \sum_{k_2' > k_F} \frac{|\vec{k}_1' \vec{k}_2'; \chi_{S m_S'}) (\vec{k}_1' \vec{k}_2'; \chi_{S m_S'} | V | \psi_{k_1 k_2}; \chi_{S m_S})}{\epsilon(k_1) + \epsilon(k_2) - \epsilon(k_1') - \epsilon(k_2')}$$

We now consider the coordinate representation of this equation. We can separate out the centre of mass motion provided V is independent of the centre of mass position coordinate of the two particles, as we now show.

We have

(4.20)

$$\begin{aligned} \psi_{k_1 k_2}(\vec{r}_1 \vec{r}_2) \chi_{S m_S} &\equiv (\vec{r}_1 \vec{r}_2 | \psi_{k_1 k_2}; \chi_{S m_S}) = \frac{1}{\Omega} e^{i\vec{k}_1 \cdot \vec{r}_1} e^{i\vec{k}_2 \cdot \vec{r}_2} \chi_{S m_S} \\ &+ \sum_{m_S' = -S}^{+S} \sum_{k_1' > k_F} \sum_{k_2' > k_F} \frac{\frac{1}{\Omega} e^{i\vec{k}_1 \cdot \vec{r}_1} e^{i\vec{k}_2 \cdot \vec{r}_2} \chi_{S m_S'}(\vec{k}_1' \vec{k}_2'; \chi_{S m_S'} | V | \psi_{k_1 k_2}; \chi_{S m_S})}{\epsilon(k_1) + \epsilon(k_2) - \epsilon(k_1') - \epsilon(k_2')} \end{aligned}$$

where we have used (4.1) to write

$$(\vec{r}_1 \vec{r}_2 | \vec{k}_1 \vec{k}_2; \chi_{S m_S}) = \frac{1}{\Omega} e^{i\vec{k}_1 \cdot \vec{r}_1} e^{i\vec{k}_2 \cdot \vec{r}_2} \chi_{S m_S}.$$

Now let us introduce centre of mass and relative coordinates by

$$(4.21) \quad \vec{k} = \frac{1}{2}(\vec{k}_1 - \vec{k}_2) \quad \vec{r} = \vec{r}_1 - \vec{r}_2$$

$$\vec{P} = \frac{1}{2}(\vec{k}_1 + \vec{k}_2) \quad \vec{R} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2)$$

$$\vec{k}_1 \cdot \vec{r}_1 + \vec{k}_2 \cdot \vec{r}_2 = \vec{k} \cdot \vec{r} + 2\vec{P} \cdot \vec{R}$$

where \vec{k} is the relative momentum, $2\vec{P}$ is the centre of mass momentum, \vec{r} is the relative coordinate, and \vec{R} is the centre of mass coordinate.

Now if V is independent of the centre of mass coordinate we can write

$$\psi_{k_1 k_2}(\vec{r}_1 \vec{r}_2) = \frac{1}{\Omega} e^{2i\vec{P} \cdot \vec{R}} \psi_{kP}(\vec{r})$$

so that

$$\begin{aligned} & (\vec{k}_1 \vec{k}_2; \chi_{Sm_S} | V | \psi_{k_1 k_2}; \chi_{Sm_S}) = \\ &= \frac{1}{\Omega^2} \int d\vec{r} d\vec{R} e^{-i(\vec{k} \cdot \vec{r} - 2\vec{P} \cdot \vec{R})} \chi_{Sm_S}^* V(\vec{r}) e^{2i\vec{P} \cdot \vec{R}} \psi_{kP}(\vec{r}) \chi_{Sm_S} \\ &= \delta_{\vec{P}\vec{P}'} \frac{1}{\Omega} (\vec{k}; \chi_{Sm_S} | V | \psi_{kP}; \chi_{Sm_S}) \end{aligned}$$

where

$$(\vec{k}; \chi_{Sm_S} | V | \psi_{kP}; \chi_{Sm_S}) = \int d\vec{r} e^{-i\vec{k} \cdot \vec{r}} \chi_{Sm_S}^* V(\vec{r}) \psi_{kP}(\vec{r}) \chi_{Sm_S}$$

and where we have used

$$\frac{1}{\Omega} \int d\vec{R} e^{2i(\vec{P} - \vec{P}') \cdot \vec{R}} = \delta_{\vec{P}\vec{P}'}$$

From (4.20) we obtain the following equation for $\psi_{kP}(\vec{r}) \chi_{Sm_S}$

$$\begin{aligned} (4.22) \quad \psi_{kP}(\vec{r}) \chi_{Sm_S} &= e^{i\vec{k} \cdot \vec{r}} \chi_{Sm_S} + \\ &+ \frac{1}{\Omega} \sum_{|\vec{P} + \vec{k}'| > k_F} \sum_{m_S = -S}^{+S} \frac{e^{i\vec{k}' \cdot \vec{r}} \chi_{Sm_S'} (\vec{k}'; \chi_{Sm_S'} | V | \psi_{kP}; \chi_{Sm_S})}{\epsilon(\vec{P} + \vec{k}) + \epsilon(\vec{P} - \vec{k}') - \epsilon(\vec{P} + \vec{k}') - \epsilon(\vec{P} - \vec{k})} \end{aligned}$$

The summation $\sum_{\substack{\vec{k}' \\ |\vec{P}+\vec{k}'| > k_F \\ |\vec{P}-\vec{k}'| > k_F}}$ means summation over all \vec{k}' wherein $|\vec{P}+\vec{k}'| > k_F$ and $|\vec{P}-\vec{k}'| > k_F$. Taking now $\Omega \rightarrow \infty$ we replace the summation by an integral using

$$(4.23) \quad \sum_{\vec{k}} \rightarrow \frac{\Omega}{(2\pi)^3} \int d\vec{k}$$

so that finally

$$(4.24) \quad \psi_{kP}(\vec{r}) \chi_{Sm_S} = e^{i\vec{k} \cdot \vec{r}} \chi_{Sm_S} + \\ + \sum_{m_S=-S}^{+S} \int d\vec{r}' \chi_{Sm_S} G_{kP}(\vec{r}, \vec{r}') \chi_{Sm_S}^* V(\vec{r}') \psi_{kP}(\vec{r}') \chi_{Sm_S}$$

where

$$G_{kP}(\vec{r}, \vec{r}') \equiv (\vec{r} | G_{kP} | \vec{r}') = \left(\frac{1}{2\pi} \right)^3 \int_{\substack{\vec{k}' \\ |\vec{P}+\vec{k}'| > k_F}} \frac{d\vec{k}' e^{i\vec{k}' \cdot (\vec{r}-\vec{r}')}}{\epsilon(\vec{P}+\vec{k}') + \epsilon(\vec{P}-\vec{k}') - \epsilon(\vec{P}+\vec{k}') - \epsilon(\vec{P}-\vec{k}')}.$$

The K matrix elements now are

$$(4.25) \quad (\vec{k}_1' \vec{k}_2'; \chi_{Sm_S} | K | \vec{k}_1 \vec{k}_2; \chi_{Sm_S}) = (\vec{k}_1' \vec{k}_2'; \chi_{Sm_S} | V | \psi_{k_1 k_2}; \chi_{Sm_S}) = \\ = \frac{1}{\Omega} (\vec{k}' ; \chi_{Sm_S} | K | \vec{k}, \vec{P}; \chi_{Sm_S})$$

where we define

$$(4.26) \quad (\vec{k}'; \chi_{Sm_S} | K | \vec{k}, \vec{P}; \chi_{Sm_S}) = (\vec{k}'; \chi_{Sm_S} | V | \psi_{kP}; \chi_{Sm_S}) = \\ = \int d\vec{r} \, e^{-i\vec{k}' \cdot \vec{r}} \, \chi_{Sm_S}^* V(\vec{r}) \psi_{kP}(\vec{r}) \chi_{Sm_S}$$

From (4.26) and (4.24) we get the following integral equation for $(\vec{k}'; \chi_{Sm_S} | K | \vec{k}, \vec{P}; \chi_{Sm_S})$

$$(4.27) \quad (\vec{k}'; \chi_{Sm_S} | K | \vec{k}, \vec{P}; \chi_{Sm_S}) = (\vec{k}'; \chi_{Sm_S} | V | \vec{k}; \chi_{Sm_S}) \\ + \sum_{m_S''=-S}^{+S} \left(\frac{1}{2\pi} \right)^3 \int \frac{d\vec{k}'' (\vec{k}'; \chi_{Sm_S} | V | \vec{k}''; \chi_{Sm_S}) (\vec{k}''; \chi_{Sm_S} | K | \vec{k}, \vec{P}; \chi_{Sm_S})}{|\vec{P} + \vec{k}''| > k_F \, \epsilon(\vec{P} + \vec{k}) + \epsilon(\vec{P} - \vec{k}) - \epsilon(\vec{P} + \vec{k}'') - \epsilon(\vec{P} - \vec{k}'')}$$

where (4.28)

$$(\vec{k}'; \chi_{Sm_S} | V | \vec{k}; \chi_{Sm_S}) = \int d\vec{r} \, e^{-i\vec{k}' \cdot \vec{r}} \, \chi_{Sm_S}^* V(\vec{r}) e^{i\vec{k} \cdot \vec{r}} \chi_{Sm_S}$$

The similarity between equations (4.24), (4.26), (4.27) and the two-body equations (3.9), (3.12), and (3.13) should be noted. In the two-body case it is possible to effect a complete separation of centre of mass coordinates but in the BG case this is not possible because the lower limit of the \vec{k}'' integrals and the energy denominators have a \vec{P} dependence. That a \vec{P} dependence arises in the BG case and not in the

two-body case should not be surprising since in the former the two particles are moving against a background of other particles, whereas in the latter the particles are isolated in space. This \vec{P} dependence complicates the problem considerably but in the next Chapter we shall introduce techniques to handle this.

We shall now put (4.17c) and (4.17d) in more convenient forms. We make the observation that the potential can be written as

$$V = {}^1V^+ + {}^1V^- + {}^3V^+ + {}^3V^-$$

where ${}^1V^+$ connects only states having singlet spin, triplet isospin, and even parity (i.e., ${}^1S_0, {}^1D_2, \dots$)

${}^1V^-$ connects only states having singlet spin, singlet isospin, and odd parity (i.e., ${}^1P_1, {}^1F_3, \dots$)

${}^3V^+$ connects only states having triplet spin, singlet isospin, and even parity (i.e., ${}^3S_1, {}^3D_1, \dots$)

${}^3V^-$ connects only states having triplet spin, triplet isospin, and odd parity (i.e., ${}^3P_0, {}^3P_1, \dots$)

When V can be written in the above form, then indeed K can be written as

$$K = {}^1K^+ + {}^1K^- + {}^3K^+ + {}^3K^-$$

where ${}^1K^+$ connects the same states as does ${}^1V^+$, etc.

Thus we have that

$$\begin{aligned}
 (4.29) \quad & \sum_{\alpha < F} \sum_{\beta < F} \left[(\alpha\beta | K | \alpha\beta) - (\alpha\beta | K | \beta\alpha) \right] = \\
 & = \sum_{k_1 < k_F} \sum_{k_2 < k_F} \left[6(\vec{k}_1 \vec{k}_2; \chi_{00} | {}^1K^+ | \vec{k}_1 \vec{k}_2; \chi_{00}) + 2(\vec{k}_1 \vec{k}_2; \chi_{00} | {}^1K^- | \vec{k}_1 \vec{k}_2; \chi_{00}) \right. \\
 & \left. + \sum_{m_S = -1}^{+1} \left\{ 2(\vec{k}_1 \vec{k}_2; \chi_{1m_S} | {}^3K^+ | \vec{k}_1 \vec{k}_2; \chi_{1m_S}) + 6(\vec{k}_1 \vec{k}_2; \chi_{1m_S} | {}^3K^- | \vec{k}_1 \vec{k}_2; \chi_{1m_S}) \right\} \right]
 \end{aligned}$$

The statistical factors can easily be understood. In the first term, for example, a factor 3 comes from summing over the 3 isospin states, and another factor 2 arises from the exchange terms.

We can write also

$$\begin{aligned}
 (4.30) \quad & \sum_{\alpha < F} \left[(\alpha\beta | K | \alpha\beta) - (\alpha\beta | K | \beta\alpha) \right] = \\
 & = \sum_{k_1 < k_F} \left[\frac{3}{2}(\vec{k}_1 \vec{k}_2; \chi_{00} | {}^1K^+ | \vec{k}_1 \vec{k}_2; \chi_{00}) + \frac{1}{2}(\vec{k}_1 \vec{k}_2; \chi_{00} | {}^1K^- | \vec{k}_1 \vec{k}_2; \chi_{00}) \right. \\
 & \left. + \sum_{m_S = -1}^{+1} \left\{ \frac{1}{2}(\vec{k}_1 \vec{k}_2; \chi_{1m_S} | {}^3K^+ | \vec{k}_1 \vec{k}_2; \chi_{1m_S}) + \frac{3}{2}(\vec{k}_1 \vec{k}_2; \chi_{1m_S} | {}^3K^- | \vec{k}_1 \vec{k}_2; \chi_{1m_S}) \right\} \right]
 \end{aligned}$$

The summations in (4.29) and (4.30) can be changed to

integrals using (4.23). In (4.29) we write

$$\sum_{k_1 < k_F} \sum_{k_2 < k_F} \rightarrow \frac{\Omega^2}{(2\pi)^6} \int_{k_1 < k_F} d\vec{k}_1 \int_{k_2 < k_F} d\vec{k}_2$$

Now from (4.21) $d\vec{k}_1 d\vec{k}_2 = 8d\vec{k} d\vec{P}$, and using (4.5) we can rewrite one of the Ω 's as $\frac{3\pi^2 A}{2k_F^3}$, so that

$$(4.31) \quad \sum_{k_1 < k_F} \sum_{k_2 < k_F} \rightarrow \frac{3\pi^2 A}{2k_F^3} \frac{8\Omega}{(2\pi)^6} \int_{P < k_F} d\vec{P} \int_{|\vec{P} + \vec{k}| < k_F} d\vec{k}$$

The integral over \vec{k} is over all values of \vec{k} wherein $|\vec{P} + \vec{k}| < k_F$ and $|\vec{P} - \vec{k}| < k_F$. The integral over \vec{P} is over all values of \vec{P} wherein $P < k_F$.

In (4.30) we write

$$\sum_{k_1 < k_F} \rightarrow \frac{\Omega}{(2\pi)^3} \int_{k_1 < k_F} d\vec{k}_1$$

But from (4.21) $d\vec{k}_1 = 8d\vec{k}$ when \vec{k}_2 is kept fixed so that we write

$$(4.32) \quad \sum_{k_1 < k_F} \rightarrow \frac{8\Omega}{(2\pi)^3} \int_{|2\vec{k} + \vec{k}_2| < k_F} d\vec{k}$$

The integral over \vec{k} is over all values of \vec{k} wherein \vec{k}_2 is fixed and $|2\vec{k} + \vec{k}_2| < k_F$.

Let us call $E-E_0$ the potential energy (PE) of the system, then using (4.17d), (4.25), (4.29) and (4.31) we have

$$(4.33) \quad PE/A = \frac{3}{16\pi^4 k_F^3} \int_{P < k_F} d\vec{P} \int_{|\vec{P}+\vec{K}| < k_F} d\vec{K} \left[3(\vec{K}; \alpha_{00} | {}^1K^+ | \vec{K}, \vec{P}; \alpha_{00}) \right. \\ \left. + (\vec{K}; \alpha_{00} | {}^1K^- | \vec{K}, \vec{P}; \alpha_{00}) \right. \\ \left. + \sum_{m_S=-1}^{+1} \left\{ (\vec{K}; \alpha_{1m_S} | {}^3K^+ | \vec{K}, \vec{P}; \alpha_{1m_S}) + 3(\vec{K}; \alpha_{1m_S} | {}^3K^- | \vec{K}, \vec{P}; \alpha_{1m_S}) \right\} \right]$$

Using (4.17c), (4.30), (4.25), and (4.32) we have

$$(4.34) \quad U(k_2) = \\ = \frac{1}{2\pi^3} \int_{|2\vec{K}+\vec{K}_2| < k_F} d\vec{K} \left[3(\vec{K}; \alpha_{00} | {}^1K^+ | \vec{K}, \vec{P}; \alpha_{00}) + (\vec{K}; \alpha_{00} | {}^1K^- | \vec{K}, \vec{P}; \alpha_{00}) \right. \\ \left. + \sum_{m_S=-1}^{+1} \left\{ (\vec{K}; \alpha_{1m_S} | {}^3K^+ | \vec{K}, \vec{P}; \alpha_{1m_S}) + 3(\vec{K}; \alpha_{1m_S} | {}^3K^- | \vec{K}, \vec{P}; \alpha_{1m_S}) \right\} \right]$$

In the above $\vec{P}=\vec{K}+\vec{K}_2$ from (4.21).

We have now put the self-consistent equations for the IPAM in a useable form. The procedure now is to solve (4.24) for the BG wave function, use (4.26) to obtain the K matrix elements (or obtain the K matrix elements directly from (4.27)), calculate the new single particle potential from (4.34), and

calculate the potential energy of the system from (4.33). Self-consistency is obtained by adjusting the energy denominators so that the single particle potential used there, is equal to the single particle potential calculated. The energy of the system will be a function of k_F . If the nucleon-nucleon potential used provides an adequate description of the nuclear force, and if the IPAM gives a good description of nuclear matter, then, when k_F is varied, E/A should attain a minimum of approximately -15 Mev when $k_F \approx 1.4 \text{ (fermi)}^{-1}$.

Attempts to improve the IPAM (or the Brueckner approximation) have been made by including the so-called "hole-hole interactions". In the virtual excitations of the system, the interactions between the two particles above the Fermi sea have been taken into account, but no interactions of the two holes below the Fermi sea have been included. It has been conjectured that a model which includes these latter interactions would give a better approximation to the many-body problem.^{6,7)} Of course, the interaction of the two holes is really a many-body interaction and thus including the hole-hole interactions means that we now include certain many-body interactions. As we pointed out in Section 4.1, the Brueckner approximation for the energy consists in picking out the "ladder" diagrams from the Goldstone series. Including hole-hole interactions means including, in addition to the ladder diagrams, diagrams containing interactions

between the hole lines on the ladders. When hole-hole interactions are included, the BG equation (4.15) is changed to

$$(4.35) \quad (H_1 + H_2 - \epsilon_\alpha - \epsilon_\beta) |\psi_{\alpha\beta}\rangle = - (Q - \bar{Q}) V |\psi_{\alpha\beta}\rangle$$

$$\bar{Q} = \sum_{\gamma < F} \sum_{\delta < F} |\gamma\delta\rangle \langle \gamma\delta|$$

and the integral equation for $|\psi_{\alpha\beta}\rangle$ now is

$$(4.36) \quad |\psi_{\alpha\beta}\rangle = |\alpha\beta\rangle + \\ + \sum_{\gamma > F} \sum_{\delta > F} \frac{|\gamma\delta\rangle \langle \gamma\delta| V |\psi_{\alpha\beta}\rangle}{\epsilon_\alpha + \epsilon_\beta - \epsilon_\gamma - \epsilon_\delta} - \sum_{\gamma < F} \sum_{\delta < F} \frac{|\gamma\delta\rangle \langle \gamma\delta| V |\psi_{\alpha\beta}\rangle}{\epsilon_\alpha + \epsilon_\beta - \epsilon_\gamma - \epsilon_\delta}$$

The principal value of the singular integral is to be taken.

The new K matrix is given by

$$(4.37) \quad (\alpha'\beta' | K | \alpha\beta) = (\alpha'\beta' | V | \alpha\beta) + \\ + \sum_{\gamma > F} \sum_{\delta > F} \frac{(\alpha'\beta' | V | \gamma\delta) \langle \gamma\delta | K | \alpha\beta \rangle}{\epsilon_\alpha + \epsilon_\beta - \epsilon_\gamma - \epsilon_\delta} - \sum_{\gamma < F} \sum_{\delta < F} \frac{(\alpha'\beta' | V | \gamma\delta) \langle \gamma\delta | K | \alpha\beta \rangle}{\epsilon_\alpha + \epsilon_\beta - \epsilon_\gamma - \epsilon_\delta}$$

and (4.24) and (4.27) become

$$(4.38) \quad \psi_{\mathbf{kP}}(\vec{r}) \chi_{S m_S} = \\ = e^{i\vec{k} \cdot \vec{r}} \chi_{S m_S} + \sum_{m'_S = -S}^{+S} \int d\vec{r}' \chi_{S m'_S} G_{\mathbf{kP}}(\vec{r}, \vec{r}') \chi_{S m'_S}^* V(\vec{r}') \psi_{\mathbf{kP}}(\vec{r}') \chi_{S m_S}$$

$$G_{\mathbf{kP}}(\vec{r}, \vec{r}') =$$

$$= \left(\frac{1}{2\pi}\right)^3 \left[\int_{|\vec{P} + \vec{k}'| > k_F} d\vec{k}' - \int_{|\vec{P} + \vec{k}'| < k_F} d\vec{k}' \right] \frac{e^{i\vec{k}' \cdot (\vec{r} - \vec{r}')}}{\epsilon(\vec{P} + \vec{k}') + \epsilon(\vec{P} - \vec{k}') - \epsilon(\vec{P} + \vec{k}') - \epsilon(\vec{P} - \vec{k}')}$$

$$\begin{aligned}
 (4.39) \quad & (\vec{k}'; \chi_{S m_S'} | K | \vec{k}, \vec{p}; \chi_{S m_S}) = \\
 & = (\vec{k}'; \chi_{S m_S'} | V | \vec{k}; \chi_{S m_S}) + \sum_{m_S''=-S}^{+S} \left(\frac{1}{2\pi} \right)^3 \left[\int_{|\vec{p}+\vec{k}''| > k_F} d\vec{k}'' \right. \\
 & \left. - \int_{|\vec{p}+\vec{k}''| < k_F} d\vec{k}'' \right] \frac{(\vec{k}'; \chi_{S m_S'} | V | \vec{k}''; \chi_{S m_S''}) (\vec{k}''; \chi_{S m_S''} | K | \vec{k}, \vec{p}; \chi_{S m_S})}{\epsilon(\vec{p}+\vec{k}) + \epsilon(\vec{p}-\vec{k}) - \epsilon(\vec{p}+\vec{k}'') - \epsilon(\vec{p}-\vec{k}'')}
 \end{aligned}$$

In the above the principal value of the singular integral is to be taken.

The wave function given by (4.35) (which we shall call the Bethe-Goldstone-Iwamoto (BGI) wave function) does not heal to the unperturbed wave function at large distance (when $k_\alpha, k_\beta < k_F$) because of the pole in the second integral. However as Iwamoto⁷⁾ has pointed out, the physical interpretation of the BGI wave function is different from that of the BG wave function. The BG wave function is the amplitude of the physical particle pair state compared to the degenerate Fermi gas ground state, but the BGI wave function is the amplitude of the physical hole pair or physical particle pair state compared to the physical A particle ground state. Thus there is no reason why the BGI wave function should heal to the unperturbed two-particle state. One can think of the BG wave function as a

Tamm-Dancoff amplitude and BGI wave function as a Tamm-Dancoff-Dyson amplitude.

This completes our formal analysis of the nuclear many-body problem. We have set up the equations for the IPAM and for the IPAM including hole-hole interactions and we shall in the next Chapter examine these equations when the potential is given by our pseudopotential (2.5).

CHAPTER V

USE OF THE PSEUDOPOTENTIAL IN THE NUCLEAR MANY-BODY PROBLEM

In this Chapter we shall apply the BCM via our pseudopotential (2.5) to the IPAM equations of the last Chapter. The pseudopotential is equivalent to the BCM and we have in Chapter III used it in a discussion of the two-body problem. We found then that because of its simple structure certain equations could be solved exactly. We investigate now what success the model has in the many-body problem, and in particular in the IPAM of nuclear matter. We shall see that the simple structure of the pseudopotential allows us to handle the IPAM equations relatively easily and without taking recourse to perturbation methods. Thus we are able to use a realistic description of nuclear forces (the BCM) and still obtain simplicity in handling the IPAM equations.

We shall be concerned with finding the K matrix elements. There are two methods we can use: We can solve (4.24) for the BG wave function then calculate the K matrix elements by (4.26), or we can solve directly the integral equation (4.27) for the K matrix elements. We shall use the latter method. (For simplicity we neglect for the moment the hole-hole

interactions since they can easily be taken into account at the end of the calculation.) It will be recalled from our discussion at the end of Section 3.2 that, in the two-body case, these two methods lead to different results since by solving directly the integral equation for the K matrix elements, it is not possible to take into account the discontinuity in the first derivative of the Schrodinger wave function. The same situation will prevail here since the first derivative of the BG wave function is also discontinuous at the core radius. We shall then do the following: We shall disregard the difference between r_0^+ and r_0^- in the pseudopotential, and then in our final results for the K matrix elements we shall interpret certain integrals similarly to the way they were interpreted when we constructed the two-body K matrix elements from the Schrodinger wave function. As a check on our results we should find in the singlet case that as $k_F \rightarrow 0$ the diagonal K matrix elements for each angular momentum state are proportional to the tangent of the phase shift. These remarks will become clearer later. The important point here is that we can disregard the difference between r_0^+ and r_0^- in the pseudopotential when we solve the integral equation (4.27) for the K matrix elements.

5.1 The K Matrix Elements for Singlet Spin States

Let us consider first the singlet spin state pseudo-potential which from (2.5) is

$$(5.1) \quad V(r) = \frac{\hbar^2}{m} \sum_{\ell=0}^{\infty} \left[\frac{f_{\ell}}{r_{o_{\ell}}} \delta(r-r_{o_{\ell}}) - \delta(r-r_{o_{\ell}}) \frac{\partial}{\partial r} \right] P_{\ell 0}$$

The integral equation we want to solve is (4.27) for $S=0$, that is,

$$(5.2a) \quad (\vec{k}' | K | \vec{k}, \vec{P}) = (\vec{k}' | V | \vec{k}) + \left(\frac{1}{2\pi} \right)^3 \int_{|\vec{P}+\vec{k}''| > k_F} d\vec{k}'' e^{-i(\vec{k}' | V | \vec{k}'')} (\vec{k}'' | K | \vec{k}, \vec{P})$$

$$(5.2b) \quad e = \varepsilon(\vec{P}+\vec{k}) + \varepsilon(\vec{P}-\vec{k}) - \varepsilon(\vec{P}+\vec{k}'') - \varepsilon(\vec{P}-\vec{k}'')$$

We neglect for the moment the hole-hole interactions since we can easily take these interactions into account at the end of the calculation.

First we need $(\vec{k}' | V | \vec{k})$ which is defined by (4.28). Using (5.1) and the decomposition (3.17) we have

$$(5.3a) \quad (\vec{k}' | V | \vec{k}) = \int d\vec{r} e^{-i\vec{k}' \cdot \vec{r}} V(r) e^{i\vec{k} \cdot \vec{r}} \\ = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} a_{\ell}(\vec{k}') Y_{\ell m}(\hat{k}') b_{\ell}(\vec{k}) Y_{\ell m}^*(\hat{k})$$

$$(5.3b) \quad a_{\ell}(\vec{k}') = \frac{(4\pi\hbar)^2 r_{o_{\ell}}}{m} j_{\ell}(k' r_{o_{\ell}})$$

$$(5.3c) \quad b_\ell(k) = f_\ell j_\ell(kr_{o_\ell}) - kr_{o_\ell} j'_\ell(kr_{o_\ell})$$

We now substitute the separable potential (5.3) into (5.2) and obtain:

$$(\vec{k}' | K | \vec{k}, \vec{P}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} a_\ell(k') Y_{\ell m}(\hat{k}') \left[b_\ell(k) Y_{\ell m}^*(\hat{k}) + \left(\frac{1}{2\pi}\right)^3 \int_{|\vec{P}+\vec{k}''| > k_F} d\vec{k}'' e^{-1} b_\ell(k'') Y_{\ell m}^*(\hat{k}'') (\vec{k}'' | K | \vec{k}, \vec{P}) \right]$$

That is

$$(5.4) \quad (\vec{k}' | K | \vec{k}, \vec{P}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} a_\ell(k') Y_{\ell m}(\hat{k}') f_{\ell m}(\vec{k}, \vec{P})$$

where

$$(5.5) \quad f_{\ell m}(\vec{k}, \vec{P}) = b_\ell(k) Y_{\ell m}^*(\hat{k}) + \left(\frac{1}{2\pi}\right)^3 \int_{|\vec{P}+\vec{k}''| > k_F} d\vec{k}'' e^{-1} b_\ell(k'') Y_{\ell m}^*(\hat{k}'') (\vec{k}'' | K | \vec{k}, \vec{P})$$

Now by substituting (5.4) into (5.5) we obtain a set of algebraic equations for the f 's:

$$(5.6) \quad f_{\ell m}(\vec{k}, \vec{P}) = b_\ell(k) Y_{\ell m}^*(\hat{k}) + \left(\frac{1}{2\pi}\right)^3 \sum_{L=0}^{\infty} \sum_{\mu=-L}^{+L} f_{L\mu}(\vec{k}, \vec{P}) \int_{|\vec{P}+\vec{k}''| > k_F} d\vec{k}'' e^{-1} b_\ell(k'') a_L(k'') Y_{\ell m}^*(\hat{k}'') Y_{L\mu}(\hat{k}'')$$

Let us call the integral occurring in (5.6)

$I_{\ell m; L\mu}(\vec{k}, \vec{P})$, i.e.,

$$(5.7) \quad I_{\ell m; L\mu}(\vec{k}, \vec{P}) = \int_{|\vec{P} + \vec{k}''| > k_F} d\vec{k}'' e^{-i\vec{k}'' \cdot \vec{r}} b_{\ell}(k'') a_L(k'') Y_{\ell m}^*(k'') Y_{L\mu}(k'')$$

In contrast to the two-body case, we cannot immediately simplify this integral by using the orthonormality of the spherical harmonics since the region of integration is not spherically symmetric. The condition $|\vec{P} + \vec{k}''| > k_F$ means that the integration is over all of \vec{k}'' space exterior to the volume of two spheres each of radius k_F , the distance between whose centres is $2P$. If the origin of \vec{k}'' is halfway along the line joining the centres of the spheres, then, for $P < k_F$, the integration is over the darkened region in the following figure:

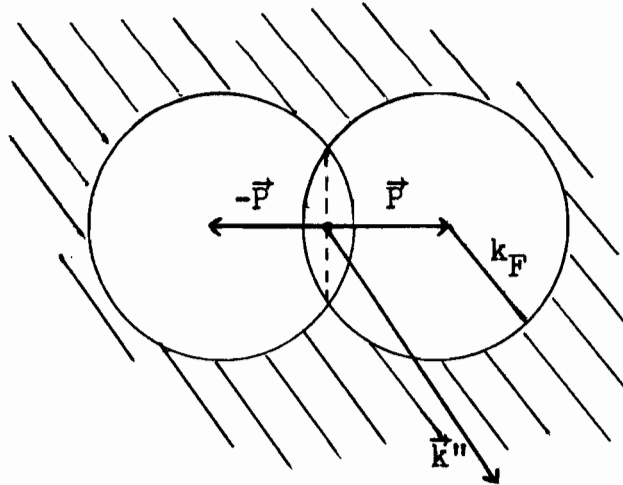


Figure A

The lack of spherical symmetry of the integration region prevents us from giving an exact expression for $(\vec{K}' | K | \vec{K}, \vec{P})$ since (5.6) is an infinite set of coupled algebraic equations. For the special case when $\vec{P} = 0$ the integration region is spherical and the equations uncouple. The difficulty then is in treating the non-zero centre of mass momentum case. In the more standard treatments of the Brueckner theory the same situation arises, of course, and what has been done is either to assume $(\vec{K}' | K | \vec{K}, \vec{P})$ is independent of \vec{P} and thus equal to the value for $\vec{P} = 0$ ³⁾, or to treat the non-zero centre of mass momentum case in some average fashion.^{1,4,5)} We shall make no such approximations. In the treatment we give below we shall assume for simplicity however that the energy denominator e is independent of the angles between the k -vectors and the P -vectors. (It is indeed independent of angles in some cases as we shall see in the next Chapter.) The methods we introduce below can be used to handle the case of angular dependent energy denominators but we do not include this most general case here.

Let us concern ourselves now with the integral $I_{l_m; L_\mu}(\vec{K}, \vec{P})$. This is a three dimensional integral but we shall introduce a technique for handling the angular integrations. We note first that

$$|\vec{P} + \vec{K}''| > k_F \quad \text{means } k''^2 - 2Pk'' |\cos\theta| + P^2 > k_F^2$$

where θ is the angle between \vec{k}'' and \vec{P} . We shall consider \vec{P} to be the z-axis for the coordinate system so that we can write (5.7) as

$$I_{\ell m; L \mu}(\vec{k}, P) = \int_{k''^2 - 2Pk'' |\cos \theta| + P^2 > k_F^2} dk'' k''^2 d(\cos \theta) d\varphi e^{-1} b_{\ell}(k'') a_L(k'') Y_{\ell m}^*(\theta, \varphi) Y_{L \mu}(\theta, \varphi)$$

Now we define a function $\zeta(k, P, |\cos \theta|)$ by

$$(5.8) \quad \zeta(k, P, |\cos \theta|) = \begin{cases} 1 & \text{if } k^2 - 2Pk |\cos \theta| + P^2 > k_F^2 \\ 0 & \text{otherwise} \end{cases}$$

so that $I_{\ell m; L \mu}$ can be written

$$(5.9) \quad I_{\ell m; L \mu}(\vec{k}, P) = \int_0^{\infty} dk'' k''^2 \int_{-1}^{+1} d(\cos \theta) \int_0^{2\pi} d\varphi e^{-1} b_{\ell}(k'') a_L(k'') Y_{\ell m}^*(\theta, \varphi) Y_{L \mu}(\theta, \varphi) \zeta(k'', P, |\cos \theta|)$$

Thus by introducing $\zeta(k'', P, |\cos \theta|)$ we can extend the \vec{k}'' integral over all of momentum space. This trick has also been used by Brueckner and Gammel¹⁾ and by Moszkowski and Scott⁴⁾ but they further replace the function they have introduced by its angular average. We shall go beyond this approximation.

We now expand $\zeta(k, P, |\cos\theta|)$ in terms of Legendre polynomials, $P_\ell(\cos\theta)$. These functions have the following properties:^{19,20)}

$$(5.10a) \quad \int_{-1}^{+1} dx P_\ell(x) P_{\ell'}(x) = \frac{2}{2\ell+1} \delta_{\ell\ell'}$$

$$(5.10b) \quad P_0(x) = 1 \quad P_1(x) = x$$

$$(5.10c) \quad P_\ell(x) = \frac{1}{2\ell+1} [P'_{\ell+1}(x) - P'_{\ell-1}(x)]$$

$$(5.10d) \quad P_\ell(-x) = (-1)^\ell P_\ell(x)$$

$$(5.10e) \quad P_\ell(\cos\theta) = \left[\frac{4\pi}{2\ell+1} \right]^{\frac{1}{2}} Y_{\ell,0}(\theta)$$

We write

$$\zeta(k, P, |\cos\theta|) = \sum_{\ell=0}^{\infty} A_\ell(k, P) P_\ell(\cos\theta)$$

where using (5.10a)

$$A_\ell(k, P) = \frac{2\ell+1}{2} \int_{-1}^{+1} d(\cos\theta) \zeta(k, P, |\cos\theta|) P_\ell(\cos\theta)$$

We are concerned with $P < k_F$. From the definition of $\zeta(k, P, |\cos\theta|)$ we have that

- 1) if $k < \sqrt{k_F^2 - P^2}$, $\zeta(k, P, |\cos\theta|) = 0$ for all θ , so that $A_\ell(k, P) = 0$
- 2) if $k > k_F + P$, $\zeta(k, P, |\cos\theta|) = 1$ for all θ , so that using (5.10a), $A_\ell(k, P) = \delta_{\ell, 0}$
- 3) if $\sqrt{k_F^2 - P^2} < k < k_F + P$ then

$$A_\ell(k, P) = \frac{2\ell+1}{2} \int_{-\kappa}^{+\kappa} d(\cos\theta) P_\ell(\cos\theta)$$

$$\kappa = \frac{k^2 + P^2 - k_F^2}{2Pk}$$

The integral can be evaluated using (5.10c). Using also (5.10d) we obtain

$$A_\ell(k, P) = \begin{cases} P_{\ell+1}(\kappa) - P_{\ell-1}(\kappa) & \text{if } \ell \text{ even } (\ell \neq 0) \\ 0 & \text{if } \ell \text{ odd} \end{cases}$$

Finally then

$$(5.11a) \quad \zeta(k, P, |\cos\theta|) = \sum_{\ell=0}^{\infty} A_{2\ell}(k, P) P_{2\ell}(\cos\theta) =$$

$$= \sum_{\ell=0}^{\infty} \left(\frac{4\pi}{4\ell+1} \right)^{\frac{1}{2}} A_{2\ell}(k, P) Y_{2\ell, 0}(\theta)$$

where for $P < k_F$

$$(5.11b) \quad A_0(k, P) = \begin{cases} 1 & \text{if } k > k_F + P \\ \kappa & \text{if } \sqrt{k_F^2 - P^2} < k < k_F + P \\ 0 & \text{if } k < \sqrt{k_F^2 - P^2} \end{cases}$$

(5.11c)

$$\ell \neq 0, \quad A_{2\ell}(k, P) = \begin{cases} 0 & \text{if } k > k_F + P \\ P_{2\ell+1}(\kappa) - P_{2\ell-1}(\kappa) & \text{if } \sqrt{k_F^2 - P^2} < k < k_F + P \\ 0 & \text{if } k < \sqrt{k_F^2 - P^2} \end{cases}$$

$$(5.11d) \quad \kappa = \frac{k^2 + P^2 - k_F^2}{2Pk}$$

Thus we have

$$(5.12) \quad \int_{|\vec{P} + \vec{k}| > k_F} dk k^2 d(\cos\theta) d\varphi =$$

$$= \sum_{\ell=0}^{\infty} \left(\frac{4\pi}{4\ell+1} \right)^{\frac{1}{2}} \int_0^{\infty} dk k^2 \int_{-1}^{+1} d(\cos\theta) \int_0^{2\pi} d\varphi A_{2\ell}(k, P) Y_{2\ell,0}(\theta).$$

In this expression, \vec{P} is considered to be the z-axis of the coordinate system.

The first term in the series (5.11) is the angular average of $\zeta(k, P, |\cos\theta|)$ and is indeed equal to the function used by Brueckner and Gammel (equation (34) in their paper¹⁾) and by Moszkowski and Scott (equation (III-7) in their paper⁴⁾). Thus their approximation consists of taking only the first term in our series (5.11).

In Figure I we have plotted $A_{2\ell}(k, P)$ for $\ell=0, 1, 2, 3$ and $P = \frac{1}{2} k_F$. We see that the A's oscillate more as ℓ increases and that their amplitudes decrease as ℓ increases.

Substituting (5.11) into (5.9) yields

$$I_{\ell m; L \mu}(\vec{k}, P) = \int_0^\infty dk'' k''^2 \int_{-1}^{+1} d(\cos\theta) \int_0^{2\pi} d\varphi e^{-i b_\ell(k'') a_L(k'') Y_{\ell m}^*(\theta, \varphi) Y_{L \mu}(\theta, \varphi) \times \\ \times \left[\sum_{a=0}^{\infty} \left(\frac{4\pi}{4a+1} \right)^{\frac{1}{2}} A_{2a}(k'', P) Y_{2a, 0}(\theta) \right]$$

If we now assume that the energy denominator $e \equiv \epsilon(\vec{P} + \vec{k}) + \epsilon(\vec{P} - \vec{k}) - \epsilon(\vec{P} + \vec{k}'') - \epsilon(\vec{P} - \vec{k}'')$ is independent of angles, the angular integrations can be performed using ^{21a)}

$$(5.13) \int_{-1}^{+1} d(\cos\theta) \int_0^{2\pi} d\varphi Y_{\ell_3 m_3}^*(\theta, \varphi) Y_{\ell_2 m_2}(\theta, \varphi) Y_{\ell_1 m_1}(\theta, \varphi) = \\ = \left[\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2\ell_3+1)} \right]^{\frac{1}{2}} (\ell_1 \ell_2 m_1 m_2 | \ell_3 m_3) (\ell_1 \ell_2 00 | \ell_3 0)$$

where $(\ell_1 \ell_2 m_1 m_2 | \ell_3 m_3)$ is a Clebsch-Gordan coefficient.

We have finally that

$$(5.14) \quad I_{\ell m; L \mu}(k, P) = \sum_{\alpha=0}^{\infty} \left(\frac{2L+1}{2\ell+1} \right)^{\frac{1}{2}} (2\alpha, L, 0, \mu | \ell, m) (2\alpha, L, 0, 0 | \ell, 0) \int_0^{\infty} dk'' k''^2 e^{-1} b_{\ell}(k'') a_L(k'') A_{2\alpha}(k'', P)$$

if e is independent of angles. ($I_{\ell m; L \mu}$ is now independent of angles because of the restriction to angular independent energy denominators.) We could, of course, let e have an arbitrary angle dependence and then expand it in terms of Legendre polynomials. For simplicity we have not included this most general case.

Let us now note that $(2\alpha, L, 0, \mu | \ell m)$ is zero unless $\mu=m$,^{21b)} and $(2\alpha, L, 0, 0 | \ell 0)$ is zero unless $L-\ell$ is an even integer.^{21c)} Thus $I_{\ell m; L \mu}$ is zero unless $\mu=m$ and $L-\ell$ is an even integer. These facts are manifestations of the two symmetries possessed by the integration region (see Fig. A): There is cylindrical symmetry about \vec{P} and reflection symmetry about a plane passing through the origin and perpendicular to \vec{P} . The first symmetry implies that $\mu=m$, and the second that $L-\ell$ is an even integer. Thus we have coupling of S states to D states, S states to G states, etc. We note that the coupled terms occur for values of α different from zero, so that taking $\alpha=0$ only (as, in effect, Brueckner and Gammel¹⁾ and Moszkowski and Scott⁴⁾ have done) means neglecting and coupling between the states.

Finally then we can write (5.6) as

$$\begin{aligned}
 (5.15) \quad f_{\ell m}(\vec{k}, P) &= b_{\ell}(k) Y_{\ell m}^*(\hat{k}) + \\
 &+ \left(\frac{1}{2\pi}\right)^3 \sum_{L=0}^{\infty} \sum_{\alpha=0}^{\infty} f_{Lm}(\vec{k}, P) \left(\frac{2L+1}{2\ell+1}\right)^{\frac{1}{2}} (2\alpha, L0m | \ell m) (2\alpha, L00 | \ell 0) \times \\
 &\times \int_0^{\infty} dk'' k''^2 e^{-1} b_{\ell}(k'') a_L(k'') A_{2\alpha}(k'', P)
 \end{aligned}$$

We are still left with an infinite set of coupled algebraic equations but we have considerably simplified the situation by reducing the integrals to a single integration over k'' . Except for the restriction to angular independent energy denominators, no other approximations have been made. We can truncate the infinite set of equations by restricting ourselves to a finite set of angular momentum states. That is, if we suppose that the potential vanishes for states with angular momentum greater than ℓ_{\max} , say, (which in effect it does because of the angular momentum barrier) then this means $a_{\ell}(k) = b_{\ell}(k) = 0$ for $\ell > \ell_{\max}$ and thus the upper limit of the summation over L in (5.15) can be replaced by ℓ_{\max} .

Up to this point in this Section we have neglected the hole-hole interactions. When these interactions are included the integral equation for $(\vec{k}' | K | \vec{k}, \vec{P})$ is, from (4.39) with

$S=0$,

$$(5.16) \quad (\vec{k}' | K | \vec{k}, \vec{P}) = (\vec{k}' | V | \vec{k}) + \\ + \left(\frac{1}{2\pi}\right)^3 \left[\int_{|\vec{P}+\vec{k}''| > k_F} d\vec{k}'' - \int_{|\vec{P}+\vec{k}''| < k_F} d\vec{k}'' \right] e^{-1} (\vec{k}' | V | \vec{k}'') (\vec{k}'' | K | \vec{k}, \vec{P})$$

rather than (5.2). Proceeding in a manner identical to that following (5.2) we find that (5.4) still holds but now the equation for the f 's is

$$(5.17) \quad f_{\ell m}(\vec{k}, \vec{P}) = b_{\ell}(k) Y_{\ell m}^*(\hat{k}) + \\ + \left(\frac{1}{2\pi}\right)^3 \sum_{L=0}^{\infty} \sum_{\mu=-L}^{+L} f_{L\mu}(\vec{k}, \vec{P}) \left[\int_{|\vec{P}+\vec{k}''| > k_F} d\vec{k}'' - \int_{|\vec{P}+\vec{k}''| < k_F} d\vec{k}'' \right] e^{-1} b_{\ell}(k'') a_L(k'') Y_{\ell m}^*(\hat{k}'') Y_{L\mu}(\hat{k}'')$$

We thus have to calculate a new integral, $J_{\ell m; L\mu}(\vec{k}, \vec{P})$:

$$J_{\ell m; L\mu}(\vec{k}, \vec{P}) = \int_{|\vec{P}+\vec{k}''| < k_F} d\vec{k}'' e^{-1} b_{\ell}(k'') a_L(k'') Y_{\ell m}^*(\hat{k}'') Y_{L\mu}(\hat{k}'')$$

We shall use a procedure identical to that used to reduce $I_{\ell m; L\mu}$ to a single integral. We note that

$$|\vec{P}+\vec{k}''| < k_F \quad \text{means} \quad k''^2 + 2Pk''|\cos \theta| + P^2 < k_F^2$$

where θ is the angle between \vec{k}'' and \vec{P} . We shall consider

\vec{P} as the z-axis for the coordinate system so that we can write $J_{\ell m; L \mu}$ as

$$J_{\ell m; L \mu}(\vec{k}, P) = \int dk "k"^2 d(\cos \theta) d\varphi e^{-1} b_{\ell}(k") a_L(k") Y_{\ell m}^*(\theta, \varphi) Y_{L \mu}(\theta, \varphi) k"^{2+2Pk"} |\cos \theta| + P^2 < k_F^2$$

Now we define a function $\eta(k, P, |\cos \theta|)$ by

$$(5.18) \quad \eta(k, P, |\cos \theta|) = \begin{cases} 1 & \text{if } k^2 + 2Pk |\cos \theta| + P^2 < k_F^2 \\ 0 & \text{otherwise} \end{cases}$$

so that $J_{\ell m; L \mu}$ can be written

$$J_{\ell m; L \mu}(\vec{k}, P) = \int_0^{\infty} dk "k"^2 \int_{-1}^{+1} d(\cos \theta) \int_0^{2\pi} d\varphi e^{-1} b_{\ell}(k") a_L(k") Y_{\ell m}^*(\theta, \varphi) Y_{L \mu}(\theta, \varphi) \eta(k", P, |\cos \theta|)$$

We now expand $\eta(k, P, |\cos \theta|)$ in terms of Legendre polynomials:

$$\eta(k, P, |\cos \theta|) = \sum_{\ell=0}^{\infty} B_{\ell}(k, P) P_{\ell}(\cos \theta)$$

$$B_{\ell}(k, P) = \frac{2\ell+1}{2} \int_{-1}^{+1} d(\cos \theta) \eta(k, P, |\cos \theta|) P_{\ell}(\cos \theta)$$

We are concerned with $P < k_F$. (It should be noted that $\eta \equiv 0$ if

$P > k_F$.) From the definition (5.18) of $\eta(k, P, |\cos\theta|)$ we have that

- 1) if $k > \sqrt{k_F^2 - P^2}$, $\eta(k, P, |\cos\theta|) = 0$ for all θ so that $B_\ell(k, P) = 0$
- 2) if $k < k_F - P$, $\eta(k, P, |\cos\theta|) = 1$ for all θ so that $B_\ell(k, P) = \delta_{\ell 0}$
- 3) if $k_F - P < k < \sqrt{k_F^2 - P^2}$ then

$$B_\ell(k, P) = \frac{2\ell+1}{2} \int_{-\kappa}^{\kappa} d(\cos\theta) P_\ell(\cos\theta)$$

$$\kappa = \frac{k^2 + P^2 - k_F^2}{2Pk}$$

This integral has been evaluated above and we have

$$B_0(k, P) = -\kappa$$

$$B_\ell(k, P) = \begin{cases} -[P_{\ell+1}(\kappa) - P_{\ell-1}(\kappa)] & \text{if } \ell \text{ even } (\ell \neq 0) \\ 0 & \text{if } \ell \text{ odd} \end{cases}$$

Finally then, for $P < k_F$, we have

$$(5.19a) \quad \eta(k, P, |\cos\theta|) = \sum_{\ell=0}^{\infty} B_{2\ell}(k, P) P_{2\ell}(\cos\theta)$$

$$(5.19b) \quad B_0(k, P) = \begin{cases} 1 & \text{if } k < k_F - P \\ -\kappa & \text{if } k_F - P < k < \sqrt{k_F^2 - P^2} \\ 0 & \text{if } k > \sqrt{k_F^2 - P^2} \end{cases}$$

(5.19c)

$$l \neq 0, B_{2l}(k, P) = \begin{cases} 0 & \text{if } k < k_F - P \\ -[P_{2l+1}(\kappa) - P_{2l-1}(\kappa)] & \text{if } k_F - P < k < \sqrt{k_F^2 - P^2} \\ 0 & \text{if } k > \sqrt{k_F^2 - P^2} \end{cases}$$

$$(5.11d) \quad \kappa = \frac{k^2 + P^2 - k_F^2}{2Pk}$$

Thus we have

$$(5.20) \quad \int_{|\vec{P} + \vec{k}| < k_F} dk k^2 d(\cos\theta) d\varphi = \\ = \sum_{l=0}^{\infty} \left(\frac{4\pi}{4l+1} \right)^{\frac{1}{2}} \int_0^{\infty} dk k^2 \int_{-1}^{+1} d(\cos\theta) \int_0^{2\pi} d\varphi B_{2l}(k, P) Y_{2l,0}(\theta)$$

In this expression, \vec{P} is considered to be the z-axis of the coordinate system.

Using (5.19) and (5.13) we have

$$(5.21) \quad J_{lm; L\mu}(k, P) = \\ = \sum_{a=0}^{\infty} \left(\frac{2L+1}{2l+1} \right)^{\frac{1}{2}} (2a, L0\mu | lm) (2a, L00 | l0) \int_0^{\infty} dk'' k''^2 e^{-1} b_l(k'') a_L(k'') B_{2a}(k'', P)$$

if e is independent of angles.

Thus, finally, we can write the equation for the f 's as

$$(5.22) \quad f_{\ell m}(\vec{k}, P) = b_{\ell}(k) Y_{\ell m}^*(\hat{k}) + \\ + \left(\frac{1}{2\pi}\right)^3 \sum_{L=0}^{\infty} \sum_{a=0}^{\infty} f_{Lm}(\vec{k}, P) \left(\frac{2L+1}{2\ell+1}\right)^{\frac{1}{2}} (2a, L0m | \ell m) (2a, L00 | \ell 0) \times \\ \times \int_0^{\infty} dk'' k''^2 e^{-1} b_{\ell}(k'') a_L(k'') C_{2a}(k'', P)$$

where

$$(5.23a) \quad C_0(k, P) = \begin{cases} -\lambda & \text{if } k < k_F - P \\ \lambda \kappa & \text{if } k_F - P < k < \sqrt{k_F^2 - P^2} \\ \kappa & \text{if } \sqrt{k_F^2 - P^2} < k < k_F + P \\ 1 & \text{if } k > k_F + P \end{cases}$$

$$(5.23b) \quad a \neq 0, C_{2a}(k, P) = \begin{cases} 0 & \text{if } k < k_F - P \\ \lambda [P_{2a+1}(\kappa) - P_{2a-1}(\kappa)] & \text{if } k_F - P < k < \sqrt{k_F^2 - P^2} \\ P_{2a+1}(\kappa) - P_{2a-1}(\kappa) & \text{if } \sqrt{k_F^2 - P^2} < k < k_F + P \\ 0 & \text{if } k > k_F + P \end{cases}$$

$$(5.11d) \quad \kappa = \frac{k^2 + P^2 - k_F^2}{2Pk}$$

(5.23c)

$$\lambda = \begin{cases} 0 & \text{when the hole-hole interactions are neglected} \\ 1 & \text{when the hole-hole interactions are included} \end{cases}$$

That is, when $\lambda=0$, $C_{2a}=A_{2a}$ and when $\lambda=1$, $C_{2a}=A_{2a}-B_{2a}$. We have introduced the quantity λ in order to keep track of the terms arising because of the hole-hole interactions

The f 's above are functions of the angles of \vec{k} . It is convenient for later work to separate out this angle dependence. To this end we write

$$(5.24) \quad f_{\ell m}(\vec{k}, P) = \sum_{\lambda=0}^{\infty} f_{\ell m}^{(\lambda)}(k, P) Y_{\ell m}^*(\hat{k})$$

Substituting (5.24) into (5.22), multiplying both sides by $Y_{\ell m}(\hat{k})$, integrating over \hat{k} , and using the orthonormality of the Y 's (3.16) yields the following equation for $f_{\ell m}^{(\lambda)}(k, P)$:

$$(5.25) \quad f_{\ell m}^{(\lambda)}(k, P) = \delta_{\lambda \ell} b_{\ell}(k) + \\ + \left(\frac{1}{2\pi}\right)^3 \sum_{L=0}^{\infty} \sum_{\alpha=0}^{\infty} f_{Lm}^{(\lambda)}(k, P) \left(\frac{2L+1}{2\ell+1}\right)^{\frac{1}{2}} (2\alpha, L0m | \ell m) (2\alpha, L00 | \ell 0) \times \\ \times \int_0^{\infty} dk'' k''^2 e^{-1} b_{\ell}(k'') a_L(k'') C_{2\alpha}(k'', P)$$

where $\delta_{\lambda \ell}$ is a Kronecker δ -function. The equation for the

K-matrix elements is then, from (5.4) and (5.24):

$$(5.26) \quad (\vec{k}' | K | \vec{k}, \vec{P}) =$$

$$= \sum_{\ell=0}^{\infty} \sum_{\lambda=0}^{\infty} \sum_{m=-\ell}^{+\ell} a_{\ell}(k') f_{\ell m}^{(\lambda)}(k, P) Y_{\ell m}(\hat{k}') Y_{\lambda m}^*(\hat{k})$$

In arriving at the above equations we have disregarded the difference between r_0^+ and r_0^- in the pseudopotential (2.5). As we pointed out at the beginning of this Chapter, we shall interpret the integrals in the equation for the f 's similarly to the way they were interpreted when we constructed the two-body K matrix elements from the Schrodinger wave function. Thus we interpret the integrals in (5.25) as follows:

$$(5.27) \quad \int_0^{\infty} dk k'^2 e^{-1} b_{\ell}(k'') a_L(k'') C_{2a}(k'', P) =$$

$$= \frac{(4\pi\hbar)^2 r_{0\ell}}{m} \int_0^{\infty} dk'' k''^2 e^{-1} j_L(k'' r_{0L}) \left[f_{\ell} j_{\ell}(k'' r_{0\ell}) - k'' r_{0\ell} j_{\ell}'(k'' r_{0\ell}^-) \right] C_{2a}(k'', P)$$

That is, the argument of j_{ℓ}' contains $r_{0\ell}^-$. (Compare this integral with the integral occurring in (5.33b).) The similarity with the two-body case can be most easily seen by considering the nuclear matter K matrix elements when $\vec{P}=0$.

We have in this case

$$(5.28a) \quad C_0(k,0) = \begin{cases} -\lambda & \text{if } k < k_F \\ +1 & \text{if } k > k_F \end{cases}$$

$$(5.28b) \quad C_{2a}(k,0) = 0 \quad \text{for } a \neq 0$$

Using (5.25) to (5.28) we have then, when $\vec{P} = 0$,

$$(5.29a) \quad (\vec{k}' | K | \vec{k}, 0) = (4\pi)^2 \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} (k' | K_{\ell} | k) Y_{\ell m}(\hat{k}') Y_{\ell m}^*(\hat{k})$$

$$(5.29b) \quad (k' | K_{\ell} | k) =$$

$$= \frac{\frac{\hbar^2 r_{0\ell}}{m} j_{\ell}(k' r_{0\ell}) \left[f_{\ell} j_{\ell}(k r_{0\ell}) - k r_{0\ell} j'_{\ell}(k r_{0\ell}) \right]}{1 - \frac{2\hbar^2 r_{0\ell}}{\pi m} \left(\int_{k_F}^{\infty} -\lambda \int_0^{k_F} \right) dk'' k''^2 e^{-1} j_{\ell}(k'' r_{0\ell}) \left[f_{\ell} j_{\ell}(k'' r_{0\ell}) - k'' r_{0\ell} j'_{\ell}(k'' r_{0\ell}) \right]}$$

Equation (5.29b) differs from the two-body equation (5.33b) in three ways: 1) by having an integral proportional to λ (i.e., the hole-hole interaction term), 2) by having the lower limit of the first integral k_F rather than zero, and 3) by having different energy denominators. If in (5.29b) we take $k_F \rightarrow 0$

and $e = \frac{\hbar^2}{m}(k^2 - k'^2)$, then (5.29b) becomes identical to the two-body equation (5.33b). In particular, we have in this limit that the diagonal elements $\langle k | K_\ell | k \rangle$ are proportional to $\tan \delta_\ell$ where δ_ℓ is the phase shift (see equation (3.24)). We wish to point out that the limit obtained in this manner will not be the low density limit of $\langle k | K_\ell | k \rangle$, this being proportional to the phase shift.²²⁾ The difference in the limits lies in the interpretation of the singular integrals. When principal values of the integrals are taken the resulting limit must be proportional to the tangent of the phase shift. To obtain the low density limit a more sophisticated handling of the singular integrals is necessary, but we shall not discuss this point further in this thesis.

This completes our discussion of the K matrix elements for singlet spin states. We have found that our pseudopotential leads to a soluble set of algebraic equations, and in addition we have shown how non-zero centre of mass momentum effects can be taken into account exactly.

5.2 The K Matrix Elements for Triplet Spin States

The triplet spin pseudopotential is, from (2.5):

$$(5.30a) \quad V(r) = \frac{\hbar^2}{m} \sum_{J=0}^{\infty} \left\{ \left[\frac{f_{JJ}}{r_{o_{JJ}}} \delta(r - r_{o_{JJ}}) - \delta(r - r_{o_{JJ}}) \frac{\partial}{\partial r} \right] P_{JJ1} \right. \\ \left. + \left[\frac{S}{r_{o_J}} \delta(r - r_{o_J}) - \delta(r - r_{o_J}) \frac{\partial}{\partial r} \right] (P_{J,J+1,1} + P_{J,J-1,1}) \right\}$$

where

$$(5.30b) \quad \mathcal{U}_{J,J-1,1}^M = f_{J,J-1} \mathcal{U}_{J,J-1,1}^M + f_J^{(t)} \mathcal{U}_{J,J+1,1}^M$$

$$(5.30c) \quad \mathcal{U}_{J,J+1,1}^M = f_J^{(t)} \mathcal{U}_{J,J-1,1}^M + f_{J,J+1} \mathcal{U}_{J,J+1,1}^M$$

$$(5.30d) \quad \mathcal{U}_{J,0}^M = 0 \quad \text{otherwise}$$

The integral equation we want to solve is, from (4.27) with $S=1$,

$$(5.31a) \quad (\vec{k}'; \chi_{1m_S} | K | \vec{k}, \vec{P}; \chi_{1m_S}) = (\vec{k}'; \chi_{1m_S} | V | \vec{k}; \chi_{1m_S}) + \\ + \left(\frac{1}{2\pi}\right)^3 \sum_{m_S''=-1}^{+1} \int_{|\vec{P}+\vec{k}''| > k_F} d\vec{k}'' e^{-1} (\vec{k}'; \chi_{1m_S} | V | \vec{k}''; \chi_{1m_S''}) (\vec{k}''; \chi_{1m_S''} | K | \vec{k}, \vec{P}; \chi_{1m_S})$$

where

$$(5.31b) \quad e = \epsilon(\vec{P}+\vec{k}) + \epsilon(\vec{P}-\vec{k}) - \epsilon(\vec{P}+\vec{k}'') - \epsilon(\vec{P}-\vec{k}'')$$

and

$$(5.32) \quad (\vec{k}'; \chi_{1m_S} | V | \vec{k}; \chi_{1m_S}) = \int d\vec{r} e^{-i\vec{k}' \cdot \vec{r}} \chi_{1m_S} V(r) e^{i\vec{k} \cdot \vec{r}} \chi_{1m_S}$$

We neglect for the moment the hole-hole interactions since we can easily take these into account at the end of the calculation.

Now,

$$e^{i\vec{k}\cdot\vec{r}} \times_{S m_S} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m_{\ell}=-\ell}^{+\ell} i^{\ell} j_{\ell}(kr) Y_{\ell m_{\ell}}^*(\hat{k}) Y_{\ell m_{\ell}}(\hat{r}) \times_{S m_S}$$

and

$$Y_{\ell m_{\ell}}(\hat{r}) \times_{S m_S} = \sum_{J=|\ell-S|}^{\ell+S} (\ell S m_{\ell} m_S | JM) \psi_{J \ell S}^M \quad (M=m_{\ell}+m_S)$$

so

$$(5.33) \quad e^{i\vec{k}\cdot\vec{r}} \chi_{\ell m_S} =$$

$$= 4\pi \sum_{\ell=0}^{\infty} \sum_{m_{\ell}=-\ell}^{+\ell} \sum_{J=|\ell-1|}^{\ell+1} i^{\ell} (\ell 1 m_{\ell} m_S | J, m_{\ell}+m_S) Y_{\ell m_{\ell}}^*(\hat{k}) j_{\ell}(kr) \psi_{J \ell 1}^{m_{\ell}+m_S}$$

We substitute (5.33) and (5.30) into (5.32) and use the orthonormality of the ψ 's to simplify the result. The calculation is very straightforward so we shall just quote the results. We suppose as a simplifying assumption that there is just one core radius r_{o_J} for each state (that is, we take $r_{o_{JJ}} = r_{o_J}$). In this case we have

$$(5.34a) \quad (\vec{k}^{\dagger}; \chi_{\ell m_S^{\dagger}} | V | \vec{k}; \chi_{\ell m_S}) =$$

$$= \sum_{J=0}^{\infty} \sum_{M=-J}^{+J} \sum_{(\xi \eta)} (J+\xi, 1, M-m_S^{\dagger}, m_S^{\dagger} | JM) (J+\eta, 1, M-m_S, m_S | JM) a_{J\xi}(k^{\dagger}) Y_{J+\xi, M-m_S^{\dagger}}(\hat{k}^{\dagger}) \times$$

$$\times b_{J\xi\eta}(k) Y_{J+\eta, M-m_S}^*(\hat{k})$$

where $\sum_{(\xi, \eta)}$ means that the pair (ξ, η) takes the values $(0,0)$, $(1,-1)$, $(1,1)$, $(-1,1)$, $(-1,-1)$ and where

$$(5.34b) \quad a_{J\xi}(k^{\dagger}) = \frac{(4\pi\hbar)^2 r_{oJ}}{m} j_{J+\xi}(k^{\dagger} r_{oJ})$$

$$(5.34c) \quad b_{J00}(k) = f_{JJ} j_J(kr_{oJ}) - kr_{oJ} j_J^{\dagger}(kr_{oJ})$$

$$b_{J11}(k) = f_{J,J+1} j_{J+1}(kr_{oJ}) - kr_{oJ} j_{J+1}^{\dagger}(kr_{oJ})$$

$$b_{J-1-1}(k) = f_{J,J-1} j_{J-1}(kr_{oJ}) - kr_{oJ} j_{J-1}^{\dagger}(kr_{oJ})$$

$$b_{J1-1}(k) = -f_J^{(t)} j_{J-1}(kr_{oJ})$$

$$b_{J-11}(k) = -f_J^{(t)} j_{J+1}(kr_{oJ})$$

We now substitute (5.34a) into (5.31). The procedure is similar to that used for the singlet spin case. We can write

$$(5.35) \quad (\vec{k}^{\dagger}; \chi_{1m_S^{\dagger}} | K | \vec{k}, \vec{P}; \chi_{1m_S}) =$$

$$= \sum_{J=0}^{\infty} \sum_{M=-J}^{+J} \sum_{(\xi, \eta)} (J+\xi, 1, M-m_S^{\dagger}, m_S^{\dagger} | J, M) a_{J\xi}(k^{\dagger}) Y_{J+\xi, M-m_S^{\dagger}}(\hat{k}^{\dagger}) f_{JM\xi\eta}(\vec{k}, \vec{P}, m_S)$$

where the f 's are solutions of the set of algebraic equations:

$$\begin{aligned}
 (5.36) \quad f_{JM\xi\eta}(\vec{k}, \vec{P}, m_S) &= (J+\eta, 1, M-m_S, m_S | JM) b_{J\xi\eta}(k) Y_{J+\eta, M-m_S}^*(\hat{k}) \\
 &+ \left(\frac{1}{2\pi}\right)^3 \sum_{m_S''=-1}^{+1} (J+\eta, 1, M-m_S'', m_S'' | JM) \sum_{L=0}^{\infty} \sum_{\mu=-L(\gamma\delta)}^{+L} (L+\gamma, 1, \mu-m_S'', m_S'' | L\mu) f_{L\mu\gamma\delta}(\vec{k}, \vec{P}, m_S) \times \\
 &\times \int_{|\vec{P}+\vec{k}''| > k_F} d\vec{k}'' e^{-1} b_{J\xi\eta}(k'') a_{L\gamma}(k'') Y_{J+\eta, M-m_S}^*(\hat{k}'') Y_{L+\gamma, \mu-m_S''}(\hat{k}'')
 \end{aligned}$$

We can use (5.8) and (5.11) to simplify the integral in (5.36). Further if e is assumed to be independent of angles then the angular integrations can easily be done using (5.13) so that we have

$$\begin{aligned}
 &\int_{|\vec{P}+\vec{k}''| > k_F} d\vec{k}'' e^{-1} b_{J\xi\eta}(k'') a_{L\gamma}(k'') Y_{J+\eta, M-m_S}^*(\hat{k}'') Y_{L+\gamma, \mu-m_S''}(\hat{k}'') = \\
 &= \sum_{\alpha=0}^{\infty} \left[\frac{2(L+\gamma)+1}{2(J+\eta)+1} \right]^{\frac{1}{2}} (2\alpha, L+\gamma, 0, \mu-m_S'' | J+\eta, M-m_S'') (2\alpha, L+\gamma, 00 | J+\eta, 0) \times \\
 &\quad \times \int_0^{\infty} dk'' k''^2 e^{-1} b_{J\xi\eta}(k'') a_{L\gamma}(k'') A_{2\alpha}(k'', P)
 \end{aligned}$$

We note that $(2\alpha, L+\gamma, 0, \mu-m_S'' | J+\eta, M-m_S'')=0$ unless $\mu=M$.^{21b)}

Thus, when e is independent of angles we can write (5.36) as

$$\begin{aligned}
 (5.37) \quad f_{JM\xi\eta}(\vec{k}, P, m_S) &= (J+\eta, 1, M-m_S, m_S | JM) b_{J\xi\eta}(k) Y_{J+\eta, M-m_S}^*(\hat{k}) \\
 &+ \left(\frac{1}{2\pi}\right)^3 \sum_{m_S''=-1}^{+1} (J+\eta, 1, M-m_S'', m_S'' | JM) \sum_{a=0}^{\infty} \sum_{L=0}^{\infty} \sum_{(\gamma\delta)} (L+\gamma, 1, M-m_S'', m_S'' | LM) \times \\
 &\times (2a, L+\gamma, 0, M-m_S'' | J+\eta, M-m_S'') (2a, L+\gamma, 0, 0 | J+\eta, 0) \left[\frac{2(L+\gamma)+1}{2(J+\eta)+1}\right]^{\frac{1}{2}} \times \\
 &\times f_{LM\gamma\delta}(\vec{k}, P, m_S) \int_0^{\infty} dk'' k''^2 e^{-1} b_{J\xi\eta}(k'') a_{L\gamma}(k'') A_{2a}(k'', P)
 \end{aligned}$$

We can perform the summation over m_S'' by making use of the following relation between Clebsch-Gordan and Racah coefficients:²³⁾

$$\begin{aligned}
 (5.38) \quad \sum_{\beta} (ab\alpha\beta | e, \alpha+\beta) (ed, \alpha+\beta, \gamma-\alpha-\beta | c\gamma) (bd\beta, \gamma-\alpha-\beta | f, \gamma-\alpha) &= \\
 &= (2e+1)^{\frac{1}{2}} (2f+1)^{\frac{1}{2}} (af\alpha, \gamma-\alpha | c\gamma) W(abcd; ef)
 \end{aligned}$$

where W is a Racah coefficient. Using this relation we have

$$(5.39) \quad \sum_{m_S''} (2\alpha, L+\gamma, 0, M-m_S'' | J+\eta, M-m_S'') (J+\eta, 1, M-m_S'', m_S'' | JM) (L+\gamma, 1, M-m_S'', m_S'' | LM) =$$

$$= [2(J+\eta)+1]^{\frac{1}{2}} (2L+1)^{\frac{1}{2}} (2\alpha, L, 0, M | JM) W(2\alpha, L+\gamma, J, 1; J+\eta, L)$$

From (5.37) and (5.39) we have the following equation for the f 's:

$$(5.40) \quad f_{JM\xi\eta}(\vec{k}, P, m_S) = (J+\eta, 1, M-m_S, m_S | JM) b_{J\xi\eta}(k) Y_{J+\eta, M-m_S}^*(\hat{k}) +$$

$$+ \left(\frac{1}{2\pi}\right)^3 \sum_{L=0}^{\infty} \sum_{(\gamma\delta)} \sum_{\alpha=0}^{\infty} (2L+1)^{\frac{1}{2}} [2(L+\gamma)+1]^{\frac{1}{2}} W(2\alpha, L+\gamma, J, 1; J+\eta, L) \times$$

$$\times (2\alpha, L, 0, M | JM) (2\alpha, L+\gamma, 0, 0 | J+\eta, 0) f_{LM\gamma\delta}(\vec{k}, P, m_S) \times$$

$$\times \int_0^{\infty} dk'' k''^2 e^{-1} b_{J\xi\eta}(k'') a_{L\gamma}(k'') A_{2\alpha}(k'', P)$$

It is clear from the last Section that in order to include the hole-hole interactions all we have to do is replace $A_{2\alpha}(k'', P)$ by $C_{2\alpha}(k'', P)$ where $C_{2\alpha}(k'', P)$, is given by (5.23).

It is interesting finally to see which states are coupled. If we neglect $C_{2a}(k'', P)$ for a different from zero then there is a considerable simplification of (5.40). For $a=0$ the Clebsch-Gordon coefficients then vanish unless $L=J$ and unless $\gamma=\eta$. In this case, there is then a single equation for f_{JM00} for each J and M and a set of coupled equations for $f_{JM\xi\eta}(\xi, \eta) \neq (0,0)$ for each J and M . Thus there is no coupling between states of different angular momentum other than the usual coupling due to the tensor force.

In the more general case ($a \neq 0$) the situation is more complicated. The second Clebsch-Gordon coefficient vanishes unless $L+\gamma+J+\eta$ ^{21c)} is even, the result being that all states with even orbital angular momentum couple together and all states with odd orbital angular momentum couple together. As in the singlet case however the infinite set of equations can be truncated by assuming the potential to vanish for states with $J > J_{\max}$.

The f 's above are functions of the angles of \vec{k} and of the quantum number m_S . It is convenient for later work to separate out these dependences. We can write

$$(5.41) \quad f_{JM\xi\eta}(\vec{k}, P, m_S) = \sum_{J'=0}^{\infty} \sum_{\sigma=-1}^{+1} (J'+\sigma, 1, M-m_S, m_S | J' M) Y_{J'+\sigma, M-m_S}^*(\hat{k}) f_{JM\xi\eta}^{(J'\sigma)}(k, P)$$

We substitute (5.41) into (5.40), multiply both sides by $(J^{\dagger} + \sigma, 1, M - m_S, m_S | J^{\dagger} M) Y_{J^{\dagger} + \sigma, M - m_S}(\hat{k})$, integrate over \hat{k} and sum over m_S , and use the orthonormality of the Y 's (3.16) and the following relationship^{21b)}

$$\sum_{\beta} (a b a \beta | c^{\dagger}, a + \beta) (a b a \beta | c, a + \beta) = \delta_{cc^{\dagger}}$$

to obtain the equation for $f_{JM\xi\eta}^{(J^{\dagger}\sigma)}(k, P)$:

$$\begin{aligned} (5.42) \quad f_{JM\xi\eta}^{(J^{\dagger}\sigma)}(k, P) &= b_{J\xi\eta}(k) \delta_{\sigma\eta} \delta_{J^{\dagger}J} + \\ &+ \left(\frac{1}{2\pi}\right)^3 \sum_{L=0}^{\infty} \sum_{(\gamma\delta)} \sum_{\alpha=0}^{\infty} (2L+1)^{\frac{1}{2}} [2(L+\gamma)+1]^{\frac{1}{2}} W(2\alpha, L+\gamma, J, 1; J+\eta, L) \times \\ &\times (2\alpha, L, 0, M | JM) (2\alpha, L+\gamma, 0, 0 | J+\eta, 0) f_{LM\gamma\delta}^{(J^{\dagger}\sigma)}(k, P) \times \\ &\times \int_0^{\infty} dk'' k''^2 e^{-1} b_{J\xi\eta}(k'') a_{L\gamma}(k'') C_{2\alpha}(k'', P) \end{aligned}$$

The equation for the K matrix elements is then, from (5.35) and (5.41),

$$\begin{aligned} (5.43) \quad (\vec{k}^{\dagger}; \chi_{1m_S^{\dagger}} | K | \vec{k}, \vec{P}; \chi_{1m_S}) &= \\ &= \sum_{J=0}^{\infty} \sum_{J^{\dagger}=0}^{\infty} \sum_{M=-J}^{+J} \sum_{(\xi\eta)} \sum_{\sigma=-1}^{+1} (J+\xi, 1, M - m_S^{\dagger}, m_S^{\dagger} | JM) (J^{\dagger} + \sigma, 1, M - m_S, m_S | J^{\dagger} M) \times \\ &\times a_{J\xi}(k^{\dagger}) f_{JM\xi\eta}^{(J^{\dagger}\sigma)}(k, P) Y_{J+\xi, M - m_S^{\dagger}}(\hat{k}^{\dagger}) Y_{J^{\dagger} + \sigma, M - m_S}^*(\hat{k}) \end{aligned}$$

In arriving at the above equations we have disregarded the difference between r_0^+ and r_0^- in the pseudopotential. We shall interpret the integrals in equation (5.42) for the f's exactly as we did for the singlet spin case of the preceding Section. That is, we shall write the argument of the derivatives of the spherical Bessel functions as $k''r_{0J}^-$.

5.3 Expressions for the Potential Energy per Particle and the Single Particle Potential using the Pseudopotential

The general expression (4.33) for PE/A involves the following integral

$$\int_{|\vec{P}+\vec{k}| < k_F} d\vec{k} (\vec{k}; \chi_{S_{m_S}} | K | \vec{k}, \vec{P}; \chi_{S_{m_S}})$$

Using (5.20) and the expressions (5.26) and (5.43) for the K matrix elements this integral can easily be reduced to a single integral over k.

For the singlet spin case the K matrix elements are given by (5.26) and using this and (5.20) we can write

$$\begin{aligned} & \int_{|\vec{P}+\vec{k}| < k_F} dk k^2 d(\cos\theta) d\varphi (\vec{k} | K | \vec{k}, \vec{P}) = \\ &= \sum_{\ell=0}^{\infty} \sum_{\lambda=0}^{\infty} \sum_{m=-\ell}^{+\ell} \sum_{\alpha=0}^{\infty} \left(\frac{4\pi}{4\alpha+1} \right)^{\frac{1}{2}} \int_0^{\infty} dk k^2 \int_{-1}^{+1} d(\cos\theta) \times \\ & \quad \times \int_0^{2\pi} d\varphi a_{\ell}(k) f_{\ell m}^{(\lambda)}(k, P) Y_{\lambda m}^*(\theta, \varphi) Y_{\ell m}(\theta, \varphi) Y_{2\alpha, 0}(\theta). \end{aligned}$$

Since the f 's are independent of angles, we can immediately perform the angular integrations using (5.13), obtaining finally,

$$\begin{aligned}
 (5.44) \quad & \int_{|\vec{P}+\vec{K}| < k_F} d\vec{K} (\vec{K}|K|\vec{K};\vec{P}) = \\
 & = \sum_{\ell=0}^{\infty} \sum_{\lambda=0}^{\infty} \sum_{m=-\ell}^{+\ell} \sum_{\alpha=0}^{\infty} \left(\frac{2\ell+1}{2\lambda+1} \right)^{\frac{1}{2}} (2\alpha, \ell, 0, m | \lambda, m) (2\alpha, \ell 0 0 | \lambda 0) \times \\
 & \quad \times \int_0^{\infty} dk k^2 a_{\ell}(k) f_{\ell m}^{(\lambda)}(k, P) B_{2\alpha}(k, P)
 \end{aligned}$$

Similarly for the triplet case we can write, using (5.43) and (5.20),

$$\begin{aligned}
 (5.45) \quad & \int_{|\vec{P}+\vec{K}| < k_F} d\vec{K} (\vec{K}; \times_{1m_S} | K | \vec{K}; P; \times_{1m_S}) = \\
 & = \sum_{J=0}^{\infty} \sum_{J'=0}^{\infty} \sum_{M=-J}^{+J} \sum_{(\xi\eta)} \sum_{\sigma=-1}^{+1} \sum_{\alpha=0}^{\infty} \left[\frac{2(J+\xi)+1}{2(J'+\sigma)+1} \right]^{\frac{1}{2}} (J+\xi, 1, M-m_S, m_S | JM) \times \\
 & \quad \times (J'+\sigma, 1, M-m_S, m_S | J'M) (2\alpha, J+\xi, 0, M-m_S | J'+\sigma, M-m_S) (2\alpha, J+\xi, 0 0 | J'+\sigma, 0) \times \\
 & \quad \times \int_0^{\infty} dk k^2 a_{J\xi}(k) f_{JM\xi\eta}^{(J'\sigma)}(k, P) B_{2\alpha}(k, P)
 \end{aligned}$$

We are interested in the summation of this expression over all values of m_S (see (4.33)). This summation can easily be performed using (5.38) and we find

$$\begin{aligned}
 (5.46) \quad & \sum_{m_S=-1}^{+1} \int_{|\vec{P}+\vec{k}| < k_F} d\vec{k}(\vec{k}; \chi_{1m_S} | K | \vec{k}, \vec{P}; \chi_{1m_S}) = \\
 & = \sum_{J=0}^{\infty} \sum_{J'=0}^{\infty} \sum_{M=-J}^{+J} \sum_{(\xi\eta)} \sum_{\sigma=-1}^{+1} \sum_{a=0}^{\infty} (2J+1)^{\frac{1}{2}} [2(J+\xi)+1]^{\frac{1}{2}} (2a, J0M | J'M) \times \\
 & \times W(2a, J+\xi, J', 1; J'+\sigma, J) (2a, J+\xi, 00 | J'+\sigma, 0) \times \\
 & \times \int_0^{\infty} dk k^2 a_{J\xi}(k) f_{JM\xi\eta}^{(J'\sigma)}(k, P) B_{2a}(k, P).
 \end{aligned}$$

The expression (4.33) for PE/A can thus be written

$$\begin{aligned}
 (5.47) \quad PE/A &= \frac{3}{4} (\pi k_F)^{-3} \int_0^{k_F} dP P^2 \left[\left(3 \sum_{\ell \text{ even}} + \sum_{\ell \text{ odd}} \right) \times \right. \\
 & \times \sum_{\lambda=0}^{\infty} \sum_{m=-\ell}^{+\ell} \sum_{a=0}^{\infty} \left(\frac{2\ell+1}{2\lambda+1} \right)^{\frac{1}{2}} (2a, \ell 0 m | \lambda m) (2a, \ell 0 0 | \lambda 0) \int_0^{\infty} dk k^2 a_{\ell}(k) f_{\ell m}^{(\lambda)}(k, P) B_{2a}(k, P) \\
 & + \left(\sum_{\substack{J(\xi\eta) \\ J+\xi \text{ even}}} + 3 \sum_{\substack{J(\xi\eta) \\ J+\xi \text{ odd}}} \right) \sum_{J'=0}^{\infty} \sum_{M=-J}^{+J} \sum_{\sigma=-1}^{+1} \sum_{a=0}^{\infty} (2J+1)^{\frac{1}{2}} [2(J+\xi)+1]^{\frac{1}{2}} \times \\
 & \times (2a, J0M | J'M) W(2a, J+\xi, J', 1; J'+\sigma, J) (2a, J+\xi, 00 | J'+\sigma, 0) \times \\
 & \times \left. \int_0^{\infty} dk k^2 a_{J\xi}(k) f_{JM\xi\eta}^{(J'\sigma)}(k, P) B_{2a}(k, P) \right]
 \end{aligned}$$

In the above we have made use of the fact that K^+ couples only states with even orbital angular momentum quantum number and K^- couples only states with odd orbital angular momentum quantum number.

Let us now consider the expression (4.34) for the single particle potential. It involves the following integral

$$\int_{|2\vec{k}+\vec{k}_2|<k_F} d\vec{k} (\vec{k}; x_{S m_S} |K| \vec{k}, \vec{p}; x_{S m_S})$$

We shall simplify the integrals using techniques similar to those introduced in Section 5.1. We notice that

$$|2\vec{k}+\vec{k}_2|<k_F \quad \text{means} \quad 4k^2+4kk_2\cos\theta+k_2^2<k_F^2$$

where θ is the angle between \vec{k} and \vec{k}_2 . We define a function $\xi(k, k_2, \cos\theta)$ by

$$(5.48) \quad \xi(k, k_2, \cos\theta) = \begin{cases} 1 & \text{if } 4k^2+4kk_2\cos\theta+k_2^2<k_F^2 \\ 0 & \text{otherwise} \end{cases}$$

so that we can write

$$(5.49) \quad \int_{|2\vec{k}+\vec{k}_2|<k_F} dk k^2 d(\cos\theta) d\varphi = \int_0^\infty dk k^2 \int_{-1}^{+1} d(\cos\theta) \int_0^{2\pi} d\varphi \xi(k, k_2, \cos\theta)$$

and

$$\begin{aligned}
 (5.50) \quad & \int d\vec{k} (\vec{k}; \times_{S_{m_S}} |K| \vec{k}, \vec{P}; \times_{S_{m_S}}) = \\
 & \int_0^\infty dk k^2 \int_{-1}^{+1} d(\cos\theta) \int_0^{2\pi} d\varphi \xi(k, k_2, \cos\theta) (\vec{k}; \times_{S_{m_S}} |K| \vec{k}, \vec{P}; \times_{S_{m_S}})
 \end{aligned}$$

where \vec{k}_2 is the z-axis of the coordinate system.

We now expand ξ in terms of Legendre polynomials, that is,

$$\begin{aligned}
 \xi(k, k_2, \cos\theta) &= \sum_{\ell=0}^{\infty} D_{\ell}(k, k_2) P_{\ell}(\cos\theta) \\
 D_{\ell}(k, k_2) &= \frac{2\ell+1}{2} \int_{-1}^{+1} d(\cos\theta) \xi(k, k_2, \cos\theta) P_{\ell}(\cos\theta)
 \end{aligned}$$

From the definition of $\xi(k, k_2, \cos\theta)$ we have

- 1) if $k > \frac{1}{2}(k_2 + k_F)$ or $k < \frac{1}{2}(k_2 - k_F)$, $\xi(k, k_2, \cos\theta) = 0$ for all θ so that $D_{\ell}(k, k_2) = 0$
- 2) if $k < \frac{1}{2}(k_F - k_2)$, $\xi(k, k_2, \cos\theta) = 1$ for all θ so that using (5.10a), $D_{\ell}(k, k_2) = \delta_{\ell,0}$
- 3) if $\frac{1}{2} |k_2 - k_F| < k < \frac{1}{2} (k_2 + k_F)$,

$$D_{\ell}(k, k_2) = \frac{2\ell+1}{2} \int_{-1}^{\kappa_2} dx P_{\ell}(x), \quad \kappa_2 = \frac{k_F^2 - 4k^2 - k_2^2}{4kk_2}$$

This integral can be evaluated using (5.10c). We find

$$D_0(k, k_2) = \frac{1}{2} (\kappa_2 + 1)$$

$$\ell \neq 0, D_\ell(k, k_2) = \frac{1}{2} [P_{\ell+1}(\kappa_2) - P_{\ell-1}(\kappa_2)]$$

Finally, then,

$$(5.51a) \quad \xi(k, k_2, \cos\theta) = \sum_{\ell=0}^{\infty} D_\ell(k, k_2) P_\ell(\cos\theta)$$

(5.51b)

$$D_0(k, k_2) = \begin{cases} 1 & \text{if } k < \frac{1}{2}(k_F - k_2) \\ \frac{1}{2}(\kappa_2 + 1) & \text{if } \frac{1}{2}|k_2 - k_F| < k < \frac{1}{2}(k_2 + k_F) \\ 0 & \text{if } k > \frac{1}{2}(k_2 + k_F) \text{ or } k < \frac{1}{2}(k_2 - k_F) \end{cases}$$

(5.51c)

$$\ell \neq 0, D_\ell(k, k_2) = \begin{cases} 0 & \text{if } k < \frac{1}{2}(k_F - k_2) \\ \frac{1}{2}[P_{\ell+1}(\kappa_2) - P_{\ell-1}(\kappa_2)] & \text{if } \frac{1}{2}|k_2 - k_F| < k < \frac{1}{2}(k_2 + k_F) \\ 0 & \text{if } k > \frac{1}{2}(k_2 + k_F) \text{ or } k < \frac{1}{2}(k_2 - k_F) \end{cases}$$

$$(5.51d) \quad \kappa_2 = \frac{k_F^2 - 4k^2 - k_2^2}{4kk_2}$$

Thus, using (5.51), we can write (5.49) and (5.50) in the following forms:

(5.52)

$$\int_{|\vec{k} + \vec{k}_2| < k_F} dk k^2 d(\cos\theta) d\varphi = \sum_{\ell=0}^{\infty} \left(\frac{4\pi}{2\ell+1} \right)^{\frac{1}{2}} \int_0^{\infty} dk k^2 \int_{-1}^{+1} d(\cos\theta) \int_0^{2\pi} d\varphi D_\ell(k, k_2) Y_{\ell,0}(\theta)$$

(5.53)

$$\int_{|\vec{k}+\vec{k}_2| < k_F} d\vec{k} (\vec{k}; \times_{S_{m_S}} |K| \vec{k}, \vec{P}; \times_{S_{m_S}}) =$$

$$= \sum_{\ell=0}^{\infty} \left(\frac{4\pi}{2\ell+1} \right)^{\frac{1}{2}} \int_0^{\infty} dk k^2 \int_{-1}^{+1} d(\cos\theta) \int_0^{2\pi} d\varphi D_{\ell}(k, k_2) Y_{\ell 0}(\theta) (\vec{k}; \times_{S_{m_S}} |K| \vec{k}, \vec{P}; \times_{S_{m_S}})$$

It should be noted that in contrast to the expansions of $\zeta(k, P, |\cos\theta|)$ and $\eta(k, P, |\cos\theta|)$ (equations (5.11) and (5.19) respectively), the expansion above of $\xi(k, k_2, \cos\theta)$ contains odd as well as even orders of spherical harmonics. The reason for this is that in this case the integration region does not possess reflection symmetry.

Now for the singlet spin case, the K matrix elements are given by (5.26) so using this and (5.53) we can write

(5.54)

$$\int_{|\vec{k}+\vec{k}_2| < k_F} d\vec{k} (\vec{k} | K | \vec{k}, \vec{P}) = \sum_{\ell=0}^{\infty} \sum_{\lambda=0}^{\infty} \sum_{m=-\ell}^{+\ell} \sum_{\alpha=0}^{\infty} \left(\frac{4\pi}{2\alpha+1} \right)^{\frac{1}{2}} \times$$

$$\times \int_0^{\infty} dk k^2 \int_{-1}^{+1} d(\cos\theta) \int_0^{2\pi} d\varphi a_{\ell}(k) f_{\ell m}^{(\lambda)}(k, P) D_{\alpha}(k, k_2) \times$$

$$\times Y_{\lambda m}^*(\theta, \varphi) Y_{\ell m}(\theta, \varphi) Y_{\alpha, 0}(\theta)$$

There is now one important point which much be noticed. There is a very complicated angle dependence of the integrand because of the P dependence of the f's. That is, we have

$$(5.55) \quad P = |\vec{k}+\vec{k}_2| = (k^2 + 2kk_2 \cos\theta + k_2^2)^{\frac{1}{2}}$$

Because of this angle dependence of P, the integral above is very messy. The angular integrations could of course be performed by an electronic computer, but for our present purposes here we shall assume that we can approximate P by \tilde{P} , where \tilde{P} is an average of P over angles or some other appropriate average. Using this assumption, the angular integration can easily be performed using (5.13) so we obtain

$$(5.56) \quad \int_{|\vec{k}+\vec{k}_2| < k_F} d\vec{k} (\vec{k} | K | \vec{k}, \vec{P}) =$$

$$= \sum_{\ell=0}^{\infty} \sum_{\lambda=0}^{\infty} \sum_{m=-\ell}^{+\ell} \sum_{\alpha=0}^{\infty} \left(\frac{2\ell+1}{2\lambda+1} \right)^{\frac{1}{2}} (a\ell 0 m | \lambda m) (a\ell 0 0 | \lambda 0) \times$$

$$\times \int_0^{\infty} dk k^2 a_{\ell}(k) f_{\ell m}^{(\lambda)}(k, \tilde{P}) D_{\alpha}(k, k_2)$$

Similarly for the triplet case we can write, using (5.43) and (5.53),

$$(5.57) \quad \int_{|\vec{k}+\vec{k}_2| < k_F} d\vec{k} (\vec{k}; \chi_{1m_S} | K | \vec{k}, \vec{P}; \chi_{1m_S}) =$$

$$= \sum_{J=0}^{\infty} \sum_{J'=0}^{\infty} \sum_{M=-J}^{+J} \sum_{(\xi\eta)} \sum_{\sigma=-1}^{+1} \sum_{\alpha=0}^{\infty} \left[\frac{2(J+\xi)+1}{2(J'+\sigma)+1} \right]^{\frac{1}{2}} (J+\xi, 1, M-m_S, m_S | JM) \times$$

$$\times (J'+\sigma, 1, M-m_S, m_S | J'M) (a, J+\xi, 0, M-m_S | J'+\sigma, M-m_S) (a, J+\xi, 0 0 | J'+\sigma, 0) \times$$

$$\times \int_0^{\infty} dk k^2 a_{J\xi}(k) f_{JM\xi\eta}^{(J'+\sigma)}(k, \tilde{P}) D_{\alpha}(k, k_2)$$

We are interested in the summation of this expression over all values of m_S (see (4.34)). This summation can easily be performed using (5.38) and we find

$$\begin{aligned}
 (5.58) \quad & \sum_{m_S=-1}^{+1} \int_{|\vec{2k}+\vec{k}_2| < k_F} d\vec{k} (\vec{k}; \times_{1m_S} |K| \vec{k}, \vec{P}; \times_{1m_S}) = \\
 & = \sum_{J=0}^{\infty} \sum_{J^{\dagger}=0}^{\infty} \sum_{M=-J}^{+J} \sum_{(\xi\eta)} \sum_{\sigma=-1}^{+1} \sum_{\alpha=0}^{\infty} (2J+1)^{\frac{1}{2}} [2(J+\xi)+1]^{\frac{1}{2}} \times \\
 & \times (\alpha, JOM | J^{\dagger}M) W(\alpha, J+\xi, J^{\dagger}, 1; J^{\dagger}+\sigma, J) (\alpha, J+\xi, 00 | J^{\dagger}+\sigma, 0) \times \\
 & \times \int_0^{\infty} dk k^2 a_{J\xi}(k) f_{JM\xi\eta}^{(J^{\dagger}\sigma)}(k, \tilde{P}) D_{\alpha}(k, k_2)
 \end{aligned}$$

The expression (4.34) for the single particle potential can then be written

$$\begin{aligned}
 (5.59) \quad & U(k_2) = \\
 & = \frac{1}{2\pi^3} \left[\left(3 \sum_{\text{even } \ell} + \sum_{\text{odd } \ell} \right) \sum_{\lambda=0}^{\infty} \sum_{m=-\ell}^{+\ell} \sum_{\alpha=0}^{\infty} \left(\frac{2\ell+1}{2\lambda+1} \right)^{\frac{1}{2}} (\alpha\ell 0m | \lambda m) (\alpha\ell 00 | \lambda 0) \times \right. \\
 & \quad \times \int_0^{\infty} dk k^2 a_{\ell}(k) f_{\ell m}^{(\lambda)}(k, \tilde{P}) D_{\alpha}(k, k_2) \\
 & + \left(\sum_{J(\xi\eta)} + 3 \sum_{J(\xi\eta)} \right) \sum_{J^{\dagger}=0}^{\infty} \sum_{M=-J}^{+J} \sum_{\sigma=-1}^{+1} \sum_{\alpha=0}^{\infty} (2J+1)^{\frac{1}{2}} [2(J+\xi)+1]^{\frac{1}{2}} \times \\
 & \quad \times (\alpha JOM | J^{\dagger}M) W(\alpha, J+\xi, J^{\dagger}, 1; J^{\dagger}+\sigma, J) (\alpha, J+\xi, 00 | J^{\dagger}+\sigma, 0) \times \\
 & \quad \times \int_0^{\infty} dk k^2 a_{J\xi}(k) f_{JM\xi\eta}^{(J^{\dagger}\sigma)}(k, \tilde{P}) D_{\alpha}(k, k_2) \Big]
 \end{aligned}$$

In the above we have made use of the fact that K^+ couples only states with even orbital angular momentum quantum number and K^- couples only states with odd orbital angular momentum quantum number.

This completes our manipulations with the formal expressions of the theory. In the next Chapter we shall apply these expressions and obtain some numerical results for the case of interactions in 1S_0 states only.

CHAPTER VI

INTERACTION IN 1S_0 STATES

We have to this point in the thesis concerned ourselves with applying the BCM, via our pseudopotential, to the IPAM (or Brueckner approximation) for nuclear matter, and with the casting of the equations involved into forms which can be used directly for numerical work. We have not as yet discussed whether our approach leads to meaningful numerical results. In this Chapter we shall discuss in some detail and give some numerical results for the case when the potential vanishes in all but 1S_0 states. Of course by using this simplified potential we cannot expect to obtain the experimental equilibrium properties of nuclear matter, that is, (see Section 4.1)

$$E/A \approx -15 \text{ Mev}$$

$$\rho \approx .18 \text{ nucleons (fermi)}^3 \text{ so } k_F \approx 1.4 \text{ fermi}^{-1}$$

$$PE/A = E/A - \frac{3}{5} \frac{\hbar^2 k_F^2}{2m} \approx -15 \text{ Mev} - 24.4 \text{ Mev}$$

$$= -39.4 \text{ Mev.}$$

We expect of course that the major contributions to the PE/A come from the interactions in 1S_0 and 3S_1 states. Indeed

Moszkowski and Scott⁵⁾ have done a calculation using the realistic two-body potential used by Brueckner and Gammel¹⁾, and have found a minimum E/A of -14.2 Mev at $k_F = 1.5$ fermi⁻¹. Table III of their paper shows that for $k_F = 1.5$ fermi⁻¹ the contribution to the PE/A by S states is -33.3 Mev, of which -18.6 Mev comes from 1S_0 states and -14.7 Mev comes from 3S_1 states. From Table II of their paper it is seen that for $k_F = 1.4$ fermi⁻¹ (the experimental value) the total S wave contribution is -31.6 Mev. Although Moszkowski and Scott have not indicated the separate 1S_0 and 3S_1 contributions at this value of k_F , we shall suppose that the 1S_0 contribution is approximately -18 Mev. Thus, by assuming an interaction in 1S_0 states only we should find that for $k_F = 1.4$ fermi⁻¹, $PE/A \approx -18$ Mev.

Assuming the interaction V to vanish in all but 1S_0 states we have then

$$(6.1) \quad (\vec{k}'; \chi_{1m_S} | V | \vec{k}; \chi_{1m_S}) = 0 \quad \text{so} \quad (\vec{k}'; \chi_{1m_S} | K | \vec{k}; \vec{P}; \chi_{1m_S}) = 0$$

and, from (5.3) and (5.26).

$$(6.2) \quad (\vec{k}' | V | \vec{k}) = \frac{1}{4\pi} a_0(k') b_0(k)$$

$$(6.3) \quad (\vec{k}' | K | \vec{k}, \vec{P}) = \frac{1}{4\pi} a_0(k') f_{00}^{(0)}(k, P)$$

where $f_{oo}^{(o)}$ satisfies (see equation (5.25)):

$$(6.4) \quad f_{oo}^{(o)}(k, P) = b_o(k) + \left(\frac{1}{2\pi}\right)^3 f_{oo}^{(o)}(k, P) \int_0^\infty dk'' k''^2 e^{-1} b_o(k'') a_o(k'') C_o(k'', P)$$

or

$$(6.5) \quad f_{oo}^{(o)}(k, P) = \frac{b_o(k)}{1 - \left(\frac{1}{2\pi}\right)^3 \int_0^\infty dk'' k''^2 e^{-1} b_o(k'') a_o(k'') C_o(k'', P)}$$

Thus, from (6.3) and (6.5), we have

$$(6.6) \quad (\vec{k}' | K | \vec{k}, \vec{P}) = \frac{\frac{1}{4\pi} a_o(k') b_o(k)}{1 - \left(\frac{1}{2\pi}\right)^3 \int_0^\infty dk'' k''^2 e^{-1} b_o(k'') a_o(k'') C_o(k'', P)}$$

Let us consider the denominator of (6.6) which we shall call $L(k, P)$:

$$(6.7) \quad L(k, P) = 1 - \left(\frac{1}{2\pi}\right)^3 \int_0^\infty dk'' k''^2 e^{-1} a_o(k'') b_o(k'') C_o(k'', P) \\ = 1 - \left(\frac{1}{2\pi}\right)^3 \frac{(4\pi\hbar)^2 r_o}{m} \int_0^\infty dk'' k''^2 e^{-1} j_o(k'' r_o) \left[f_o j_o(k'' r_o) - k'' r_o j_o'(k'' r_o) \right] C_o(k'', P)$$

using (5.27). f_o and r_o are the $1S_o$ boundary condition parameters.

In order to evaluate the above integral we shall use the effective mass approximation. That is, we assume that the single particle potential is of the form

$$(6.8) \quad U(k_2) = U_o + U_1 k_2^2$$

where U_0 and U_1 are independent of k_2 , so that the single particle energy $\epsilon(k_2)$ can be written

$$(6.9) \quad \epsilon(k_2) = \frac{\hbar^2 k_2^2}{2m} + U(k_2) = U_0 + \left(\frac{\hbar^2}{2m} + U_1\right) k_2^2$$

Defining the effective mass m^* by

$$(6.10) \quad \frac{\hbar^2}{2m^*} = \frac{\hbar^2}{2m} + U_1$$

we have then

$$(6.11) \quad U(k_2) = U_0 + \frac{1}{2} \hbar^2 \left(\frac{1}{m^*} - \frac{1}{m}\right) k_2^2$$

$$(6.12) \quad \epsilon(k_2) = U_0 + \frac{\hbar^2 k_2^2}{2m^*}$$

The effective mass m^* must be determined self-consistently. (There is no self-consistency requirement on U_0 since it will not appear in the K matrix elements.) The single particle potential which we calculate from (5.59) will not have a simple quadratic k_2 dependence, so that we shall find that m^* is really k_2 dependent. The self-consistency lies in finding a k_2 independent value of m^* for which the calculated single particle potential is as close as possible (at least for values of k_2 of the order of k_F) to the assumed single particle potential (6.11).

Using the effective mass approximation, the energy denominator e (see (5.2b)) can be written

$$(6.13) \quad e = \varepsilon(\vec{P} + \vec{k}) + \varepsilon(\vec{P} - \vec{k}) - \varepsilon(\vec{P} + \vec{k}'') - \varepsilon(\vec{P} - \vec{k}'') = \frac{\hbar^2}{m^*} (k^2 - k''^2)$$

so that (6.7) becomes

$$(6.14) \quad L(k, P) = \\ = 1 + \frac{2}{\pi} \frac{m^* r_0}{m} \int_0^\infty \frac{dk'' k''^2}{k''^2 - k^2} \left[f_0 j_0(k'' r_0) - k'' r_0 j_0'(k'' r_0) \right] j_0(k'' r_0) C_0(k'', P)$$

By defining new quantities x , x_F , y , y^- , ρ , a , β , and γ by

$$(6.15) \quad \begin{aligned} x &= k r_0 & x_F &= k_F r_0 \\ y &= k'' r_0 & y^- &= k'' r_0^- \\ \rho &= \frac{P}{k_F} \\ a &= r_0 \sqrt{k_F^2 - P^2} = x_F \sqrt{1 - \rho^2} \\ \beta &= r_0 (k_F + P) = x_F (1 + \rho) \\ \gamma &= r_0 (k_F - P) = x_F (1 - \rho) \end{aligned}$$

and using (5.23a), we can write

$$(6.16) \quad L(k, P) = \\ = 1 + \frac{2m^*}{\pi m} \left\{ \left(\int_a^\beta + \lambda \int_\gamma^a \right) \frac{dy y^2}{y^2 - x^2} \left[f_0 j_0(y) - y j_0'(y^-) \right] j_0(y) \left(\frac{y^2 - a^2}{2\rho x_F y} \right) \right. \\ \left. + \left(\int_\beta^\infty - \lambda \int_0^\gamma \right) \frac{dy y^2}{y^2 - x^2} \left[f_0 j_0(y) - y j_0'(y^-) \right] j_0(y) \right\}$$

It will be recalled from (5.23c) that λ takes the values 0 and 1. When $\lambda = 0$ only particle-particle interactions are counted, and when $\lambda=1$ both particle-particle and hole-hole interactions are counted. We recall further that the principal value of all integrals is to be taken.

Now $j_0(y) = \frac{\sin y}{y}$ so that

$$(6.17a) \quad L(k,P) =$$

$$= 1 + \frac{2}{\pi} \frac{m^*}{m} \left\{ \left(\int_a^\beta + \lambda \int_\gamma^a \right) \frac{dy \sin y}{y^2 - x^2} \left[(f_0 + 1) \sin y - y \cos y \right] \left(\frac{y^2 - a^2}{2\rho x_F y} \right) \right.$$

$$\left. + \left(\int_\beta^\infty - \lambda \int_0^\gamma \right) \frac{dy \sin y}{y^2 - x^2} \left[(f_0 + 1) \sin y - y \cos y \right] \right\}$$

The above integrals are elementary but tedious so we shall only quote the results. We have

$$(6.17b) \quad \int_a^b \frac{dy \sin^2 y}{y^2 - x^2} \left(\frac{y^2 - a^2}{2\rho x_F y} \right) = \frac{x}{4x} \left[\ln \left| \frac{b^2 - x^2}{a^2 - x^2} \right| - 2 \left(\ln \frac{b}{a} + Ci2a - Ci2b \right) \right.$$

$$\left. - (\phi_b^+ - \phi_a^+) \cos 2x - (\theta_b^- - \theta_a^-) \sin 2x \right] + \frac{1}{4\rho x_F} \left[\ln \frac{b}{a} + Ci2a - Ci2b \right]$$

$$(6.17c) \quad \int_a^b \frac{dy y \sin y \cos y}{y^2 - x^2} \left(\frac{y^2 - a^2}{2\rho x_F y} \right) = \frac{x}{4} \left[+ (\phi_b^+ - \phi_a^+) \sin 2x - (\theta_b^- - \theta_a^-) \cos 2x \right]$$

$$- \frac{1}{8\rho x_F} \left[\cos 2b - \cos 2a \right]$$

$$(6.17d) \int_{\beta}^{\infty} \frac{dy \sin^2 y}{y^2 - x^2} = \frac{1}{4x} \left[\ln \left| \frac{\beta+x}{\beta-x} \right| - \phi_{\beta}^{-} \cos 2x + (\pi - \theta_{\beta}^{+}) \sin 2x \right]$$

$$(6.17e) \int_{\beta}^{\infty} \frac{dy y \sin y \cos y}{y^2 - x^2} = + \frac{\pi}{4} + \frac{1}{4} \left[+ \phi_{\beta}^{-} \sin 2x + (\pi - \theta_{\beta}^{+}) \cos 2x \right]$$

$$(6.17f) \int_0^{\gamma} \frac{dy \sin^2 y}{y^2 - x^2} = \frac{1}{4x} \left[\ln \left| \frac{\gamma-x}{\gamma+x} \right| + \phi_{\gamma}^{-} \cos 2x + \theta_{\gamma}^{+} \sin 2x \right]$$

$$(6.17g) \int_0^{\gamma} \frac{dy y \sin y \cos y}{y^2 - x^2} = - \frac{1}{4} \left[\phi_{\gamma}^{-} \sin 2x - \theta_{\gamma}^{+} \cos 2x \right]$$

where, as in Jahnke and Emde²⁴⁾,

$$(6.18a) \quad \text{Si} x = \int_0^x \frac{dt \sin t}{t}$$

$$(6.18b) \quad \text{Ci} x = - \int_x^{\infty} \frac{dt \cos t}{t}$$

and where we have introduced the notation

$$(6.19a) \quad \theta_a^{\pm} = \text{Si} 2(a+x) \pm \text{Si} 2(a-x)$$

$$(6.19b) \quad \phi_a^{\pm} = \text{Ci} 2(a+x) \pm \text{Ci} 2(a-x)$$

Further

$$(6.20) \quad X = \frac{x^2 - a^2}{2\rho x_F x}$$

A word about the importance of using $\cos y^-$ rather than $\cos y$ or $\cos y^+$ in (6.17e) is in order. A straightforward calculation of the following integrals (compare (6.17e)):

$$\int_a^b \frac{dk "k" \text{sink} "r \cos k "r^i}{k^{''2} - k^2} \quad , \quad \int_a^b \frac{dk "k" \text{sink} "r \cos k "r}{k^{''2} - k^2}$$

shows that

1) If both a and b are not infinite, then

$$\lim_{r \rightarrow r^i} \int_a^b \frac{dk "k" \text{sink} "r \cos k "r^i}{k^{''2} - k^2} = \int_a^b \frac{dk "k" \text{sink} "r \cos k "r}{k^{''2} - k^2}$$

2) If $r > r^i$, then

$$\lim_{r \rightarrow r^i} \int_a^\infty \frac{dk "k" \text{sink} "r \cos k "r^i}{k^{''2} - k^2} = + \frac{\pi}{4} + \int_a^\infty \frac{dk "k" \text{sink} "r \cos k "r}{k^{''2} - k^2}$$

3) If $r < r^i$, then

$$\lim_{r \rightarrow r^i} \int_a^\infty \frac{dk "k" \text{sink} "r \cos k "r^i}{k^{''2} - k^2} = - \frac{\pi}{4} + \int_a^\infty \frac{dk "k" \text{sink} "r \cos k "r}{k^{''2} - k^2}$$

That is, when one limit of integration is infinite the limiting operation and the integration cannot be interchanged.

In particular then,

$$\int_{\beta}^{\infty} \frac{dy \sin y \cos y}{y^2 - x^2} = \frac{\pi}{4} + \int_{\beta}^{\infty} \frac{dy \sin y \cos y}{y^2 - x^2}$$

Thus finally we have

$$(6.21a) \quad L(k, P) =$$

$$= 1 + \frac{2m^*}{\pi m} \left[(f_0 + 1) I_1(k, P) + I_2(k, P) + \lambda \left\{ (f_0 + 1) I_3(k, P) + I_4(k, P) \right\} \right]$$

$$(6.21b) \quad I_1(k, P) =$$

$$= \frac{X}{4x} \left[\ln \left| \frac{\beta^2 - x^2}{a^2 - x^2} \right| - 2 \left(\ln \frac{\beta}{a} + \text{Ci} 2a - \text{Ci} 2\beta \right) - (\phi_{\beta}^+ - \phi_a^+) \cos 2x - (\theta_{\beta}^- - \theta_a^-) \sin 2x \right]$$

$$+ \frac{1}{4\rho x_F} \left[\ln \frac{\beta}{a} + \text{Ci} 2a - \text{Ci} 2\beta \right] + \frac{1}{4x} \left[\ln \left| \frac{\beta + x}{\beta - x} \right| - \phi_{\beta}^- \cos 2x + (\pi - \theta_{\beta}^+) \sin 2x \right]$$

$$(6.21c) \quad I_2(k, P) = -\frac{\pi}{4} + \frac{X}{4} \left[-(\phi_{\beta}^+ - \phi_a^+) \sin 2x + (\theta_{\beta}^- - \theta_a^-) \cos 2x \right]$$

$$+ \frac{1}{8\rho x_F} \left[\cos 2\beta - \cos 2a \right] - \frac{1}{4} \left[\phi_{\beta}^- \sin 2x + (\pi - \theta_{\beta}^+) \cos 2x \right]$$

$$(6.21d) \quad I_3(k, P) =$$

$$= \frac{X}{4x} \left[\ln \left| \frac{a^2 - x^2}{\gamma^2 - x^2} \right| - 2 \left(\ln \frac{a}{\gamma} + \text{Ci} 2\gamma - \text{Ci} 2a \right) - (\phi_a^+ - \phi_{\gamma}^+) \cos 2x - (\theta_a^- - \theta_{\gamma}^-) \sin 2x \right]$$

$$+ \frac{1}{4\rho x_F} \left[\ln \frac{a}{\gamma} + \text{Ci} 2\gamma - \text{Ci} 2a \right] + \frac{1}{4x} \left[\ln \left| \frac{\gamma + x}{\gamma - x} \right| - \phi_{\gamma}^- \cos 2x - \theta_{\gamma}^+ \sin 2x \right]$$

$$(6.21e) \quad I_4(k, P) = \frac{X}{4} \left[-(\phi_a^+ - \phi_\gamma^+) \sin 2x + (\theta_a^- - \theta_\gamma^-) \cos 2x \right] \\ + \frac{1}{8\rho x_F} \left[\cos 2\alpha - \cos 2\gamma \right] - \frac{1}{4} \left[\phi_\gamma^- \sin 2x - \theta_\gamma^+ \cos 2x \right]$$

Recall from (5.3) that

$$(6.22) \quad (\vec{k} | V | \vec{k}) = \\ = \frac{1}{4\pi} a_o(k') b_o(k) = \frac{4\pi\hbar^2 r_o}{m} j_o(k' r_o) \left[f_o j_o(kr_o) - kr_o j_o'(kr_o) \right] \\ = \frac{4\pi\hbar^2 r_o}{m} \frac{\sin k' r_o}{k' r_o} \left[(f_o + 1) \frac{\sin kr_o}{kr_o} - \cos kr_o \right]$$

so that, from (6.6) and (6.7),

$$(6.23) \quad (\vec{k}' | K | \vec{k}, \vec{P}) = \frac{\frac{4\pi\hbar^2 r_o}{m} \frac{\sin k' r_o}{k' r_o} \left[(f_o + 1) \frac{\sin kr_o}{kr_o} - \cos kr_o \right]}{L(k, P)}$$

with $L(k, P)$ given by (6.21) when the effective mass approximation for the energy denominator is used.

As a check on our results we should find on taking $k_F \rightarrow 0$ and $m^* \rightarrow m$ in (6.21) that $(\vec{k} | K | \vec{k}, \vec{P})$ given by (6.23) reduces to the S wave part of the two-body reactance matrix, that is, it should be proportional to the tangent of the S wave phase shift (3.24). In Appendix III we show that this is indeed the case.

In order to proceed with numerical calculations we need the values of the 1S_0 boundary condition parameters, f_0 and r_0 . Feshbach and Lomon⁹⁾ have found these parameters by relating them to the scattering length and effective range, that is, from the low energy scattering data. They find

$$\left. \begin{array}{ll} (6.24a) & f_0 = 0.082 \text{--} 1 \\ (6.24b) & r_0 = 1.32 \text{ fermi} \end{array} \right\} \text{ for proton-proton scattering}$$

$$\left. \begin{array}{ll} (6.25a) & f_0 = 0.053 \text{--} 1 \\ (6.25b) & r_0 = 1.32 \text{ fermi} \end{array} \right\} \text{ for neutron-proton scattering}$$

These values are not sufficient for our purposes. We need parameters which fit the data over an energy range much larger than can be covered by the effective range approximation formulae. Indeed, the maximum energy of a nucleon in the ground state of nuclear matter is $\frac{\hbar^2 k_F^2}{2m} \approx 40 \text{ Mev}$ so that collision energies of the order of 80 Mev in the centre of mass system are possible. Thus we need values of the boundary condition parameters which fit the two body scattering data for laboratory energies up to 160 Mev. Clearly at these energies the effective approximation formulae cannot be used.

In order to determine the parameters we have used the proton-proton scattering data of Breit et al.²⁵⁾ Figure 1 of this paper shows a plot of the 1S_0 phase shift vs. lab energy. Now (2.4) gives the relationship with the phase shifts and the boundary condition parameters so we have, with $\ell = 0$,

$$(6.26) \quad \tan \delta_0 = \frac{f_0 j_0(kr_0) - kr_0 j_0'(kr_0)}{f_0 n_0(kr_0) - kr_0 n_0'(kr_0)} = \frac{(f_0+1) \sin kr_0 - kr_0 \cos kr_0}{-(f_0+1) \cos kr_0 - kr_0 \sin kr_0}$$

using $j_0(x) = \frac{\sin x}{x}$ and $n_0(x) = -\frac{\cos x}{x}$. We have adjusted f_0 and r_0 so that this expression gives an over-all fit to the YLAM 1S_0 phase shift data of Breit et al for lab energies up to 160 Mev. We find a fit with

$$(6.27a) \quad f_0 = - .89$$

$$(6.27b) \quad r_0 = .95 \text{ fermi}$$

In Figure II we have plotted Breit's YLAM 1S_0 phase shift data and δ_0 from (6.26) for these values of f_0 and r_0 . For comparison, we have also plotted δ_0 using the Feshbach and Lomon values of the parameters (6.24).

Using the values of the parameters given by (6.27) we have plotted, as functions of k , $(\vec{k}|\mathbf{V}|\vec{k})$ and $(\vec{k}|\mathbf{K}|\vec{k}, \vec{P})$ as given by (6.22) and (6.23). In Figure III we show $(\vec{k}|\mathbf{K}|\vec{k}, \vec{P})$ with $m^*=m$ for $P=0$, $P=.5k_F$, $P=.9k_F$ for the case when the hole terms are neglected ($\lambda=0$), and in Figure IV

we show $(\vec{k}|K|\vec{k},\vec{P})$ with $m^*=m$ for $P=0$, $P=.5k_F$, $P=.9k_F$ when the hole terms are included ($\lambda=1$). On each Figure we have also plotted $(\vec{k}|V|\vec{k})$, which incidentally is the value of $(\vec{k}|K|\vec{k},\vec{P})$ when $m^*=0$. We notice in both cases that $(\vec{k}|K|\vec{k},\vec{P})$ is quite different from $(\vec{k}|V|\vec{k})$, especially near the bottom of the Fermi sea (that is, for $k \ll k_F$). Thus, the first Born approximation is a rather poor approximation for the K matrix elements. We notice in the case when the hole terms are neglected (see Fig. III) that for fixed k less than k_F , $(\vec{k}|K|\vec{k},\vec{P})$ decreases monotonically as P increases to k_F . What is perhaps most striking however is the great effect the inclusion of the hole terms has on the K matrix elements. We see in particular that when the hole terms are included the K matrix elements become rather strongly dependent on P .

Let us consider now the potential energy per particle, PE/A . From (5.47) we have for this case

$$(6.28) \quad PE/A = \frac{9}{4} (\pi k_F)^{-3} \int_0^{k_F} dP P^2 \int_0^\infty dk k^2 a_o(k) f_{oo}^{(o)}(k,P) B_o(k,P)$$

which using (6.3) and (5.19b), can be written

$$(6.29) \quad PE/A = \frac{9}{\pi^2 k_F^3} \int_0^{k_F} dP P^2 \left[\int_0^{k_F-P} dk k^2 (\vec{k}|K|\vec{k},\vec{P}) \frac{1}{\sqrt{k_F^2 - P^2}} - \int_{k_F-P}^{k_F} dk k^2 (\vec{k}|K|\vec{k},\vec{P}) \frac{k^2 - k_F^2 + P^2}{2Pk} \right]$$

We have integrated this expression numerically using the plots of $(\vec{k}|K|\vec{k},\vec{P})$ vs. k (Figs. III and IV). To test the effect of various approximations we have performed the integration for the following five cases:

- 1) hole terms neglected ($\lambda=0$), $m^*=m$
- 2) hole terms included ($\lambda=1$), $m^*=m$
- 3) hole terms neglected, $m^*=m$, $(\vec{k}|K|\vec{k},\vec{P}) = (\vec{k}|K|\vec{k},0)$
- 4) hole terms included, $m^*=m$, $(\vec{k}|K|\vec{k},\vec{P}) = (\vec{k}|K|\vec{k},0)$
- 5) $(\vec{k}|K|\vec{k},\vec{P}) = (\vec{k}|V|\vec{k})$ (i.e., $m^*=0$)

In the first two cases above no approximation on P are made but in cases 3) and 4) we perform the integration assuming that $(\vec{k}|K|\vec{k},\vec{P})$ is independent of P and equal to its value when $P=0$. In the last case we take $(\vec{k}|K|\vec{k},\vec{P})=(\vec{k}|V|\vec{k})$ which is equivalent to assuming $m^*=0$. The results are given in Table I.

TABLE I

PE/A for 1S_0 states (Mev)	Form of $(\vec{k} K \vec{k},\vec{P})$
-17.62	hole terms neglected, $m^*=m$
-17.63	hole terms included, $m^*=m$
-16.6	hole terms neglected, $m^*=m$, $P=0$
-13.3	hole terms included, $m^*=m$, $P=0$
-10.4	$m^*=0$

What is perhaps most remarkable about these results is that the PE/A for cases 1) and 2) are almost identical, that is, including the hole terms does not make a significant change to the PE/A despite the large change it makes to the K matrix elements (compare Figs. III and IV). Moszkowski and Sessler⁸⁾ have also arrived at this conclusion by performing a crude calculation using an interaction containing a repulsive core and an exponential well.

Our results indicate that non-zero centre of mass momentum effects are most important when the hole terms are included. Indeed, the difference between the results of cases 1) and 3) is about 1 Mev as compared with 4.3 Mev for the difference between the results of cases 2) and 4). Our results also indicate that the PE/A depends fairly critically on the value of m^* .

Let us consider now the calculation of the single particle potential $U(k_2)$. From (5.59) we have for this case

$$(6.30) \quad U(k_2) = \frac{3}{2\pi^3} \int_0^\infty dk k^2 a_0(k) f_{00}^{(0)}(k, \tilde{P}) D_0(k, k_2)$$

which using (6.3) and (5.51b) can be written

$$(6.31) \quad U(k_2) = \frac{6}{\pi^2} \left[\int_0^{\frac{1}{2}(k_F - k_2)} dk k^2 (\vec{k} | K | \vec{k}, \vec{P}) + \int_{\frac{1}{2}|k_F - k_2|}^{\frac{1}{2}(k_F + k_2)} dk k^2 (\vec{k} | K | \vec{k}, \vec{P}) \frac{k_F^2 - (k_2 - 2k)^2}{8kk_2} \right]$$

Recall that \vec{P} denotes some average of P . We have integrated this expression numerically using the plots of $(\vec{k} | K | \vec{k}, \vec{P})$ vs. k (Figs. III and IV). We have arbitrarily taken $\vec{P} = 0$ (this is of course not the best choice of \vec{P}), and Figure V shows $U(k_2)$ vs. k_2 for the cases when the hole terms are neglected and included. Figure VI shows $U(k_2)$ for the case when $(\vec{k} | K | \vec{k}, \vec{P}) = (\vec{k} | V | \vec{k})$ (i.e. when $m^* = 0$).

The validity of the effective mass approximation (6.11) can be checked by fitting parabolas to the calculated single particle potentials. On Figs. V and VI we have drawn parabolas fitted at $k_2 = 0$ and $k_2 = k_F$, and an effective mass m^* , which represents an average effective mass over the Fermi sea, can be determined from these parabolas. From (6.11) we have

$$(6.32) \quad \frac{m^*}{m} = \frac{\frac{\hbar^2 k_F^2}{2m}}{U(k_F) - U(0) + \frac{\hbar^2 k_F^2}{2m}}$$

The values of m^* calculated in this manner are as follows:

TABLE II

$\frac{m^*}{m}$	Form of $(\vec{k} K \vec{k},\tilde{P})$
.63	hole terms neglected, $\tilde{P} = 0$
.90	hole terms included, $\tilde{P} = 0$
.74	$m^* = 0$

It is noted from Figure V that the single particle potential calculated with hole terms neglected can be fitted quite well with a parabola, indicating that in this case the effective mass approximation is quite good. The single particle potential calculated with hole terms included cannot be fitted to a parabola quite as well however, indicating that in this case it may be necessary to use a more elaborate energy denominator to obtain accurate results.

Another value for the effective mass, namely the value at the top of the Fermi sea, can be calculated from the slope of $U(k_2)$ at $k_2 = k_F$. It is worth noting from Figure V that in the case when the hole terms are neglected this effective mass will not differ much from the average

effective mass calculated above, but that for the case when the hole terms are included the value will be somewhat less than the average value. Both methods of calculating the effective mass indicate that its value is greater when the hole terms are included.

We have not attempted to calculate m^* self-consistently and thus obtain an accurate value of the PE/A. From Table I however we see that the value must lie between - 17.6 Mev ($m^*=m$ value) and -10.4 Mev ($m^*=0$ value) and probably will be closer to the larger value. We note further that if the self-consistent value of m^* calculated including hole terms is larger than that calculated neglecting hole terms (as is indicated above) then the inclusion of the hole terms decreases the PE/A somewhat. At any rate, the calculated PE/A will not be far from the value of -18 Mev which the results of Moszkowski and Scott⁵⁾ indicate for the approximate contribution by 1S_0 states to the total PE/A. The numerical results which we have given above are very encouraging and indicate the validity of our approach to the many-body problem.

CHAPTER VII

SINGULARITIES IN THE K MATRIX AND THE ENERGY GAP

We have to this point assumed that the ground state properties of nuclear matter can be described adequately in terms of the IPAM. Bohr, Mottelson, and Pines²⁶⁾ have observed however that the energy difference between the ground state and the first excited intrinsic states of even-even spheroidal nuclei is much larger than that which one would expect if the intrinsic nuclear structure could be described adequately in terms of independent particle motion. On the basis of this observation, they have suggested that the ground state of nuclear matter is a highly correlated state (similar to the superconducting state of metals) separated from the normal states by an energy gap.

Such an energy gap between the ground state and the first intrinsic excitations of nuclear matter indicates an important departure from independent particle motion. Indeed one may wonder whether the IPAM approach to the discussion of the ground state properties of nuclear matter is valid, since it would seem that one must use instead an approach similar to that used by Bardeen, Cooper, and Schrieffer (BCS)²⁷⁾

in their discussion of the superconducting state of metals. (It should be mentioned that Katz²⁸⁾ has indeed developed an approach to the nuclear matter problem which combines both the IPAM and BCS model.) However the basic feature of the BCS theory of superconductivity is that the highly correlated states only involve single particle states in a small shell near the top of the Fermi sea and wherein the total spin and momentum of the correlated pair are each zero. Thus if the energy gap is small we should expect that the IPAM approach gives an adequate description of states far away from the Fermi surface or of pair states with total momentum much different from zero, so that it should be able to produce many of the gross properties of the system (e.g., the average energy per particle).

For states near the top of the Fermi sea having total spin zero and small total momentum, we expect the IPAM (or K matrix) approach to be inadequate, this inadequacy manifesting itself by an abnormal behavior of the K matrix for these states (e.g., it may be singular) since the K matrix represents an effective potential between the particles. Indeed, Emery¹⁰⁾ has proven that the K matrix for an infinite system of fermions (when hole-hole interactions are included) is singular if and only if there exists a gap in the energy spectrum of the system. (Emery has also shown that the presence of a singularity in the K matrix with the hole-hole

interactions excluded is only a sufficient condition for the existence of a gap.)

In this Chapter we shall examine the K matrix for the case when the interaction vanishes in all but 1S_0 states in order to see if indeed it has a singularity. This case was discussed in the last Chapter and the K matrix is given by (6.6). It is convenient here to write

$$(7.1) \quad a_0(k')b_0(k) = (k'|V|k)$$

$$(7.2) \quad a_0(k')f_{00}^{(0)}(k,P) = (k'|K|k,P)$$

so that from (6.2), (6.3), and (6.6) we have

$$(7.3) \quad (\vec{k}'|V|\vec{k}) = \frac{1}{4\pi}(k'|V|k)$$

$$(7.4) \quad (\vec{k}'|K|\vec{k},\vec{P}) = \frac{1}{4\pi}(k'|K|k,P)$$

$$(7.5) \quad (k'|K|k,P) = \frac{(k'|V|k)}{1 - \left(\frac{1}{2\pi}\right) \int_0^\infty dk'' k''^2 e^{-\frac{1}{2}k''^2} (k''|V|k'') C_0(k'',P)}$$

We have called the denominator of this expression $L(k,P)$ (see (6.7)). In our discussion here, as in the preceding Chapter, we shall use the effective mass approximation (6.13) so we have

$$(7.6) \quad L(k,P) = 1 + \left(\frac{1}{2\pi}\right) \frac{3m^*}{\hbar^2} \int_0^\infty \frac{dk' k'^2 (k'|V|k') C_0(k',P)}{k'^2 - k^2}$$

which becomes, when (5.23a) is used,

$$\begin{aligned}
 (7.7a) \quad L(k, P) = & 1 + \left(\frac{1}{2\pi}\right) \frac{3m^*}{\hbar^2} \left[\int_{k_F+P}^{k_F+P} \frac{dk' k'^2 (k' |V| k') \kappa'}{k'^2 - k^2} + \right. \\
 & \left. + \int_{k_F+P}^{\infty} \frac{dk' k'^2 (k' |V| k')}{k'^2 - k^2} + \lambda \int_{k_F-P}^{\sqrt{k_F^2 - P^2}} \frac{dk' k'^2 (k' |V| k') \kappa'}{k'^2 - k^2} \right. \\
 & \left. - \lambda \int_0^{k_F-P} \frac{dk' k'^2 (k' |V| k')}{k'^2 - k^2} \right]
 \end{aligned}$$

$$(7.7b) \quad \kappa' = \frac{k'^2 - P^2 + k_F^2}{2Pk'}$$

Recall that $k \leq \sqrt{k_F^2 - P^2}$ for the initial particles to be inside the Fermi sea.

The K matrix has a singularity when $L(k, P)$ vanishes. For most values of k and P , $L(k, P)$ is positive and does not differ too much from unity (see Figs. III and IV). For some values of k and P however, the integrals in (7.7a) can become large enough so that $L(k, P)$ vanishes as we shall now see. (Later in this Chapter we shall find these values of k and P : it turns out that $L(k, P)$ vanishes only for P extremely close to zero and k extremely close to k_F , and it is for this reason that plots of $(\vec{k} | K | \vec{k}, \vec{P})$

(Figs. III and IV) do not show any singularities.)

Let us consider first the case of $P=0$. We have from (7.7)

$$(7.8) \quad L(k,0) = \\ = 1 + \left(\frac{1}{2\pi}\right)^3 \frac{m^*}{\hbar^2} \left[\int_{k_F}^{\infty} \frac{dk' k'^2 (k' | V | k')}{k'^2 - k^2} - \lambda \int_0^{k_F} \frac{dk' k'^2 (k' | V | k')}{k'^2 - k^2} \right]$$

We note that poles can occur in these integrals only when $k=k_F$. We write

$$L(k,0) = \\ = 1 + \left(\frac{1}{2\pi}\right)^3 \frac{m^*}{\hbar^2} \left[k_F^2 (k_F | V | k_F) \int_{k_F}^{\infty} \frac{dk'}{k'^2 - k^2} + \int_{k_F}^{\infty} \frac{dk' \{k'^2 (k' | V | k') - k_F^2 (k_F | V | k_F)\}}{k'^2 - k^2} \right. \\ \left. - \lambda k_F^2 (k_F | V | k_F) \int_0^{k_F} \frac{dk'}{k'^2 - k^2} - \lambda \int_0^{k_F} \frac{dk' \{k'^2 (k' | V | k') - k_F^2 (k_F | V | k_F)\}}{k'^2 - k^2} \right]$$

where now poles can occur only in the first and third integrals.

Now

$$\int_{k_F}^{\infty} \frac{dk'}{k'^2 - k^2} = - \frac{1}{2k} \ln \left| \frac{k_F - k}{k_F + k} \right|$$

$$\int_0^{k_F} \frac{dk'}{k'^2 - k^2} = + \frac{1}{2k} \ln \left| \frac{k_F - k}{k_F + k} \right|$$

so that

$$(7.9) \quad L(k,0) =$$

$$= 1 + \left(\frac{1}{2\pi}\right)^3 \frac{m^*}{\hbar^2} \frac{k_F}{2} (1+\lambda) (k_F |V| k_F) \ln |k_F - k| +$$

+ terms which are finite when $k \rightarrow k_F$.

Now as $k \rightarrow k_F$, $\ln(k_F - k) \rightarrow -\infty$ so that since $(k_F |V| k_F) < 0$ (i.e., the potential is attractive at the Fermi surface. See (Fig. III or IV)), $L(k,0) \rightarrow \infty$. Thus there exists a value of k close to k_F (call it k_λ^*) for which $L(k_\lambda^*, 0) = 0$ and hence $(\vec{K} | K | \vec{K}, \vec{P})$ is singular. It should be noted that the zero of $L(k,0)$ occurs closer to k_F when $\lambda=0$ than when $\lambda=1$ (i.e., $k_1^* < k_0^*$).

It is of interest to see what happens to the singularity when P is different from zero. To the author's knowledge, no one has discussed this point quantitatively before, although a qualitative discussion has been given by Gottfried.²⁹⁾ The conclusions we reach below are in agreement with his.

We shall follow the same procedure as above. We write

(7.7) as

$$(7.10a) \quad L(k, P) =$$

$$= 1 + \left(\frac{1}{2\pi}\right)^3 \frac{m^*}{\hbar^2} \left[k^2 (k|V|k) \kappa \int_{\sqrt{k_F^2 - P^2}}^{k_F + P} \frac{dk'}{k'^2 - k^2} + \int_{\sqrt{k_F^2 - P^2}}^{k_F + P} \frac{dk' \{k'^2 (k'|V|k') \kappa' - k^2 (k|V|k) \kappa\}}{k'^2 - k^2} \right. \\ + k^2 (k|V|k) \int_{\sqrt{k_F^2 - P^2}}^{k_F + P} \frac{dk'}{k'^2 - k^2} + \int_{\sqrt{k_F^2 - P^2}}^{k_F + P} \frac{dk' \{k'^2 (k'|V|k') - k^2 (k|V|k)\}}{k'^2 - k^2} \\ + \lambda k^2 (k|V|k) \kappa \int_{k_F - P}^{\sqrt{k_F^2 - P^2}} \frac{dk'}{k'^2 - k^2} + \lambda \int_{k_F - P}^{\sqrt{k_F^2 - P^2}} \frac{dk' \{k'^2 (k'|V|k') \kappa' - k^2 (k|V|k) \kappa\}}{k'^2 - k^2} \\ \left. - \lambda k^2 (k|V|k) \int_0^{k_F - P} \frac{dk'}{k'^2 - k^2} - \lambda \int_0^{k_F - P} \frac{dk' \{k'^2 (k'|V|k') - k^2 (k|V|k)\}}{k'^2 - k^2} \right]$$

$$(7.10b) \quad \kappa = \frac{k^2 - P^2 + k_F^2}{2Pk}$$

Poles can occur only in the first integrals on each line of

(7.10a). Let us here write

$$(7.11) \quad \alpha' = \sqrt{k_F^2 - P^2} \quad \beta' = k_F + P \quad \gamma' = k_F - P$$

We can evaluate the first integrals on each line of (7.10a) to obtain

$$(7.12) \quad L(k, P) = \\ = 1 - \left(\frac{1}{2\pi}\right)^3 \frac{m^*}{\hbar^2} \frac{1}{2} k(k|V|k) \left[(1-\kappa) \ln|\beta^i - k| + (1-\lambda) \kappa \ln|\alpha^i - k| + \lambda(1+\kappa) \ln|\gamma^i - k| \right] + \\ + \text{finite terms}$$

We notice that for $k \rightarrow \gamma^i$, $(1+\kappa) \ln|\gamma^i - k| \rightarrow 0$

$$k \rightarrow \alpha^i, \quad \kappa \ln|\alpha^i - k| \rightarrow 0$$

$$k \rightarrow \beta^i, \quad (1-\kappa) \ln|\beta^i - k| \rightarrow 0$$

Thus, when $P \neq 0$ $L(k, P)$ is finite for all values of k . For small values of P however $L(k, P)$ can vanish as we shall now see. Let us consider $k \rightarrow \alpha^i$ which is the maximum value k can obtain for the initial particles to be inside the Fermi sea. We have then

$$\kappa \ln|\alpha^i - k| \rightarrow 0$$

$$(1-\kappa) \ln|\beta^i - k| \rightarrow \ln|\beta^i - \alpha^i|$$

$$(1+\kappa) \ln|\gamma^i - k| \rightarrow \ln|\alpha^i - \gamma^i|$$

Now

$$\begin{aligned} \beta^i - \alpha^i &= k_F + P - \sqrt{k_F^2 - P^2} = k_F + P - k_F \left(1 - \frac{1}{2} \frac{P^2}{k_F^2} + \dots\right) \\ &= P \left[1 + O\left(\frac{P}{k_F}\right)\right] \end{aligned}$$

and similarly

$$a^{\dagger} - \gamma^{\dagger} = P \left[1 + O\left(\frac{P}{k_F}\right) \right]$$

so that

$$(7.13) \quad L(a^{\dagger}, P) = 1 - \left(\frac{1}{2\pi}\right)^3 \frac{m^*}{\hbar^2} \frac{a^{\dagger}}{2} (1+\lambda) (a^{\dagger} | V | a^{\dagger}) \ln\left(\frac{P}{k_F}\right) + \text{finite terms}$$

Now since $(a^{\dagger} | V | a^{\dagger}) < 0$ (see Fig. III or IV), a non-zero value of P (call it P_{λ}^*) can be found so that $L(a^{\dagger}, P_{\lambda}^*) = 0$. Further, if $P > P_{\lambda}^*$ then $L(a^{\dagger}, P) > 0$ and if $P < P_{\lambda}^*$ then $L(a^{\dagger}, P) < 0$.

We are thus led to the following picture. A zero of $L(k, P)$ (and hence a singularity of $(\vec{k} | K | \vec{k}, \vec{P})$) will exist for some value of k when $P=0$. As P increases from zero, the value of k for which $L(k, P)$ vanishes moves closer to the maximum of the k values (i.e., to $\sqrt{k_F^2 - P^2}$) reaching this maximum when $P = P_{\lambda}^*$. When P exceeds P^* , $(\vec{k} | K | \vec{k}, \vec{P})$ will no longer have any singularities. As we shall see later in this Chapter the numerical value of P_{λ}^* is extremely small.

We note that the conclusions reached above are valid for any potential which vanishes in all but 1S_0 states, is separable in momentum space, and which is attractive (i.e., has negative matrix elements) in the region of the Fermi surface. It is worth noting also that the pair states which give rise to the singularity in the K matrix (i.e., pairs of particles near the top of the Fermi sea

with total spin zero and small centre of mass momentum and whose mutual interaction is attractive) are those which also are considered responsible for superconducting phenomena.

Let us turn now to equation (6.21) for $L(k,P)$ in order to obtain some numerical results. We are particularly interested in the location of the zero of $L(k,0)$ and the maximum value of P for which a zero of $L(k,P)$ exists. We shall later relate the position of the zero of $L(k,0)$ to the size of the energy gap. Let us consider first the $P=0$ case. From (6.21) we have

$$\begin{aligned}
 (7.14) \quad L(k,0) &= \\
 &= 1 + \frac{2}{\pi} \frac{m^*}{m} \left[(f_0+1)I_1(k,0) + I_2(k,0) + \lambda \left\{ (f_0+1)I_3(k,0) + I_4(k,0) \right\} \right] \\
 I_1(k,0) &= \frac{1}{4x} \left[\ln \left| \frac{x_F+x}{x_F-x} \right| - \left\{ \text{Ci}2(x_F+x) - \text{Ci}2(x_F-x) \right\} \cos 2x + \right. \\
 &\quad \left. + \left\{ \pi - \text{Si}2(x_F+x) - \text{Si}2(x_F-x) \right\} \sin 2x \right] \\
 I_2(k,0) &= -\frac{\pi}{4} - \frac{1}{4} \left[\left\{ \text{Ci}2(x_F+x) - \text{Ci}2(x_F-x) \right\} \sin 2x + \right. \\
 &\quad \left. + \left\{ \pi - \text{Si}2(x_F+x) - \text{Si}2(x_F-x) \right\} \cos 2x \right] \\
 I_3(k,0) &= \frac{1}{4x} \left[\ln \left| \frac{x_F+x}{x_F-x} \right| - \left\{ \text{Ci}2(x_F+x) - \text{Ci}2(x_F-x) \right\} \cos 2x - \right. \\
 &\quad \left. - \left\{ \text{Si}2(x_F+x) - \text{Si}2(x_F-x) \right\} \sin 2x \right] \\
 I_4(k,0) &= -\frac{1}{4} \left[\left\{ \text{Ci}2(x_F+x) - \text{Ci}2(x_F-x) \right\} \sin 2x - \right. \\
 &\quad \left. - \left\{ \text{Si}2(x_F+x) + \text{Si}2(x_F-x) \right\} \cos 2x \right]
 \end{aligned}$$

We want to find the value of k for which this expression vanishes. We expect it to be very close to k_F and thus we shall put $k = k_F$ in all but the singular terms. We shall see that this is indeed justified. Making use of the fact that²⁴⁾

$$(7.15) \quad \text{when } y \rightarrow 0 \quad \left\{ \begin{array}{l} \text{Si } y \rightarrow 0 \\ \text{Ci } y \rightarrow \ln \gamma y \end{array} \right.$$

where $\ln \gamma = .577216$ (Euler-Mascheroni constant), we have then for $k \approx k_F$:

$$I_1(k, 0) = - \frac{1}{4x_F} (1 - \cos 2x_F) \ln(1 - \frac{k}{k_F}) + \\ + \frac{1}{4x_F} \left[\ln 2 + (\ln 2 \gamma x_F - \text{Ci } 4x_F) \cos 2x_F - (\pi - \text{Si } 4x_F) \sin 2x_F \right]$$

$$I_2(k, 0) = \frac{1}{4} \sin 2x_F \ln(1 - \frac{k}{k_F}) + \\ + \frac{1}{4} \left[-\pi + \ln 2 + (\ln 2 \gamma x_F - \text{Ci } 4x_F) \sin 2x_F - (\pi - \text{Si } 4x_F) \cos 2x_F \right]$$

$$I_3(k, 0) = - \frac{1}{4x_F} (1 - \cos 2x_F) \ln(1 - \frac{k}{k_F}) + \\ + \frac{1}{4x_F} \left[\ln 2 + (\ln 2 \gamma x_F - \text{Ci } 4x_F) \cos 2x_F + \text{Si } 4x_F \sin 2x_F \right]$$

$$I_4(k, 0) = \frac{1}{4} \sin 2x_F \ln(1 - \frac{k}{k_F}) \\ + \frac{1}{4} \left[\ln 2 + (\ln 2 \gamma x_F - \text{Ci } 4x_F) \sin 2x_F + \text{Si } 4x_F \cos 2x_F \right]$$

so that (7.14) becomes

$$(7.16a) \quad L(k,0) = 1 - \frac{x_F m^*}{\pi m} w(x_F) (1+\lambda) \ln(1 - \frac{k}{k_F}) + F(x_F)$$

$$(7.16b) \quad w(x_F) = \frac{1}{2x_F} \left[(f_0 + 1) (1 - \cos 2x_F) - x_F \sin 2x_F \right]$$

$$(7.16c) \quad F(x_F) = - \frac{m^*}{2m} +$$

$$+ \frac{m^*}{2\pi m} (1+\lambda) \left[\frac{(f_0+1)}{x_F} \left\{ \ln 2 + (\ln 2 \gamma x_F - \text{Ci} 4x_F) \cos 2x_F + (\pi - \text{Si} 4x_F) \sin 2x_F \right\} \right. \\ \left. + (\ln 2 \gamma x_F - \text{Ci} 4x_F) \sin 2x_F - (\pi - \text{Si} 4x_F) \cos 2x_F \right] \\ - \frac{\lambda m^*}{2m} \left[\frac{(f_0+1)}{x_F} \sin 2x_F - \cos 2x_F \right]$$

We observe that

$$(7.17) \quad (k_F |V| k_F) = \frac{(4\pi\hbar)^2 r_0}{m} j_0(k_F r_0) \left[f_0 j_0(k_F r_0) - k_F r_0 j_0'(k_F r_0) \right] \\ \equiv \frac{(4\pi\hbar)^2 r_0}{m} w(x_F)$$

so that (7.16) can be written (compare (7.9))

$$(7.18) \quad L(k,0) = 1 - \left(\frac{1}{2\pi}\right)^3 \frac{m^*}{\hbar^2} \frac{k_F}{2} (1+\lambda) (k_F |V| k_F) \ln(1 - \frac{k}{k_F}) + F(x_F)$$

Now we have defined k_λ^* by $L(k_\lambda^*, 0) = 0$, so that from (7.16)

$$(7.19) \quad 1 - \frac{k_\lambda^*}{k_F} = \exp \left\{ \frac{\pi m [1 + F(x_F)]}{x_F m^* w(x_F) (1+\lambda)} \right\}$$

Using the values of the parameters f_0 and r_0 given by (6.27) we have for $k_F = 1.4 \text{ fermi}^{-1}$

$$(7.20a) \quad w(x_F) = - .1155$$

$$(7.20b) \quad F(x_F) = - .5000 \frac{m^*}{m} + .3661 (1+\lambda) \frac{m^*}{m} - .4621\lambda \frac{m^*}{m} \\ = - (.1339 + .0960 \lambda) \frac{m^*}{m}$$

Taking $m^* = m$ and neglecting the hole terms ($\lambda=0$) we find from (7.19) and (7.20)

$$(7.21) \quad 1 - \frac{k_0^*}{k_F} = 4.70 \times 10^{-7}$$

Taking $m^*=m$ and including the hole terms ($\lambda=1$) we find

$$(7.22) \quad 1 - \frac{k_1^*}{k_F} = 2.63 \times 10^{-3}$$

In both cases the zero of $L(k,0)$ is indeed close to k_F , justifying our taking $k=k_F$ in all but the singular terms above. We point out further that when $m < m^*$ the position of the zero is closer to k_F in each case.

Emery¹⁰⁾ has shown that the size of the energy gap Δ is related to the position of the pole of the K matrix with hole terms included (which we call k_1^*) by

$$(7.23) \quad \Delta = \frac{2\hbar^2}{m^*} (k_F^2 - k_1^{*2})$$

We can use (7.19) to obtain an expression for Δ . We remark that in the spirit of the approximation leading to (7.19) we

write $k_F^2 - k_1^{*2} = 2k_F^2(1 - \frac{k_1^*}{k_F})$ so that

$$(7.24) \quad \Delta = \frac{4\hbar^2 k_F^2}{m^*} \left(1 - \frac{k_1^*}{k_F} \right)$$

The expression for Δ is then from (7.19)

$$(7.25) \quad \Delta = \frac{4\hbar^2 k_F^2}{m^*} \exp \left\{ \frac{\pi m [1 + F(x_F)]}{2x_F m^* w(x_F)} \right\}$$

Using (7.20) and taking $m^*=m$, we find

$$(7.26) \quad \Delta = .8 \text{ Mev}$$

There have been no estimates of the energy gap of infinite nuclear matter so it is difficult to say whether our number is satisfactory. However we can use Figure 1. in the paper by Bohr, Mottelson, and Pines²⁶⁾ to obtain a very rough idea of this quantity. This Figure shows the energy of the first excited intrinsic states of even-even spheroidal nuclei vs. A , the atomic number. Bohr, Mottelson, and Pines point out that if the intrinsic structure could adequately be described by an independent particle model, the first intrinsic excitations of even-even nuclei would have on the average an energy of about $25 A^{-1}$ Mev. Their Figure 1 shows this function and it is seen that it lies well below the data. Let us assume then that the energy of the first

intrinsic excitations of even-even nuclei is given by $\Delta + 25 A^{-1}$ where Δ is the energy gap. We take Δ to be independent of A , so that it is then the energy gap of infinite nuclear matter. It is seen from Figure 1 of Bohr, Mottelson, and Pines that in order to fit the data, a value of Δ between .7 Mev and .9 Mev is needed. Our value of .8 Mev seems not unsatisfactory then. Of course to obtain an accurate value of Δ , the self-consistent value of m^* at the Fermi surface would have to be known very accurately (see equation (7.25)).

Let us finally see what is the maximum value of P for which a singularity of $(\vec{k}|\vec{k}|\vec{k},\vec{P})$ can exist, that is, what is the value of P_λ^* . In the expression (6.21) for $L(k,P)$ we put k equal to its maximum value (i.e. to $\sqrt{k_F^2 - P^2} \equiv \alpha'$ from (7.11)) and, anticipating that P_λ^* is small, put $P=0$ in all terms but those which go to infinity when this is done. We further use the asymptotic forms of the sine and cosine integrals as given by (7.15). We find finally for $P \approx 0$

$$(7.27) \quad L(\alpha', P) = 1 - \frac{x_F m^* w(x_F) (1+\lambda)}{\pi m} \ln \frac{P}{k_F} + F(x_F)$$

where $w(x_F)$ and $F(x_F)$ are given by (7.16b) and (7.16c). We note the similarity between (7.27) and (7.16a). The only difference on the RHS of the two equations is that where $\ln(1 - \frac{k}{k_F})$ appears in (7.16a), $\ln \frac{P}{k_F}$ appears in

(7.27). The value of P_{λ}^* , which is the maximum value of P for which a singularity of $(\vec{k}|K|\vec{k},\vec{P})$ can occur, is given by $L(\alpha^*, P_{\lambda}^*) = 0$ so that from (7.27),

$$(7.28) \quad \frac{P_{\lambda}^*}{k_F} = \exp \left\{ \frac{\pi m [1+F(x_F)]}{x_F m^* w(x_F) (1+\lambda)} \right\}$$

The RHS of (7.28) is identical to the RHS of (7.19). Thus using the values of f_0 and r_0 as given by (6.27), we have for $k_F = 1.4 \text{ fermi}^{-1}$ and $m^* = m$

$$(7.29) \quad \frac{P_0^*}{k_F} = 4.70 \times 10^{-7}$$

$$(7.30) \quad \frac{P_1^*}{k_F} = 2.63 \times 10^{-3}$$

The value of P_{λ}^* is indeed close to zero justifying our setting $P=0$ in all terms except those which are singular when this is done. We note, perhaps superfluously, that for $m^* < m$ the value of P_{λ}^* is smaller in each case.

We have seen that our pseudopotential does indeed lead to a singular K matrix and thus predicts an energy gap for infinite nuclear matter. We have seen further that the singularity occurs only for particles whose

relative momentum is very close to k_F and whose centre of mass momentum is almost zero. This is an important prediction of our work since no other calculation, based on a nucleon-nucleon interaction potential which fits the scattering data for energies up to those relevant in the many-body case, has yielded this result.

CHAPTER VIII

CONCLUSIONS

We shall conclude this thesis by giving in this Chapter a brief summary of some of the main points we have discussed.

In this thesis we have applied the Feshbach-Lomon Boundary Condition Model for nucleon-nucleon interactions to the Independent Pair Model (or Brueckner theory) of an infinite nucleus. The main step in the process has been the construction of a nucleon-nucleon interaction pseudopotential which is equivalent to the Boundary Condition Model. Because of the simple structure of this pseudopotential the integral equation for the nuclear matter K matrix can be solved without resorting to the perturbative approaches used in the usual treatments. We would like to stress that we have used a realistic model of the nuclear force (in that it gives a reasonable fit to the two-nucleon data in the energy range which is important for the many-body problem), and with it we have been able to handle the equations of the Brueckner theory quite simply.

One of the important points in our discussion has been the treatment of the centre of mass momentum dependence of the nuclear matter K matrix. Our treatment involves certain

expansions only the first term of which has been given before. It should be stressed that the expansions we have introduced can also be used in the more standard treatments of the many-body problem.

We have performed numerical calculations for the case when the interaction potential vanishes in all but 1S_0 states. Our results for the potential energy per particle are very encouraging and indicate the validity of our approach to the many-body problem. In all of our discussion we have included the hole-hole interactions and our numerical results for the K matrix show the large effect on this quantity by these interactions. Our results show further that, in agreement with the rough calculations of Moszkowski and Sessler⁸⁾, the inclusion of the hole-hole interactions has a small effect on the potential energy per particle.

As a final point we have investigated the singularity of the nuclear matter K matrix for the case when the interaction vanishes in all but 1S_0 states. We have been particularly concerned with the effect on the existence and position of the singularity when the interacting particles have a center of mass momentum different from zero. These points can be investigated easily because we can solve the K matrix exactly. Previous investigations of the nuclear matter problem, using an interaction potential which fits the two-body scattering data for energies up to those relevant in the many-body case, have not included this

point because in these cases the integral equation for the K matrix can only be solved approximately by perturbative methods. We have shown that the K matrix does have a singularity and therefore that infinite nuclear matter does have energy gap in its excitation spectrum. Further the singularity occurs only for those particles whose relative momentum is close to the Fermi momentum and whose center of mass momentum is almost zero.

At present, a fit to the two-body scattering data is being performed by Feshbach and Lomon using an improvement over the simple Boundary Condition Model, namely, using the Boundary Condition Model plus an exterior potential tail obtained from the meson theory of nuclear forces. This model will be applied to the nuclear matter problem with the potential tail being handled by perturbative methods. All of the techniques used in this thesis can be used in the application of this improved model for the nucleon-nucleon interaction to the nuclear many-body problem, but a detailed discussion of this application is beyond the scope of this thesis.

APPENDIX I

In this Appendix we shall show that

$$(I.1) \quad \wp \int_0^{\infty} \frac{dk' k'^2 j_{\ell}(k'r) j_{\ell}(k'r')}{k'^2 - k^2} = -\frac{\pi k}{2} \begin{cases} j_{\ell}(kr) n_{\ell}(kr') & \text{if } r' \geq r \\ j_{\ell}(kr') n_{\ell}(kr) & \text{if } r \geq r' \end{cases}$$

where \wp means that the principal value of the integral is to be taken.

We note first the following properties of the spherical Bessel and Neumann functions¹⁵⁾:

$$j_{\ell}(-z) = (-)^{\ell} j_{\ell}(z) \quad n_{\ell}(-z) = (-)^{\ell+1} n_{\ell}(z)$$

so that

$$j_{\ell}(-z) j_{\ell}(-z') = j_{\ell}(z) j_{\ell}(z')$$

$$j_{\ell}(-z) n_{\ell}(-z') = - j_{\ell}(z) n_{\ell}(z')$$

Thus we can write

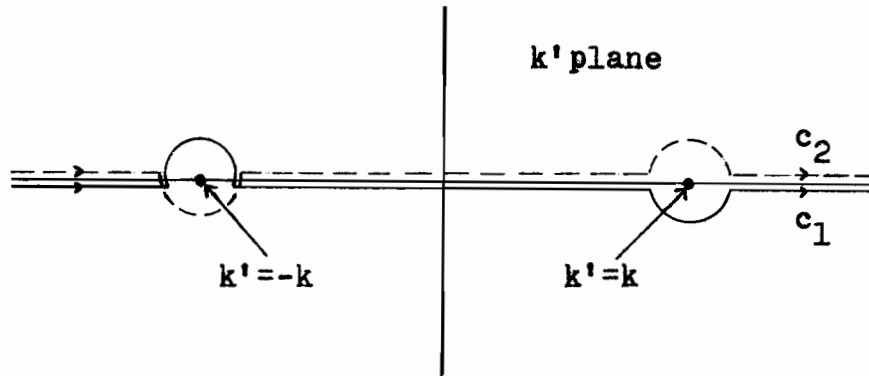
$$(I.2a) \quad \wp \int_0^{\infty} \frac{dk' k'^2 j_{\ell}(k'r) j_{\ell}(k'r')}{k'^2 - k^2} = \frac{1}{2} \wp \int_{-\infty}^{+\infty} \frac{dk' k'^2 h_{\ell}(k'r) j_{\ell}(k'r')}{k'^2 - k^2}$$

or, alternatively,

$$(I.2b) \quad \oint_0^{\infty} \frac{dk' k'^2 j_{\ell}(k'r) j_{\ell}(k'r')}{k'^2 - k^2} = \frac{1}{2} \oint_{-\infty}^{+\infty} \frac{dk' k'^2 j_{\ell}(k'r) h_{\ell}(k'r')}{k'^2 - k^2}$$

where $h_{\ell}(z) = j_{\ell}(z) + in_{\ell}(z)$.

We evaluate the integrals occurring on the RHS of (I.2) by contour integrations. Consider the following contours c_1 and c_2 :



where we suppose that both c_1 and c_2 are closed in the upper half plane by a semi-circle at infinity. We note now the following asymptotic forms of $j_{\ell}(z)$ and $h_{\ell}(z)$ ¹⁵⁾:

when $z \rightarrow \infty$, $j_{\ell}(z) \rightarrow \frac{1}{z} \cos \left[z - \frac{1}{2} \pi(\ell+1) \right]$ and $h_{\ell}(z) \rightarrow \frac{1}{z} e^{+i \left[z - \frac{\pi}{2}(\ell+1) \right]}$

so that when

$$z_1, z_2 \rightarrow \infty, h_\ell(iz_1)j_\ell(iz_2) \rightarrow \frac{1}{-2z_1z_2} (e^{-(z_1+z_2)-i\pi(\ell+1)} + e^{-(z_1-z_2)})$$

Therefore, if z_1 and z_2 are both real, then if $z_1 \geq z_2$, $h_\ell(iz_1)j_\ell(iz_2) \rightarrow 0$ at least as fast as $\frac{1}{z_1z_2}$.

Thus if $r \geq r'$ we have

$$(I.3) \quad \oint_{-\infty}^{+\infty} \frac{dk' k'^2 h_\ell(k'r) j_\ell(k'r')}{k'^2 - k^2} = \frac{1}{2} \left(\oint_{c_1} + \oint_{c_2} \right) \frac{dk' k'^2 h_\ell(k'r) j_\ell(k'r')}{k'^2 - k^2}$$

That is, because of the above arguments there is no contribution to the contour integrals from the semi-circles at infinity. The integrals occurring on the RHS of (I.3) can easily be evaluated using Cauchy's integral formula.³⁰⁾ We find

$$(I.4a) \quad \oint_{c_1} \frac{dk' k'^2 h_\ell(k'r) j_\ell(k'r')}{k'^2 - k^2} = -\pi k n_\ell(kr) j_\ell(kr') + i\pi k j_\ell(kr) j_\ell(kr')$$

$$(I.4b) \quad \oint_{c_2} \frac{dk' k'^2 h_\ell(k'r) j_\ell(k'r')}{k'^2 - k^2} = -\pi k n_\ell(kr) j_\ell(kr') - i\pi k j_\ell(kr) j_\ell(kr')$$

so that from (I.3)

$$(I.5) \quad \oint_{-\infty}^{+\infty} \frac{dk' k'^2 h_\ell(k'r) j_\ell(k'r')}{k'^2 - k^2} = -\pi k n_\ell(kr) j_\ell(kr') \text{ when } r \geq r'$$

Thus from (I.2) and (I.5) it follows that

$$\rho \int_0^{\infty} \frac{dk' k'^2 j_{\ell}(k'r) j_{\ell}(k'r')}{k'^2 - k^2} = - \frac{\pi k}{2} \begin{cases} j_{\ell}(kr) n_{\ell}(kr') & \text{if } r' \geq r \\ j_{\ell}(kr') n_{\ell}(kr) & \text{if } r \geq r' \end{cases}$$

which is equation (I.1).

It is of interest finally to differentiate both sides of (I.1) with respect to r . We find

$$\rho \int_0^{\infty} \frac{dk' k'^3 j'_{\ell}(k'r) j_{\ell}(k'r')}{k'^2 - k^2} = - \frac{\pi k^2}{2} \begin{cases} j'_{\ell}(kr) n_{\ell}(kr') & \text{if } r' > r \\ j_{\ell}(kr') n'_{\ell}(kr) & \text{if } r > r' \end{cases}$$

and

$$\rho \int_0^{\infty} \frac{dk' k'^3 j'_{\ell}(k'r) j_{\ell}(k'r)}{k'^2 - k^2} = - \frac{\pi k^2}{4} \left\{ j'_{\ell}(kr) n_{\ell}(kr) + j_{\ell}(kr) n'_{\ell}(kr) \right\}$$

Thus

$$\lim_{r' \rightarrow r} \rho \int_0^{\infty} \frac{dk' k'^3 j'_{\ell}(k'r) j_{\ell}(k'r')}{k'^2 - k^2} \neq \rho \int_0^{\infty} \frac{dk' k'^3 j'_{\ell}(k'r) j_{\ell}(k'r)}{k'^2 - k^2}$$

that is, the limit operation and the integration cannot be interchanged. We have, in fact,

$$\begin{aligned} \int_0^\infty \frac{dk' k'^3 j_\ell'(k'r) j_\ell(k'r)}{k'^2 - k^2} &= \\ &= \frac{1}{2} \left(\lim_{\substack{r' \rightarrow r \\ (r' > r)}} + \lim_{\substack{r' \rightarrow r \\ (r' < r)}} \right) \int_0^\infty \frac{dk' k'^3 j_\ell'(k'r) j_\ell(k'r')}{k'^2 - k^2} \end{aligned}$$

APPENDIX II

In this Appendix we shall show that the wave function (see (4.9))

$$(II.1) \quad |\Psi\rangle = G |\psi_{\alpha\beta}\rangle_{12} |\gamma\rangle_3 |\delta\rangle_4 \dots |\omega\rangle_A \quad k_\alpha, k_\beta, k_\gamma, \dots, k_\omega < k_F$$

is an eigenfunction of the IPAM Hamiltonian (4.6) when the Bethe-Goldstone equation (4.11) is satisfied. This has been shown by Gomes, Walecka, and Weisskopf³⁾ and in more detail by Walecka³¹⁾, but we include this point here for the sake of completeness.

We consider the eigenvalue equation

$$(II.2) \quad \left(\sum_{i=1}^A H_i - E \right) |\Psi\rangle = - U_{\alpha\beta} |\Psi\rangle$$

where H_i and $U_{\alpha\beta}$ are given by (4.6). We obtain an equation for $|\psi_{\alpha\beta}\rangle$ by multiplying both sides of (II.2) on the left by the bra

$$\frac{1}{\sqrt{2}} \quad {}_3\langle\gamma| \quad {}_4\langle\delta| \dots {}_A\langle\omega|$$

(The factor $\frac{1}{\sqrt{2}}$ has been included for convenience as will be

seen below.) That is, from (II.2) we have

$$(II.3) \quad \frac{1}{\sqrt{2}} {}_3(\gamma|_4(\delta|\dots_A(\omega|(\sum_{i=1}^A H_i - E)|\Psi) =$$

$$= - \frac{1}{\sqrt{2}} {}_3(\gamma|_4(\delta|\dots_A(\omega|u_{\alpha\beta}|\Psi)$$

Now by using the orthonormality of the single particle wave functions $|\gamma\rangle$ and the equation $H_i|\gamma\rangle_i = \epsilon_\gamma|\gamma\rangle_i$, we have

$$\text{LHS of (II.3)} = [H_1 + H_2 - (E - \sum_{\gamma \neq \alpha, \beta} \epsilon_\gamma)] \frac{1}{\sqrt{2}} [|\psi_{\alpha\beta}\rangle_{12} - |\psi_{\beta\alpha}\rangle_{12}]$$

If we write

$$\epsilon_{\alpha\beta} = E - \sum_{\gamma \neq \alpha, \beta} \epsilon_\gamma \quad \text{and} \quad |\psi_{\alpha\beta}^A\rangle_{12} = \frac{1}{\sqrt{2}} [|\psi_{\alpha\beta}\rangle_{12} - |\psi_{\beta\alpha}\rangle_{12}]$$

then

$$(II.4) \quad \text{LHS of (II.3)} = (H_1 + H_2 - \epsilon_{\alpha\beta}) |\psi_{\alpha\beta}^A\rangle_{12}$$

Let us now consider the RHS of (II.3). By using the definition (4.6) of $u_{\alpha\beta}$ and the properties of the projection operators occurring in $u_{\alpha\beta}$ (see equation (4.7)), we can write

$$\text{RHS of (II.3)} = -V_{12} |\psi_{\alpha\beta}^A\rangle_{12} + \sum_{\gamma} |\gamma\rangle_1 {}_3(\gamma|V_{32}|\psi_{\alpha\beta}^A\rangle_{32} +$$

$$+ \sum_{\delta} |\delta\rangle_2 {}_3(\delta|V_{13}|\psi_{\alpha\beta}^A\rangle_{13} - \sum_{\gamma} \sum_{\delta} |\gamma\rangle_1 |\delta\rangle_2 (\gamma\delta|V|\psi_{\alpha\beta}^A)$$

where \sum_{γ}' means summation over all states $|\gamma\rangle$ ($\gamma \neq \alpha$ or β)

where $k_{\gamma} < k_F$, and where we now write $|\gamma\rangle_1 |\delta\rangle_2 = |\gamma_1 \delta_2\rangle$.

In order to simplify this expression we use the following identities obtained from the completeness of the single particle states $|\gamma\rangle$:

$$\begin{aligned} V_{12} |\psi_{\alpha\beta}^A\rangle_{12} &= \sum_{\gamma} \sum_{\delta} |\gamma_1 \delta_2\rangle (\gamma\delta | V | \psi_{\alpha\beta}^A) \\ |\gamma\rangle_1 {}_3 \langle \gamma | V_{32} | \psi_{\alpha\beta}^A \rangle_{32} &= \sum_{\delta} |\gamma_1 \delta_2\rangle (\gamma\delta | V | \psi_{\alpha\beta}^A) \\ |\delta\rangle_2 {}_3 \langle \delta | V_{13} | \psi_{\alpha\beta}^A \rangle_{13} &= \sum_{\gamma} |\gamma_1 \delta_2\rangle (\gamma\delta | V | \psi_{\alpha\beta}^A) \end{aligned}$$

where \sum_{γ} means summation over all states $|\gamma\rangle$. Thus we have

$$\text{RHS of (II.3)} = \left(- \sum_{\gamma} \sum_{\delta} + \sum_{\gamma} \sum_{\delta}' + \sum_{\gamma} \sum_{\delta}' - \sum_{\gamma} \sum_{\delta}' \right) |\gamma_1 \delta_2\rangle (\gamma\delta | V | \psi_{\alpha\beta}^A)$$

$$\text{or RHS of (II.3)} = - \sum_{\gamma}'' \sum_{\delta}'' |\gamma_1 \delta_2\rangle (\gamma\delta | V | \psi_{\alpha\beta}^A)$$

where \sum_{γ}'' means summation over all states $|\gamma\rangle$ where $k_{\gamma} > k_F$

and also over both α and β . We can write now

$$\begin{aligned} \text{RHS of (II.3)} = & - \sum_{\gamma > F} \sum_{\delta > F} |\gamma_1 \delta_2\rangle (\gamma \delta | V | \psi_{\alpha\beta}^A) \\ & - |\alpha_1 \beta_2\rangle (\alpha \beta | V | \psi_{\alpha\beta}^A) - |\beta_1 \alpha_2\rangle (\beta \alpha | V | \psi_{\alpha\beta}^A) \\ & + \left\{ \text{terms involving matrix elements of the forms} \right. \\ & \quad (\alpha \delta | V | \psi_{\alpha\beta}^A) \text{ and } (\gamma \beta | V | \psi_{\alpha\beta}^A) \text{ } (\gamma, \delta \neq \alpha \text{ or } \beta) \text{ which} \\ & \quad \text{vanish for an infinite nucleus since they do} \\ & \quad \left. \text{not conserve centre of mass momentum} \right\} \end{aligned}$$

But since $V | \psi_{\alpha\beta}^A \rangle$ is antisymmetric, only the antisymmetric parts of the vectors $(\alpha \beta |$ and $(\beta \alpha |$ contribute to the above matrix elements, so that for an infinite nucleus,

$$\begin{aligned} \text{(II.5) RHS of (II.3)} = & - \sum_{\gamma > F} \sum_{\delta > F} |\gamma_1 \delta_2\rangle (\gamma \delta | V | \psi_{\alpha\beta}^A) \\ & - |\alpha_1 \beta_2^A\rangle (\alpha \beta^A | V | \psi_{\alpha\beta}^A) \end{aligned}$$

where

$$|\alpha_1 \beta_2^A\rangle = \frac{1}{\sqrt{2}} \left[|\alpha_1 \beta_2\rangle - |\beta_1 \alpha_2\rangle \right] .$$

Thus, if we are considering an infinite nucleus, we have from (II.4) and (II.5)

$$(H_1 + H_2 - \epsilon_{\alpha\beta}) | \psi_{\alpha\beta}^A \rangle_{12} = - \sum_{\gamma > F} \sum_{\delta > F} |\gamma_1 \delta_2\rangle (\gamma \delta | V | \psi_{\alpha\beta}^A) - |\alpha_1 \beta_2^A\rangle (\alpha \beta^A | V | \psi_{\alpha\beta}^A)$$

This is the Bethe-Goldstone equation (see (4.11)).

APPENDIX III

In this Appendix we shall consider

$$\lim_{k_F \rightarrow 0} L(k, P)$$

where (6.21a)

$$L(k, P) = 1 + \frac{2}{\pi} \frac{m^*}{m} \left[(f_0 + 1) I_1(k, P) + I_2(k, P) + \lambda \{ (f_0 + 1) I_3(k, P) + I_4(k, P) \} \right]$$

and where the I 's are given by (6.21b) to (6.21e). To investigate this limit, we expand various functions in Taylor series about $k_F = 0$. We note first the following expansions of the cosine and sine integrals (equation (6.18))

$$Ci2(a \pm x) = Ci2x \pm \frac{a}{x} \cos 2x + O(a^2)$$

$$Si2(a \pm x) = \pm Si2x + \frac{a}{x} \sin 2x + O(a^2)$$

where we have used $Si(-x) = -Si x$ and $Ci(-x) = Cix$. Thus using the definition of ϕ_a^\pm and θ_a^\pm (equation (6.19)) we have

$$\begin{aligned} \phi_a^+ &= 2Ci2x + O(a^2) & \phi_a^- &= O(a) \\ \theta_a^+ &= O(a) & \theta_a^- &= 2Si2x + O(a^2) \end{aligned}$$

From the definition of α, β , and γ (equation (6.15)) we have then

$$\phi_{\beta}^{+} - \phi_{\alpha}^{+} = O(k_F^2) \qquad \phi_{\alpha}^{+} - \phi_{\gamma}^{+} = O(k_F^2)$$

$$\theta_{\beta}^{-} - \theta_{\alpha}^{-} = O(k_F^2) \qquad \theta_{\alpha}^{-} - \theta_{\gamma}^{-} = O(k_F^2)$$

Further,

$$\ln \left| \frac{\beta^2 - x^2}{\alpha^2 - x^2} \right| = O(k_F^2) \qquad \ln \left| \frac{\gamma^2 + x^2}{\gamma^2 - x^2} \right| = O(k_F^2)$$

$$\ln \left| \frac{\beta + x}{\beta - x} \right| = O(k_F) \qquad \ln \left| \frac{\gamma + x}{\gamma - x} \right| = O(k_F)$$

$$\ln \frac{\beta}{\alpha} + C_{12}\alpha - C_{12}\beta = O(k_F^2) \qquad \ln \frac{\alpha}{\gamma} + C_{12}\gamma - C_{12}\alpha = O(k_F^2)$$

where we have used also (7.15).

Thus using the above results, equation (6.20), and equations (6.21b) to (6.21e) we have, when $k_F \rightarrow 0$,

$$I_1(k, P) \rightarrow \frac{\pi}{4x} \sin 2x$$

$$I_2(k, P) \rightarrow -\frac{\pi}{4} - \frac{\pi}{2} \cos 2x$$

$$I_3(k, P) \rightarrow 0$$

$$I_4(k, P) \rightarrow 0$$

Taking also $m^*=m$, the expression (6.21a) for $L(k,P)$ thus becomes, when $k_F \rightarrow 0$,

$$\begin{aligned} L(k,P) &\rightarrow 1 + \frac{2}{\pi} \left[(f_0+1) \frac{\pi}{4x} \sin 2x - \frac{\pi}{4} (1 + \cos 2x) \right] \\ &= \frac{\sin x}{x} \left[(f_0+1) \cos x + x \sin x \right] \\ &= \frac{\sin k r_0}{k r_0} \left[(f_0+1) \cos k r_0 + k r_0 \sin k r_0 \right] \end{aligned}$$

using $x = k r_0$ (6.15).

Finally from equation (6.23), which is the expression for the K matrix when the interaction potential vanishes in all but 1S_0 states, we have, for $k_F \rightarrow 0$,

(III.1)

$$\begin{aligned} \langle \vec{k} | K | \vec{k}, \vec{p} \rangle &\rightarrow \frac{4\pi\hbar^2}{mk} \left\{ \frac{(f_0+1) \sin k r_0 - k r_0 \cos k r_0}{(f_0+1) \cos k r_0 + k r_0 \sin k r_0} \right\} \\ &= - \frac{4\pi\hbar^2}{mk} \tan \delta_0 \quad \text{using (6.26)} \end{aligned}$$

Thus in the limit as $k_F \rightarrow 0$, the diagonal element of the K matrix for 1S_0 states is proportional to the tangent of the 1S_0 state phase shift, as indeed it should be by (3.24).

FIGURE CAPTIONS

- Fig. I. The function $A_{2\ell}(k,P)$ (see equation (5.11)) as a function of k for $P = .5 k_F$ and $\ell=0,1,2,3$.
- Fig. II. Comparison of the energy-dependence of the YLAM 1S_0 phase shift of Breit et al.²⁵⁾ with that of the 1S_0 phase shift given by equation (6.26) using our boundary condition parameters (6.27) and those of Feshbach and Lomon (6.25).
- Fig. III. The nuclear matter $(\vec{k}|K|\vec{k},\vec{P})$ as a function of k for various values of P for the case when the interaction potential vanishes in all but 1S_0 states and when the hole-hole interactions are neglected (see equation (6.23)). Also included are the first Born approximation (i.e. $(\vec{k}|V|\vec{k})$. See equation (6.22)) and the two-body $(\vec{k}|K|\vec{k})$ (see equation (III.1)).
- Fig. IV. The nuclear matter $(\vec{k}|K|\vec{k},\vec{P})$ as a function of k for various values of P for the case when the interaction potential vanishes in all but 1S_0 states and when the hole-hole interactions

are included (see equation (6.23)). Also included are the first Born approximation (i.e., $(\vec{k}|V|\vec{k})$. See equation (6.22)) and the two-body $(\vec{k}|K|\vec{k})$ (see equation (III.1)).

Fig. V. The single particle potential $U(k_2)$ as a function of k_2 for the case when the interaction potential vanishes in all but 1S_0 states (see equation (6.31)). Also included are parabolas fitted at $k_2 = 0$ and $k_2 = k_F$.

Fig. VI. The single particle potential $U(k_2)$ as a function of k_2 for the case when the interaction potential vanishes in all but 1S_0 states (see equation (6.31)) and when the first Born approximation for the K matrix elements (i.e., $(\vec{k}|V|\vec{k})$) is used. Also included is a parabola fitted at $k_2 = 0$ and $k_2 = k_F$.

Figure I

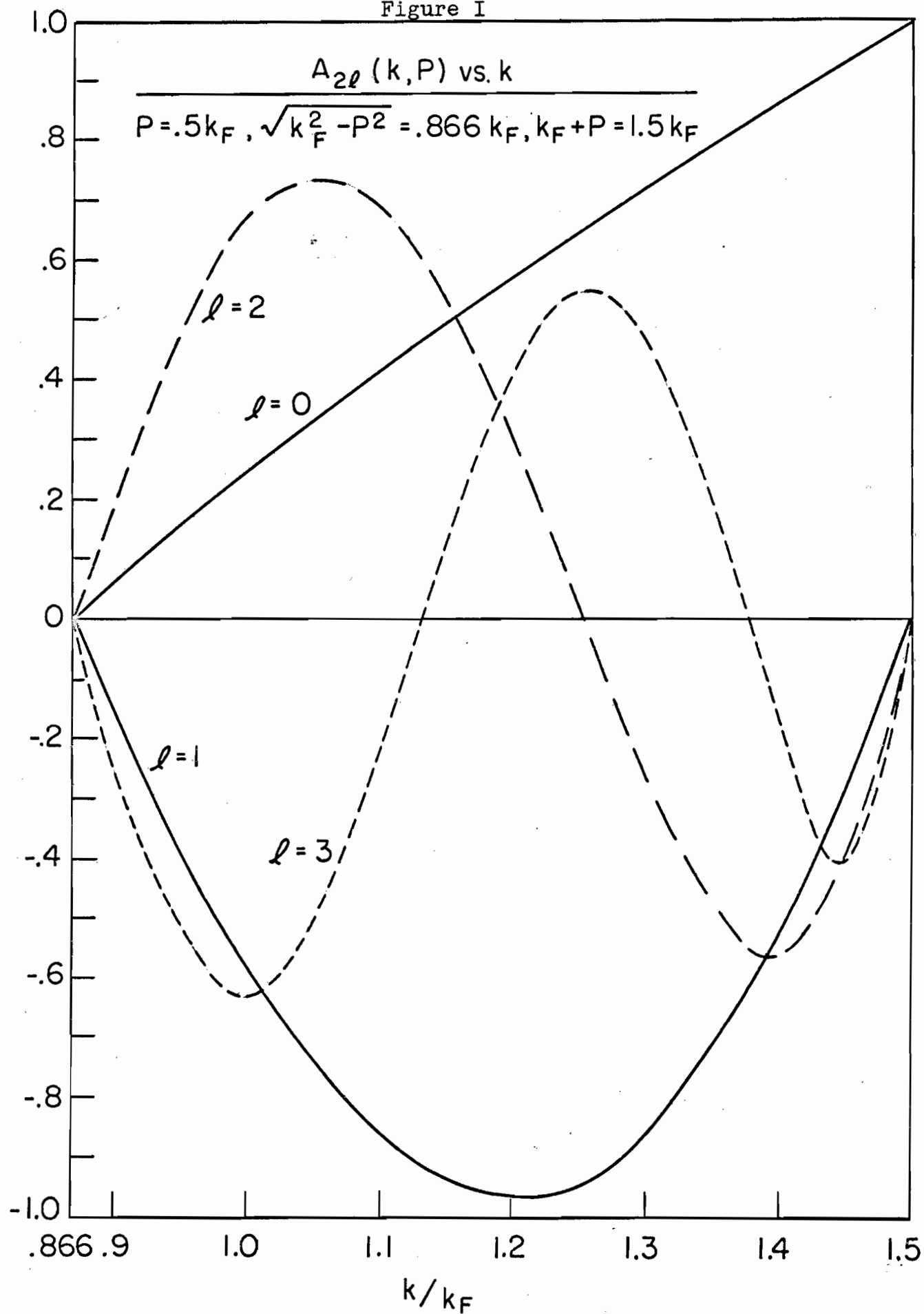


Figure II

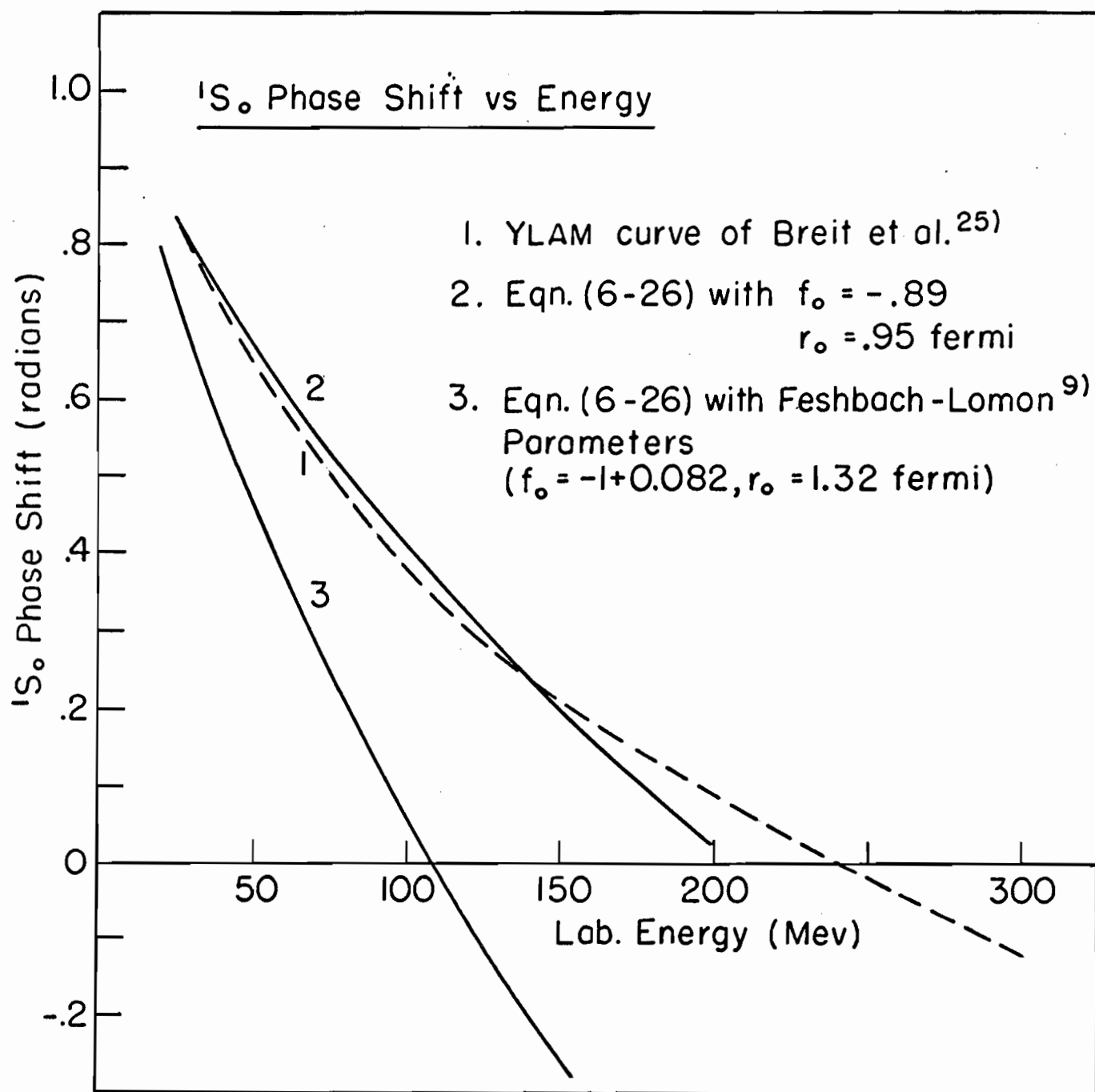


Figure III

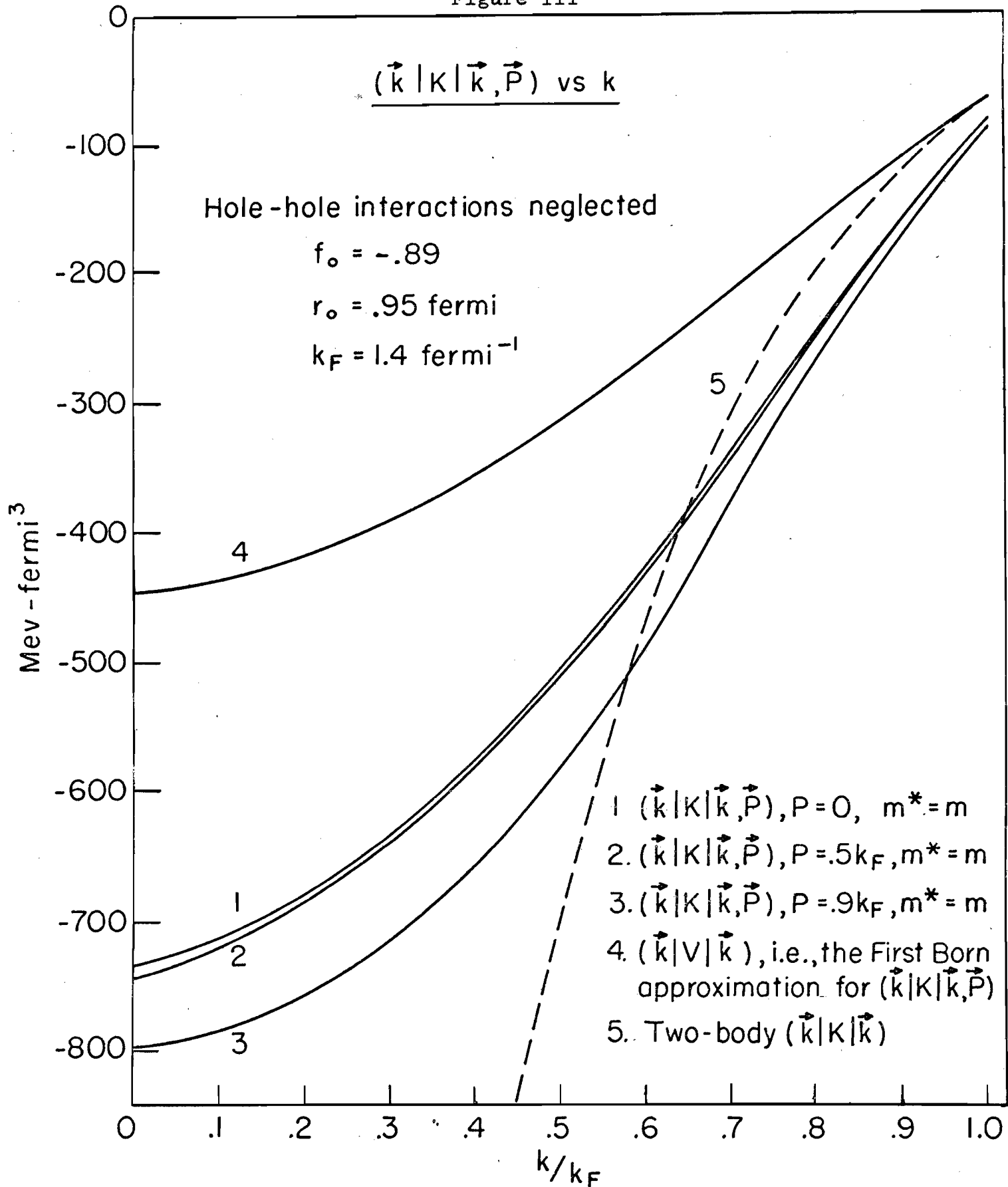
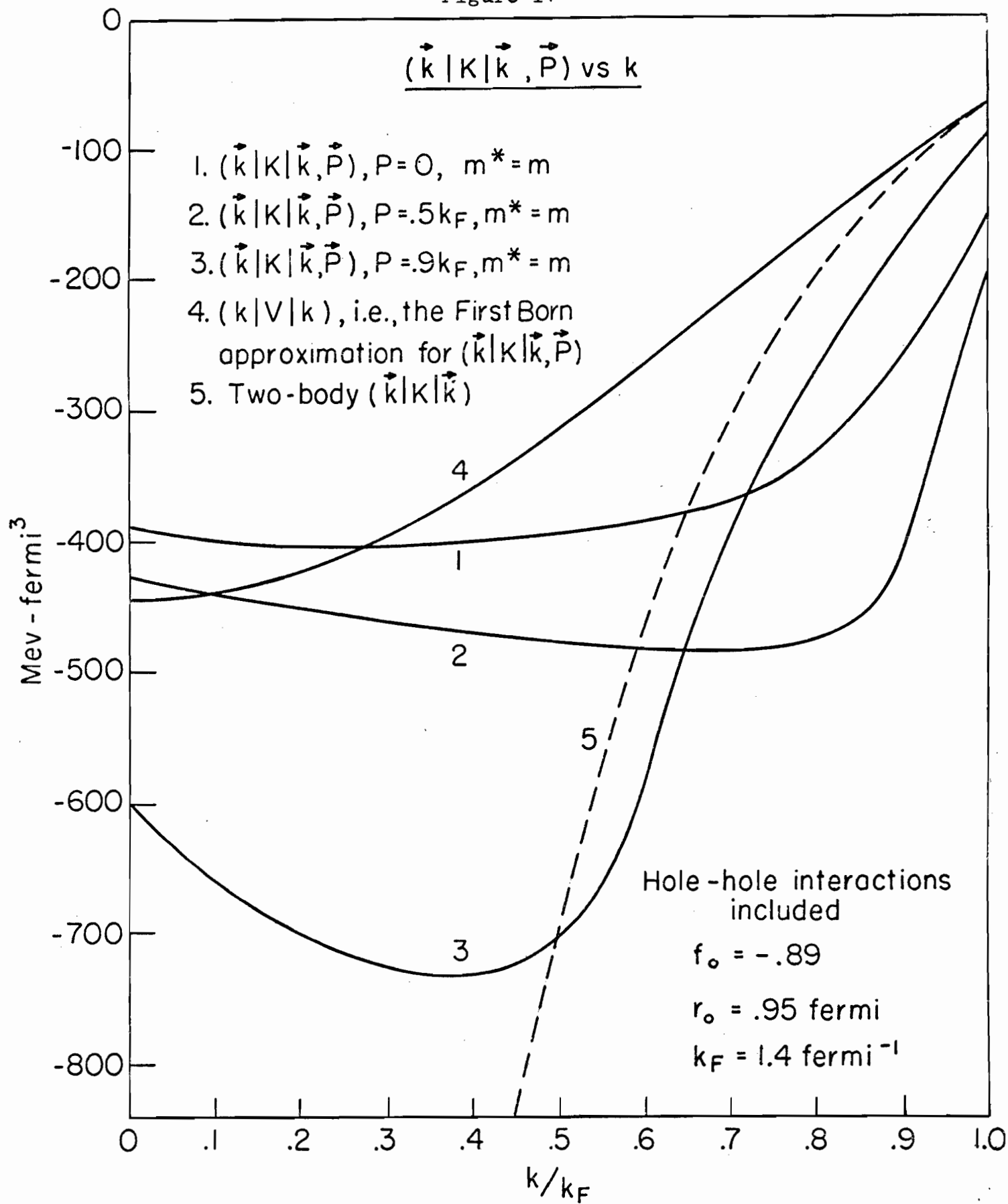


Figure IV



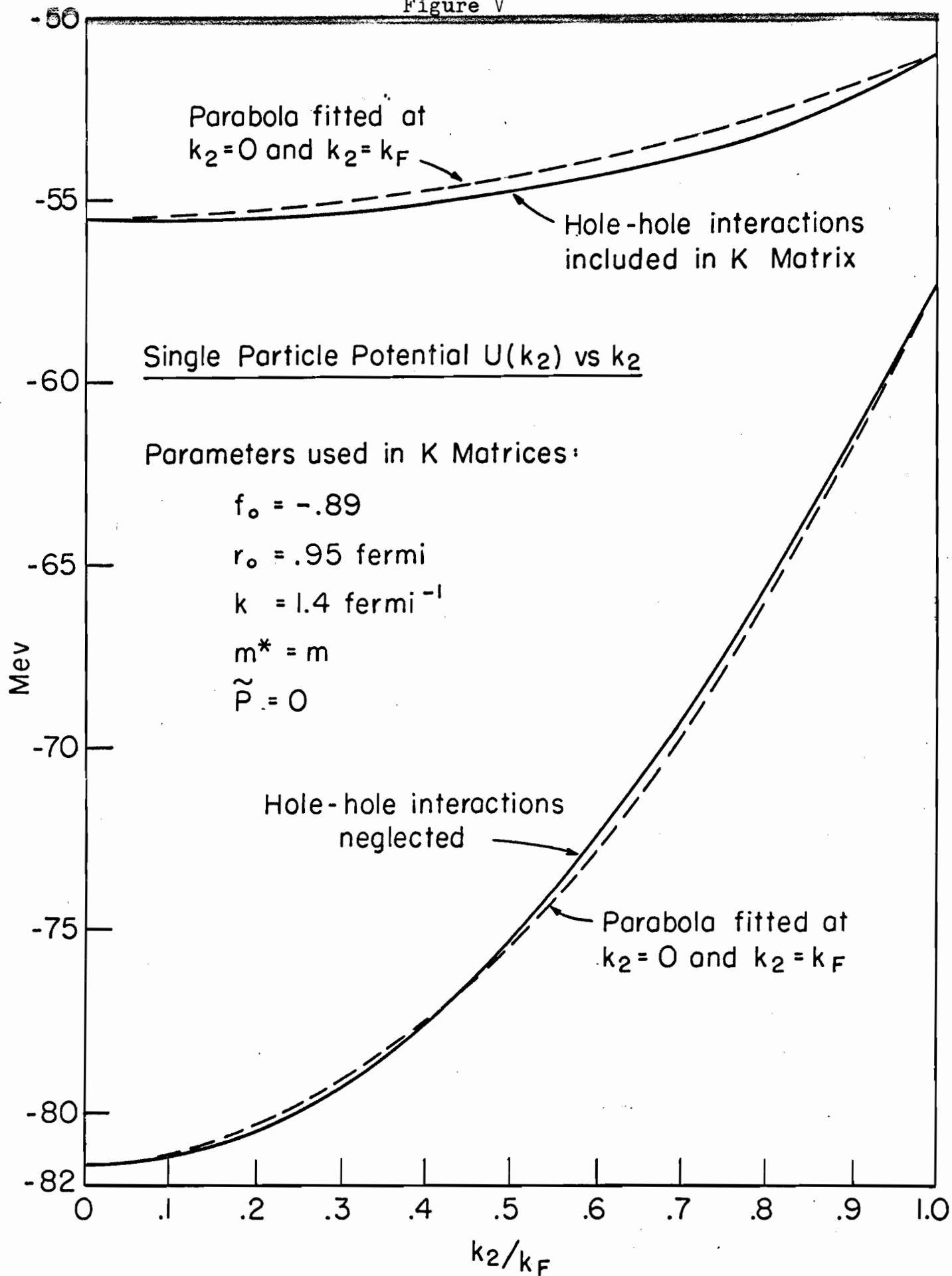
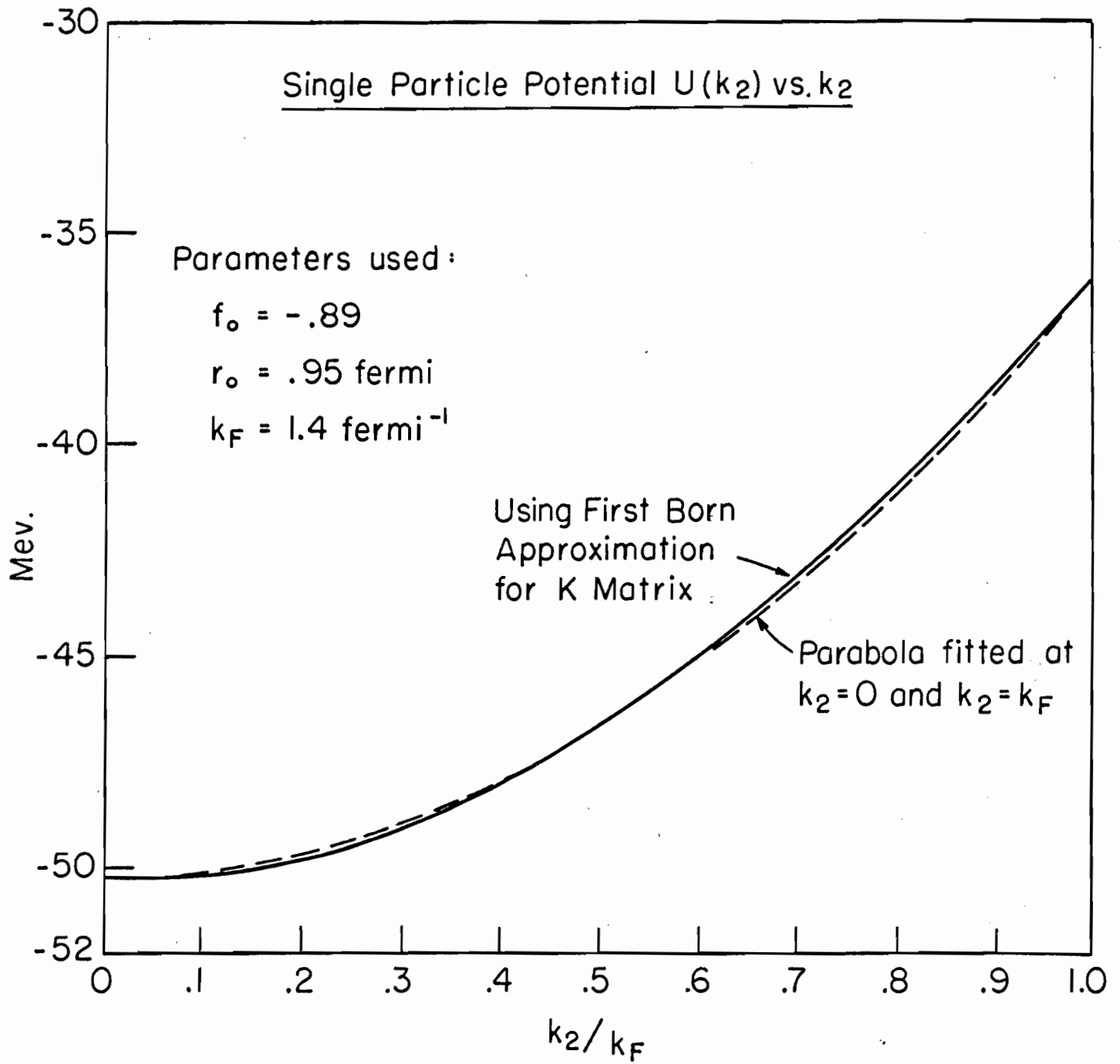


Figure VI



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