Prediction of the Effect of Streamline Curvature on Turbulence

by

H.P.A.H. Irwin

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McGill University

Montreal, Quebec, Canada

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Summary

It is shown that the observed effects of streamline curvature on turbulence are accounted for by the small curvature terms which appear in the Reynolds stress equations and in the turbulence model of Launder, Reece and Rodi (1973). Because of the smallness of these terms this is a surprising result but their effect is shown to appear in a magnified form in the effective eddy viscosity. A comparison is made between computed solutions of the modelled Reynolds stress, mean momentum and dissipation rate equations and experimental data for curved wall jets and a curved free jet. The curvature effects are accurately predicted when the small curvature terms are included. The turbulence model implies that the effective eddy viscosity is relatively insensitive to longitudinal acceleration which seems to be in reasonable agreement with experiment.

1. Introduction

There is now abundant evidence that turbulence is very sensitive to small amounts of curvature of mean streamlines and this has been emphasized recently by the review of Bradshaw (1973). The effect of curvature tends to increase the magnitude of the turbulence shear stress when $\frac{U}{R+y} \Big/ \frac{\partial U}{\partial y}$ is negative, where R is the radius of curvature of the mean streamlines (see fig. 1), and to decrease it when this quantity is positive. This in turn affects the development of the mean flow and, as a typical example, Guitton (1970) found that the fractional change in the rate of growth $\frac{dy_0}{dx}$ of a wall jet on a convex (R positive) logarithmic spiral surface, for which $\frac{x}{R}$ is constant, is roughly eleven times the ratio y_0/R where y_0 is defined in fig. 1.

In curved flows the Reynolds stress equations contain extra production terms compared to the equations for uncurved flow so it is natural to examine them as a possible explanation of the observed curvature effects. When this is done they are found to be small, e.g. a 1% change due to curvature occurs in turbulence production for an approximately 10% change in turbulence shear stress, and for this reason it may be thought possible to dismiss them as indeed Bradshaw (1973) does. However, it is shown in this paper that despite their small size they, in combination with terms of similar order which arise in the turbulence model of

Launder, Reece and Rodi (1973), do in fact account for the observed curvature effects. The changes in turbulence structure implied by Launder et al's model are investigated in the hypothetical case of equilibrium flow, in which diffusion and advection are negligible, and an expression for the effective eddy viscosity is obtained which is very sensitive to curvature. In addition, the model, with the additional terms which arise in curved flow, has been incorporated into the finite difference scheme of Spalding and Patankar (1967, 1969) and the development of curved wall jets and a curved free jet computed. The pressure gradient normal to the mean streamlines is neglected in the computations thus restricting the validity of the results to small curvature. Within the range of validity the computed results are in good agreement with experiment.

2. The Reynolds Stress Equations and Turbulence Model for Flow with Small Curvature

Consideration is restricted to two-dimensional incompressible flows in which the curvature of the mean streamlines is small i.e. R = 0(x). Using the assumption of local isotropy the Reynolds stress equations can then be written, to order $\frac{U^3}{x} \cdot \left(\frac{y_0}{x}\right)$, as

$$\frac{D \frac{1}{2}\overline{u^2}}{Dt} = -\overline{u^2}\frac{\partial U}{\partial x} - \overline{uv}\frac{\partial U}{\partial y} - \left\{\overline{uv}\frac{U}{R}\right\} + \frac{1}{(1+y/R)}\overline{\frac{p}{\rho}\frac{\partial u}{\partial x}} - \frac{\partial \frac{1}{2}\overline{vu^2}}{\partial y} - \frac{\varepsilon}{3} \qquad (1)$$

$$\frac{D\frac{1}{2}\overline{v^2}}{Dt} = \left\{2\overline{uv}\frac{U}{R}\right\} + \overline{v^2}\frac{\partial U}{\partial x} + \overline{\frac{p}{\rho}\frac{\partial v}{\partial y}} - \frac{\partial \overline{\frac{pv}{\rho}}}{\partial y} - \frac{\partial \frac{1}{2}\overline{v^3}}{\partial y} - \frac{\varepsilon}{3} \qquad (2)$$

$$\frac{D\frac{1}{2}\overline{w^2}}{Dt} = \overline{\frac{p}{\rho}\frac{\partial w}{\partial z}} - \frac{\partial \frac{1}{2}\overline{vw^2}}{\partial y} - \frac{\varepsilon}{3} \qquad (3)$$

$$\frac{D\overline{uv}}{Dt} = -\overline{v^2}\frac{\partial U}{\partial y} + \left\{\left(2\overline{u^2} - \overline{v^2}\right)\frac{U}{R}\right\} + \overline{\frac{p}{\rho}\left(\frac{1}{(1+y/R)}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)} - \frac{\partial \overline{\frac{pu}{\rho}}}{\partial y} - \frac{\partial \overline{uv^2}}{\partial y} \qquad (4)$$

where ε is the dissipation rate, p is the fluctuating part of the pressure, ρ is the density and $\frac{D}{Dt} \equiv U\frac{\partial}{\partial x} + V\frac{\partial}{\partial y}$. In (1) to (4) the assumption has been used that the Reynolds stresses $\overline{u_i u_j}$ are of order $\left(\frac{y_0}{x}\right) U^2$ and that the triple correlations $\overline{u_i u_j u_k}$ are of order $\left(\frac{y_0}{x}\right)^2 U^3$ which is approximately what is found experimentally. Arnot Smith (1973) gives the full equations for curved flow. The wall jet data of Irwin (1973) show that the correlations between pressure and instantaneous velocity gradients are of the same order as the main production term $\overline{uv}\frac{\partial U}{\partial y}$ which is of order $\frac{U^3}{x}$; hence the (1 + y/R) denominator in (1) and (4) is retained. Adding (1) to (3) gives the turbulence energy equation

$$\frac{D\frac{1}{2}\overline{q^2}}{Dt} = -\overline{uv}\frac{\partial U}{\partial y} - \left(\overline{u^2} - \overline{v^2}\right)\frac{\partial U}{\partial x} + \left\{\overline{uv}\frac{U}{R}\right\} - \frac{\partial \overline{\frac{pv}{\rho}}}{\partial y} - \frac{\partial \frac{1}{2}\overline{vq^2}}{\partial y} - \varepsilon \qquad (5)$$

where $q^2 = u^2 + v^2 + w^2$. In all the above equations and those that follow the extra terms arising in curved flow are enclosed in braces. It may be noted that the terms involving $\frac{\partial U}{\partial x}$ are of the same order as those involving $\frac{U}{R}$ so that inclusion of curvature effects implies that those due to acceleration should also be included. However, assuming that there is no significant coupling between the two types of effect, which would seem valid for small curvature, the production terms involving $\frac{\partial U}{\partial x}$ will be neglected until the discussion at the end of the paper.

Launder et al (1973) provide a means of modelling the various correlations of higher order than the Reynolds stresses in (1) to (4). It is essentially the same as that proposed by Hanjalić and Launder (1972) except for the modelling of the pressure-velocity gradient correlations which are now expressed as a <u>linear</u> combination of the Reynolds stresses. Assuming the model is valid in curved flow it gives

$$\frac{1}{1+y/R}\overline{\frac{p}{\rho}\frac{\partial u}{\partial x}} = -C_1\frac{\varepsilon}{k}\left(\frac{\overline{u^2}}{2}-\frac{k}{3}\right) + (\alpha+2\beta)\overline{uv}\frac{\partial U}{\partial y} - \left\{(c_2+2\beta)\overline{uv}\frac{U}{R}\right\} \quad (6)$$

$$\overline{\frac{p}{\rho}\frac{\partial v}{\partial y}} = -C_1\frac{\varepsilon}{k}\left(\frac{\overline{v^2}}{2}-\frac{k}{3}\right) + (c_2+2\beta)\overline{uv}\frac{\partial U}{\partial y} - \left\{(\alpha+2\beta)\overline{uv}\frac{U}{R}\right\} \quad (7)$$

$$\overline{\frac{p}{\rho}\frac{\partial w}{\partial z}} = -C_1\frac{\varepsilon}{k}\left(\frac{\overline{w^2}}{2}-\frac{k}{3}\right) + \beta\overline{uv}\frac{\partial U}{\partial y} - \left\{\beta\overline{uv}\frac{U}{R}\right\} \quad (8)$$

$$\overline{\frac{p}{\rho}\left(\frac{1}{1+y/R}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)} = -C_1\frac{\varepsilon}{k}\overline{uv} + \left[(c_2+\beta)\overline{u^2} + (\alpha+\beta)\overline{v^2} + (\nu+\eta)k\right]\frac{\partial U}{\partial y}$$

$$- \left\{\left[(\alpha+\beta)\overline{u^2} + (c_2+\beta)\overline{v^2} + (\nu+\eta)k\right]\frac{U}{R}\right\} \quad (9)$$

where $k = \frac{1}{2}\overline{q^2}$, $\alpha = (4C_2 + 10)/11$, $\beta = -(2 + 3C_2)/11$, $\eta = -(50C_2 + 4)/55$ and $\nu = (20C_2 + 6)/55$. Launder et al suggest the values 1.5 and 0.4 for the constants c_1 and c_2 respectively. When a wall is nearby they proposed correction terms for (6) to (9) but the form of these described in their paper has since been modified (private communication). In the present work the wall correction takes the form of an additional term in equation (9) equal to $C_w\left(\frac{\ell_\varepsilon}{y}\right) k \frac{\partial U}{\partial y}$ where $\ell_\varepsilon = \frac{k^{3/2}}{\varepsilon}$ and C_w is constant. Details of how this expression is arrived at are to be reported, Irwin (1974). Good results in both boundary layer and wall jet computations without curvature have been obtained with $C_w = 0.03$. For curved flows the wall correction is not modified because it is a fairly small term and, in any event, is not expected to have the same invariant properties as the remaining part of the model. A wall correction is necessary in the first place because a wall appears to have an appreciable influence on pressure-velocity gradient correlations implying that the surface integral in Chou's (1945) exact expression for these correlations cannot be neglected. The surface integral can be converted into a volume integral by the method of images, Irwin (1974), which may be more convenient for estimating the size of this effect.

If the diffusion terms, i.e. those involving triple velocity correlations and pressure-velocity correlations, are small compared to the main production terms $\overline{uv}\,\dfrac{\partial U}{\partial y}$, $\overline{v^2}\dfrac{\partial U}{\partial y}$ and the pressure-velocity gradient correlations, then it is reasonable in the first instance to neglect the effect of curvature on them. The pressure-velocity correlations are at any rate neglected in the model. The data of Irwin (1973) and Bradbury (1965) indicate that diffusion is indeed relatively small in the region of maximum shear stress in a wall jet and free jet respectively and it is these types of flow which will be considered. The only modification made to Launder et al's transport equation for the dissipation rate is to include the extra curvature contribution to the 'generation' term.

3. The Magnitude of the Curvature Effects

To gain an insight into the likely magnitude of the curvature effects implied by the Reynolds stress equations (1) to (4) and the model of the pressure-velocity gradient correlations (6) to (9) it is instructive to consider the hypothetical case of an equilibrium flow, defined as one in which diffusion and advection are negligible. Equations (1) to (5) then become, omitting $\dfrac{\partial U}{\partial x}$ production terms,

$$\frac{1}{1+y/R}\,\overline{\frac{p}{\rho}\frac{\partial u}{\partial x}} - \overline{uv}\,\frac{\partial U}{\partial y} - \left\{\overline{uv}\,\frac{U}{R}\right\} - \frac{\varepsilon}{3} = 0 \qquad (10)$$

$$\overline{\frac{p}{\rho}\frac{\partial v}{\partial y}} + \left\{2\,\overline{uv}\,\frac{U}{R}\right\} - \frac{\varepsilon}{3} = 0 \qquad (11)$$

$$\overline{\frac{p}{\rho}\frac{\partial w}{\partial z}} - \frac{\varepsilon}{3} = 0 \tag{12}$$

$$\overline{\frac{p}{\rho}\left(\frac{1}{1+y/R}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)} - \overline{v^2}\frac{\partial U}{\partial y} + \left\{(2\overline{u^2} - \overline{v^2})\frac{U}{R}\right\} = 0 \tag{13}$$

$$\varepsilon = -\overline{uv}\frac{\partial U}{\partial y} + \left\{\overline{uv}\frac{U}{R}\right\} \tag{14}$$

Substituting (6) to (9) in (10) to (14) we find after rearrangement

$$\frac{\overline{u^2}}{2k} = \frac{1}{3} + \frac{1}{C_1}\left[\frac{2}{3} - (\alpha + 2\beta) + \left(C_2 + 2\beta + \frac{4}{3}\right)F_c\right]\frac{1}{1-F_c} \tag{15}$$

$$\frac{\overline{v^2}}{2k} = \frac{1}{3} + \frac{1}{C_1}\left[-\frac{1}{3} - (C_2 + 2\beta) + \left(\alpha + 2\beta - \frac{5}{3}\right)F_c\right]\frac{1}{1-F_c} \tag{16}$$

$$\frac{\overline{w^2}}{2k} = \frac{1}{3} + \frac{1}{C_1}\left[-\frac{1}{3} - \beta\right] \tag{17}$$

$$\left|\frac{\overline{uv}}{2k}\right| = \left[\frac{1}{2C_1(1-F_c)}\left(-(C_2+\beta)\frac{\overline{u^2}}{2k} + (1-\alpha-\beta)\frac{\overline{v^2}}{2k} - \frac{1}{2}(\eta+\nu) + \left\{(\alpha+\beta-2)\frac{\overline{u^2}}{2k} + (1+C_2+\beta)\frac{\overline{v^2}}{2k} + \frac{1}{2}(\eta+\nu)\right\}F_c\right)\right]^{\frac{1}{2}} \tag{18}$$

where $F_C = \dfrac{U/R}{\dfrac{\partial U}{\partial y}}$. For small F_C and the above values of C_1 and C_2 (15) to (18) become

$$\frac{\overline{u^2}}{2k} = 0.463\ (1 + 1.9\ F_C) \tag{19}$$

$$\frac{\overline{v^2}}{2k} = 0.232\ (1 - 3.9\ F_C) \tag{20}$$

$$\frac{\overline{w^2}}{2k} = 0.305 \tag{21}$$

$$\frac{\overline{uv}}{2k} = 0.178\ (1 - 3.2\ F_C) \tag{22}$$

The wall correction has been omitted in the above equations. If it is included it lowers the 0.178 factor in (22) slightly and raises the 3.2 factor on F_C but these changes are fairly small and do not effect the order of magnitude of the predicted curvature effects. It can be seen that the ratio $\frac{\overline{v^2}}{\overline{u^2}}$ is predicted to be quite sensitive to curvature, fractional changes in its value being equal to almost -6 F_C. This is an interesting result because Guitton (1970) found it was precisely this quantity which varied greatly with curvature in his experiments on self-preserving curved wall jets in still air. In fig. 2 $\frac{\overline{v^2}}{\overline{u^2}}$ and $\frac{\overline{uv}}{2k}$, calculated from (15) to (18), are compared with his hot wire data at $y = .75\, y_0$ (see fig. 1), where $\overline{uv}$ is at its maximum value, and it can be seen that the predictions are in fair accord with experiment bearing in mind the likely accuracy of hot wire measurements in high intensity turbulence. The predicted changes in $\frac{\overline{uv}}{2k}$ are probably too small to be detected by the measurements for $F_C < 0$. For the purposes of this comparison F_C was calculated using $F_C = \frac{U}{R+y} \Big/ \frac{\partial U}{\partial y}$ because y/R reached values of nearly 0.3 in the experiments. The hot wire data of Giles, Hays and Sawyer (1966) for the same kind of wall jets are not shown because they may be inaccurate (Guitton (1970)) but they do indicate the same trend for $\frac{\overline{v^2}}{\overline{u^2}}$. Thus, if we overlook the fact that the outer part of a wall jet is not quite an equilibrium flow, the above results can be taken as an encouraging indication that the predicted changes in turbulence structure are qualitatively correct and of the right order of magnitude. In fact, in the outer part of Irwin's (1973) self-preserving wall jet convection and diffusion were found to be fairly small and tended to cancel. Thus the equilibrium flow approximation may not be too inaccurate.

The above analysis predicts the structural changes in the turbulence caused by curvature but does not give the changes in the effective eddy viscosity or the mixing length. To investigate these the energy equation (14) for equilibrium flow is written as

$$-\overline{uv}\frac{\partial U}{\partial y}\left(1-F_c\right) = \varepsilon = \frac{k^{3/2}}{\ell_\varepsilon} \qquad (23)$$

where ℓ_ε is the dissipation length scale. In an uncurved equilibrium flow ℓ_ε is proportional to the conventional mixing length. Substituting for k using (22) it is found that

$$\frac{|\overline{uv}|}{U_o^2} = 0.045\left(\frac{\ell_\varepsilon}{y_o}\right)^2\left(1-F_c\right)^2\left(1-3.2F_c\right)^3\left(\frac{\partial\left(\frac{U}{U_o}\right)}{\partial(y/y_o)}\right)^2$$
$$\simeq 0.045\left(\frac{\ell_\varepsilon}{y_o}\right)^2\left(1-11.6F_c\right)\left(\frac{\partial(U/U_o)}{\partial(y/y_o)}\right)^2 \qquad (24)$$

since F_C is small. Thus the effective viscosity contains a -11.6 factor on F_C according to the above model which is in approximate agreement with what is observed experimentally, Bradshaw (1973). It is possible that ℓ_ε/y_0 is also affected by curvature but (24) combined with the experimental observations requires that it is not affected very much. This is in contrast to Bradshaw's (1973) length scale L, defined as $L = \frac{|\overline{uv}|^{3/2}}{\varepsilon}$, which if used in a similar way to ℓ_ε above would give a factor of only -2 on F_C. In order to agree with observed curvature effects it is therefore necessary to assume that L/y_0 is also affected by curvature, whereas this is not so in the case of ℓ_ε/y_0 implying that, of the two, ℓ_ε may be the more convenient and useful length scale to use.

Thus it looks as though it may be possible to account for curvature simply by including the extra curvature terms which arise naturally in the Reynolds stress equations and Launder et al's model. The large predicted change in effective eddy viscosity occurs because the fractional change in $\frac{\overline{uv}}{2k}$ is equal to about $-3.2\ F_C$ which is then magnified by a further factor of about 3 in the eddy viscosity. It is worth noting that in fig. 2 the slope of $\frac{\overline{uv}}{2k}$ versus F_C falls off as F_C becomes increasingly negative being zero at about $F_C = -.15$ i.e. the turbulence becomes less sensitive to curvature with increasing curvature. For positive F_C the opposite is true and there appears to be an upper limit of $F_C \simeq 0.1$ beyond which the turbulence can no longer sustain itself. This upper limit is in quite good agreement with the data of So and Mellor (1973) for a boundary layer on a convex surface. In such a boundary layer F_C increases with distance from the wall. The measurements showed that for F_C greater than about 0.15 the value of $\frac{\overline{uv}}{2k}$ was essentially zero or, in the words of the authors, the shear stress was 'turned off'. In the inner half of the boundary layer F_C was appreciably less than 0.15 and the shear stress was not 'turned off' so it is reasonable to assume that diffusion of $\overline{uv}$ outwards occured. This would tend to maintain $\frac{\overline{uv}}{2k}$ at non zero values for somewhat greater F_C than would be the case without diffusion which is consistent with the difference between the measured and predicted critical values for F_C.

4. Computed Results Compared with Experiments

The turbulence model described in section 2, including the wall correction, has been incorporated into the finite difference scheme of Spalding and Patankar (1967, 1969). Six transport equations are solved; the mean momentum equation, the four Reynolds stress equations for two-dimensional flow, and the equation for the dissipation. The details are to be reported in Irwin (1974). The mean momentum equation is left unchanged from that for uncurved flow, thus restricting the validity of the results to small curvature. This procedure is permissible because for small curvature the extra terms arising in the mean momentum equation are small compared to the changes due to curvature in the shear stress term, Bradshaw (1973). The normal stress terms are omitted from the mean momentum equation.

The boundary conditions are applied in the same way as described by Hanjalić and Launder (1972) with the following exceptions at a wall: the turbulence shear stress is matched to that given by the skin friction which is in turn calculated from the computed velocity profile and Patel's (1965) law of the wall; the three normal stresses are matched to $\overline{u^2} = 5.2\, u_\tau^{\,2}$, $\overline{v^2} = 0.61\, u_\tau^{\,2}$ and $\overline{w^2} = 2.9\, u_\tau^{\,2}$ where u_τ is the skin friction velocity. The data of Guitton (1970) and Irwin (1973) show that Patel's law of the wall is valid in wall jets at the Reynolds number of the present computations and the above relations for the normal stresses near the wall are in fair agreement with wall jet data. The computed results are relatively insensitive to changes in the wall boundary condition on the normal stresses.

In fig. 3 the predicted rate of growth of wall jets on logarithmic spirals is compared with the data of Guitton (1970) and Giles, Hays and Sawyer (1966). On a logarithmic spiral of given $\frac{x}{R}$ the value of $\frac{dy_0}{dx}$ is constant to a close approximation so that each point in fig. 3 represents a different wall jet. Guitton took great pains to establish two-dimensionality thus tending to make his data more reliable and at zero y_0/R the computed results agree very well with his measured rate of growth, that of Giles et al being somewhat higher. Of principal importance is the slope of the computed curve at $y_0/R = 0$ which is in very good agreement with the data and gives convincing evidence that inclusion of the curvature terms as described in section 2 can give realistic results without the need of an empirical constant. Agreement is good up to y_0/R of about 0.05 beyond which the computed $\frac{dy_0}{dx}$ tends to fall below the data. This is what might be expected since in practice the pressure rise across the wall jet, which is ignored in the computations, decreases with increasing x thus tending to augment

$\frac{dy_0}{dx}$ and this effect is greater for high y_0/R.

In figure 4 computed results are compared with empirical laws for the growth of a wall jet on a circular cylinder and for a free jet describing a circular arc. They are plotted in the form of fractional changes in y_0 due to curvature and $y_{0\infty}$ is the value of y_0 when $R = \infty$. In the case of the free jet y is measured from the velocity maximum and $(y_0 - y_{0\infty})/y_{0\infty}$ is calculated from

$$(y_0 - y_{0\infty})/y_{0\infty} = (y_{0x} - y_{0v})/(y_{0x} + y_{0v}) \tag{25}$$

where subscripts x and v stand for convex and concave sides respectively. This formula can be used because Arnot Smith's (1973) data shows that the sum of the rates of growth of the two sides of the jet is almost exactly equal to that for the uncurved case due to the opposite effect of curvature on each side. Also shown in fig. 4 are results for the wall jet predicted by the model of Hanjalić and Launder (1972) with the curvature terms included. Their full model was used as opposed to the simplified version described in the latter part of their paper. The wall jet on a circular cylinder is not self-preserving because y_0/R changes with x and thus the good agreement of the model of section 2 with Guitton's empirical law indicates that 'history' effects are well modelled. The Hanjalić and Launder model gives somewhat less curvature effect though still of the same order of magnitude. Its inferior agreement with experiment may be attributable to the fact that it takes no explicit account of the influence of a wall on the turbulence and is thus less physically realistic than the model of section 2. In the case of the free jet Arnot Smith found very little effect of curvature and this is also found in the computed results. This lends support to the assumption that the model of turbulence diffusion need not include curvature terms since, as it stands, it correctly accounts for the diffusion of properties from one side of the jet to the other when an assymetry is present.

Guitton's empirical law is based mainly on data at high values of y_0/R so, as supplementary evidence, fig. 5 shows a direct comparison with data from Guitton (1970) and Fekete (1963). It is seen that agreement is good up to values of $y_0/R \simeq 0.15$ beyond which the computed value of $\frac{y_0}{x}$ tends to be low, again as might be expected because of the neglect of the pressure gradient normal to the surface.

Incorporation of curvature effects into the present computer program was begun by Arnot Smith (1973) who reported some tentative results when the work was in its early stages for the curved free jet and wall jet on a circular cylinder. In addition to the curvature terms described in this paper he also made corrections to turbulence diffusion which are not used here. However, an error has since been found in the correction to diffusion and also a programming error; thus, his results, although similar to some of those described above, are superseded by the present ones.

5. Discussion and Conclusions

It has been shown in the above analysis and comparisons with experiment that the curvature terms in the Reynolds stress equations, although small, have an unexpectedly large effect. It is therefore worth returning to the neglected production terms involving $\frac{\partial U}{\partial x}$, which are of the same order, to see if this is also true for them. For uncurved flow but when $\frac{\partial U}{\partial x}$ (implying also $\frac{\partial V}{\partial y}$) is not negligible the terms in braces in (6) to (9) are zero and are replaced by

$$\left[(\alpha + 3\beta + c_2)\overline{u^2} - \beta\overline{v^2} + (\nu+1)k\right]\frac{\partial U}{\partial x} \qquad \text{in (6)}$$

$$\left[\beta\overline{u^2} - (\alpha + 3\beta + c_2)\overline{v^2} - (\nu+1)k\right]\frac{\partial U}{\partial x} \qquad \text{in (7)}$$

$$\left[\beta\left(\overline{u^2} - \overline{v^2}\right)\right]\frac{\partial U}{\partial x} \qquad \text{in (8)}$$

and by zero in (9). Then for the hypothetical equilibrium flow in which $\frac{U}{R}$ is zero but $\frac{\partial U}{\partial x}$ terms are retained it is possible to obtain the following expressions for small F_a where

$F_a = \frac{\partial U}{\partial x} \Big/ \left|\frac{\partial U}{\partial y}\right|$,

$$\frac{\overline{u^2}}{2k} = .463\ (1 - 0.8\ F_a) \qquad (26)$$

$$\frac{\overline{v^2}}{2k} = .232\ (1 + 1.6\ F_a) \qquad (27)$$

$$\frac{\overline{w^2}}{2k} = .305 \qquad (28)$$

$$\frac{\overline{uv}}{2k} = 0.178\ (1 + 1.3\ F_a) \qquad (29)$$

Comparing (26) to (29) with (19) to (22) it can be seen that the modelled turbulence structure is less sensitive to F_a than to F_c. Using the energy equation as before for small F_a

$$\frac{|\overline{uv}|}{U_o^2} = 0.045 \left(\frac{\ell_\varepsilon}{y_o}\right)^2 (1 - 1.3F_a)^2 (1 + 1.3F_a)^3 \left(\frac{\partial(U/U_o)}{\partial(y/y_o)}\right)^2$$

$$\doteq 0.045 \left(\frac{\ell_\varepsilon}{y_o}\right)^2 (1 + 1.3F_a) \left(\frac{\partial(U/U_o)}{\partial(y/y_o)}\right)^2 \qquad (30)$$

Thus the effective eddy viscosity contains a factor of only 1.3 on F_a compared to the -11.6 factor on F_c. This is in agreement with experiment to the extent that there does not appear to be any data that convincingly demonstrates that turbulence is sensitive to flow acceleration in the same way as it is sensitive to curvature (see Bradshaw's (1973) review).

In conclusion, it has been shown that the small terms in the Reynolds stress equations and in Launder et al's model for the pressure-velocity gradient correlations satisfactorily account for observed curvature effects in jet flows. By examination of equilibrium flow it is also shown how fairly small structural changes in the turbulence can have a magnified effect on the effective eddy viscosity without any change being required in $\frac{\ell_\varepsilon}{y_o}$. Similar methods applied to the production terms involving $\frac{\partial U}{\partial x}$ indicate that, provided $\frac{\ell_\varepsilon}{y_o}$ does not change, the effective viscosity is relatively insensitive to flow acceleration which is what seems to be found experimentally.

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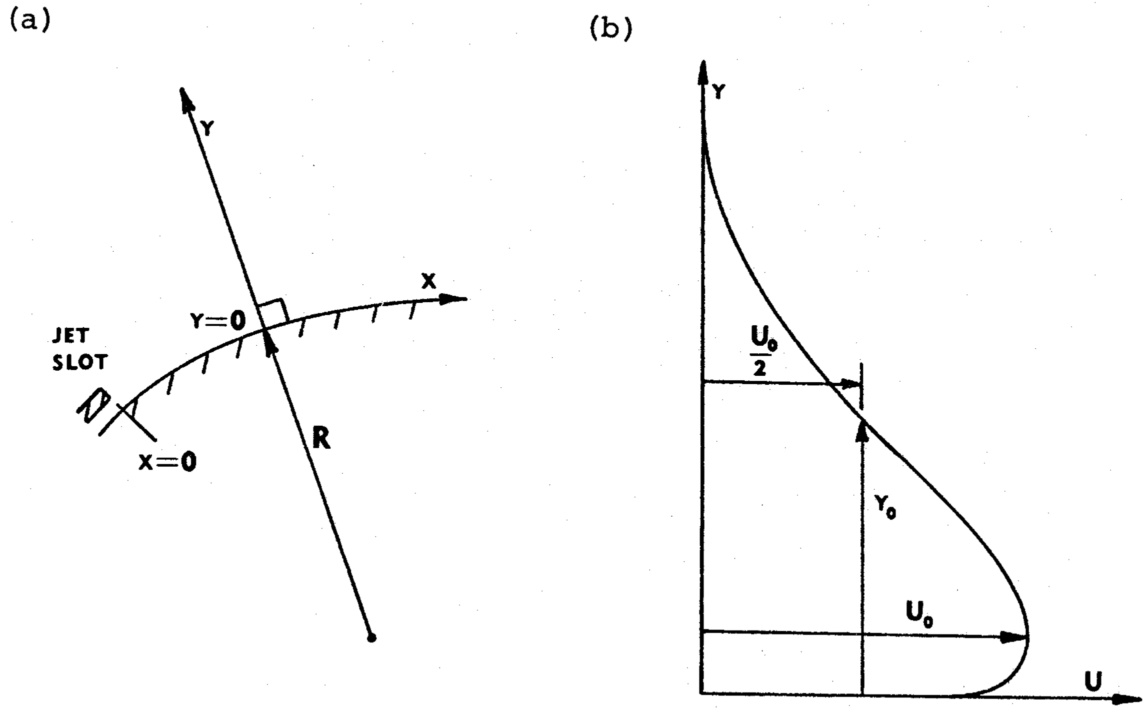

Fig. 1 (a) Coordinate system. (b) Wall jet notation.

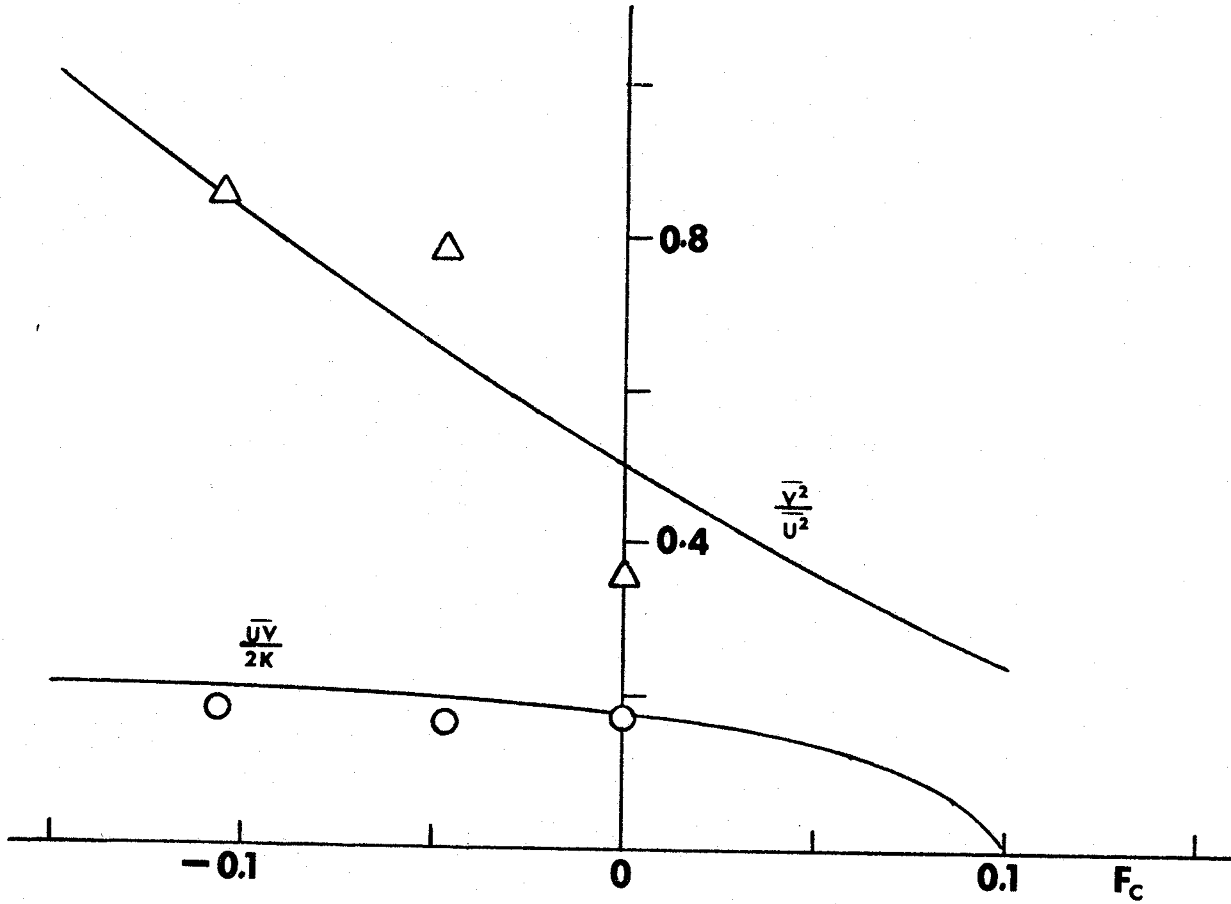

Fig. 2 Effect of curvature on the turbulence structure. ——— calculated; △ $\frac{\overline{v^2}}{\overline{u^2}}$, O $\frac{\overline{uv}}{2k}$ data of Guitton (1970).

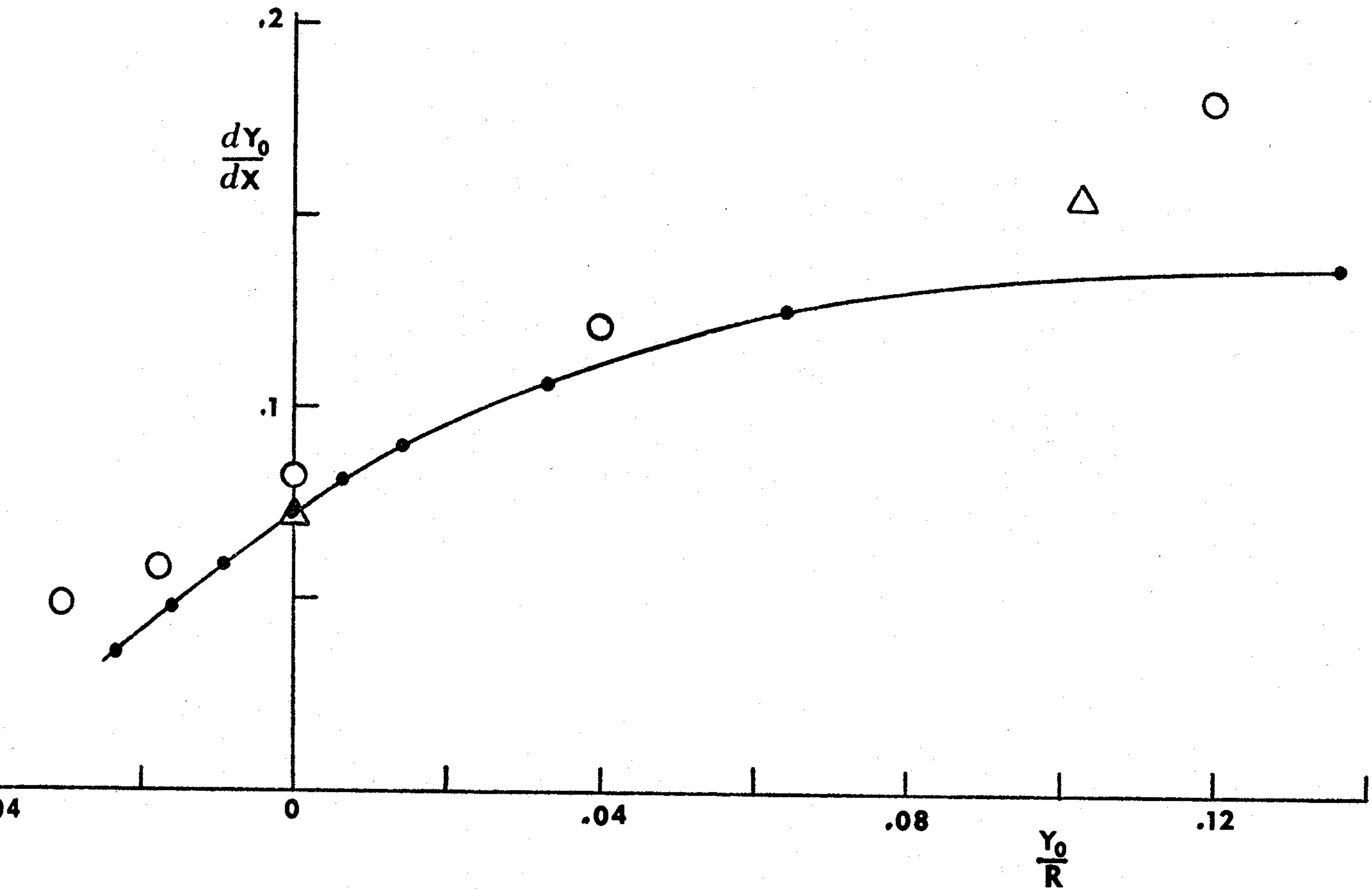

Fig. 3 Wall jets on logarithmic spirals. —●— computed, ○, △ data of Giles et al (1966) and Guitton (1970) respectively.

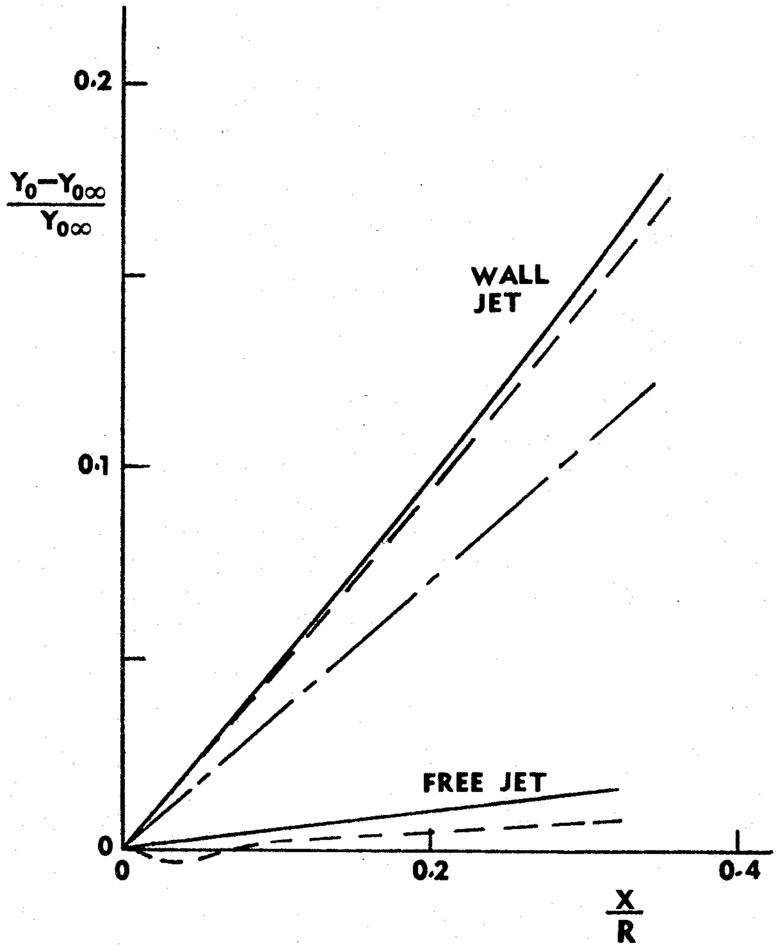

Fig. 4 Wall jet on a circular cylinder and a free jet describing a circular arc. ——— empirical laws for the wall jet, Guitton (1970), and the free jet, Arnot Smith (1973); ----- computed using model of section 2; ——— – ——— wall jet computation using model of Hanjalić and Launder (1972).

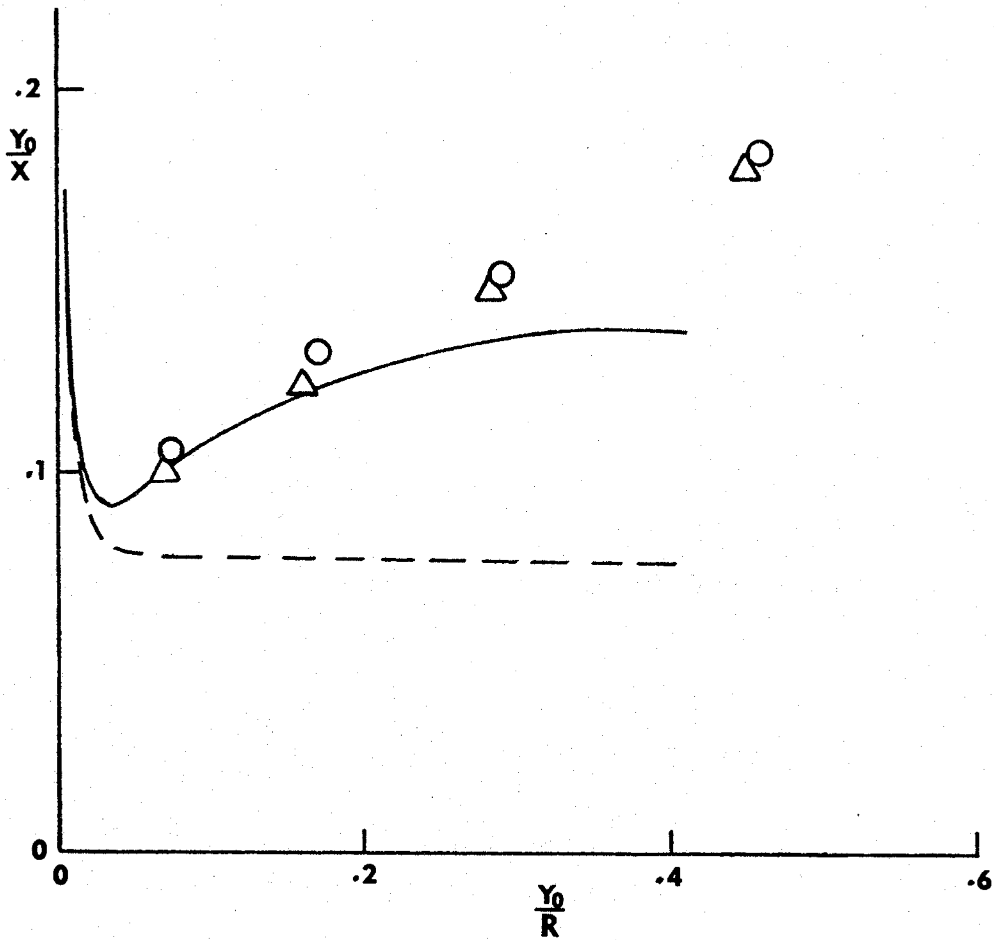

Fig. 5 Direct comparison with data for wall jet on a circular cylinder. ———, ------ computed with and without curvature terms respectively; ○, △ data of Fekete (1963) and Guitton (1970) respectively. Reynolds number $\simeq 10^4$, (slot height)/R = 0.0074.