Asymptotic Behavior of Stochastic Systems Possessing Markovian Realizations

Sean P. Meyn

M.Eng. electrical engineering, McGill University, 1984

B.A. mathematics, U.C.L.A., 1982

Department of Electrical Engineering

McGill University

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Abstract

The asymptotic properties of discrete time stochastic systems operating under feedback is addressed. It is assumed that a Markov chain Φ evolving on Euclidean space exists, and that the input and output processes appear as functions of Φ. The main objectives of the thesis are (i) to extend various asymptotic properties of Markov chains to hold for arbitrary initial distributions; and (ii) to develop a robustness theory for Markovian systems.

A condition called *local stochastic controllability*, a generalization of the concept of controllability from linear system theory, is introduced and is shown to be sufficient to ensure that the first objective is met. The second objective is explored by introducing a notion of convergence for stochastic systems and investigating the behavior of the invariant probabilities corresponding to a convergent sequence of stochastic systems.

These general results are applied to two previously unsolved problems: The asymptotic behavior of linear state space systems operating under nonlinear feedback, and the stability and asymptotic behavior of a class of random parameter $AR(p)$ stochastic systems under optimal control.

Résumé

Nous étudions les propriétés asymptotiques de systèmes stochastiques à temps discret soumis à une contre-réaction. Nous supposerons que les processus d'entrée et de sortie sont fonction d'une chaîne de Markov Φ à valeur dans un espace Euclidien. Les principaux objectifs de cette thèse sont: (*i*) de généraliser diverses propriétés asymptotiques des chaînes de Markov à des distributions initiales arbitraires; (*ii*) de développer une théorie de la robustesse pour les systèmes Markoviens.

Nous introduirons la condition de commandabilité stochastique locale, qui généralise le concept de commandabilité pour les systèmes linéaires, et nous montrerons qu'elle est suffisante pour assurer que notre premier objectif est atteint. Le second objectif sera examiné grâce à l'introduction d'une notion de convergence pour les systèmes stochastiques, et l'étude du comportement des mesures de probabilités invariantes correspondant à des suites convergentes de systèmes stochastiques.

Ces résultats généraux sont appliqués à deux problèmes jusqu'ici non résolus: Le comportement asymptotique des systèmes linéaires à états soumis à une contre-réaction non linéaire, ainsi que la stabilité et le comportement asymptotique d'une classe de systèmes stochastiques *A.R.* d'ordre p à paramètres aléatoires soumis à une commande optimale.

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Contents

Chapter 1 PRELIMINARIES

In Part I of this chapter we present a brief summary of some basic facts from probability theory and Markov chains, and in Part II we give a short review of some ideas from stochastic system theory. This chapter is intentionally biased to illustrate the close connection between stochastic systems and the theory of Markov chains.

PART I
MARKOV CHAINS

Markov chains have been called "a basic model for many types of statistical and control problems" [Kushner, 1971] and have already played a large role in stochastic system theory. In this thesis we will find that a Markov state processes may often be generated from an input-output stochastic system once a time invariant feedback control law has been assigned. This *Markovianization* will, in many cases, allow the asymptotic analysis of the distributions, and the sample path averages of the input and output processes. In this section we will give a review of some of the key results from the theory of Markov chains evolving on Euclidean space and bring together many of the important ergodic properties of these processes. This theory forms the foundation of the rest of this work.

1.1 Weak and Vague Convergence

The theory of the weak convergence of probability measures on metric spaces has been investigated in detail in [Parthasarathy, 1967], and [Billingsley, 1968]. In [Kushner, 1984] and [Ethier and Kurtz, 1986] this theory has been extended and applied to form a useful and interesting theory for the approximation of continuous time stochastic systems. In this section we present the results on weak and vague convergence of measures which will be needed later in the thesis. For further information on vague convergence of sub-probabilities see [Chung, 1974], and [Loève, 1977].

Let $\mathbf{X}$ be an open subset of $\mathbb{R}^M$. By *measure* we will always mean a positive σ-finite measure on $\mathbf{X}$. A *probability (sub-probability)* is a measure whose total mass is 1 (less than or equal to 1). The set of all bounded Borel measurable functions on $\mathbf{X}$ forms a Banach space $\mathbf{B}$ where

$$\|f\|_\infty \stackrel{\Delta}{=} \sup_{x \in \mathbf{X}} |f(x)|.$$

Letting $\mathbf{C}$ denote the set of continuous $f \in \mathbf{B}$, and $\mathcal{M}$ the set of probability measures on $\mathcal{B}(\mathbf{X})$, it is easy to see that $\mathcal{M} \subset \mathbf{C}^*$, the dual of C, and hence $\mathcal{M}$ together with the relative topology of $\mathbf{C}^*$ on $\mathcal{M}$ is a topological space (as a notational convenience, and to emphasize the duality between $\mathcal{M}$ and C we will often write $\langle \mu, f \rangle$ for $\int f \, d\mu$.) In this topology a sequence $\{\nu_k\}_{k=1}^\infty$ converges to ν if and only if $\langle \nu_k, f \rangle \to \langle \mu, f \rangle$ for every $f \in \mathbf{C}$. A subset $A \subset \mathcal{M}$ is open if and only if for each $\nu \in A$ and every sequence $\{\nu_k\}_{k=1}^\infty$ converging weakly to ν, there exists an $N \in \mathbb{Z}_+$ such that $\{\nu_k\}_{k=N}^\infty \subset A$. $\mathcal{M}$ together with this topology is *metrizable*. In fact, if the norm $\|\cdot\|_w$ is defined by

$$\|\mu\|_w \stackrel{\Delta}{=} \sum_1^\infty |\langle \mu, f_k \rangle|$$

where μ is a finite signed measure and $\{f_k\}_{k=1}^{\infty} \subset \mathbf{C}$ is a suitably defined set of functions (see [Parthasarathy, 1967]), then the topology on $\mathcal{M}$ generated by this norm is equivalent to the topology of weak convergence. The set of probabilities $\mathcal{A} \subset \mathcal{M}$ is precompact if and only if it is tight, where we say a set of probabilities $\{\nu_\alpha\}_{\alpha \in \mathcal{A}} \subseteq \mathcal{M}$ is *tight* if for every $\varepsilon \geq 0$ there exists a compact set $C \subset \mathbf{X}$ for which

$$\nu_\alpha\{C\} \geq 1 - \varepsilon \qquad \text{for every } \alpha \in \mathcal{A}.$$

The following is taken from [Billingsley, 1968] and [Parthasarathy, 1967].

Theorem 1.1.1. *The following are equivalent for a sequence $\{\nu_k\}_{k=1}^{\infty} \subset \mathcal{M}$*

(i) $\{\nu_k\}_{k=1}^{\infty} \xrightarrow{\text{weakly}} \nu$

(ii) for all open sets $\mathcal{O} \subset \mathbf{X}$, $\liminf_{k \to \infty} \nu_k\{\mathcal{O}\} \geq \nu\{\mathcal{O}\}$

(iii) for all closed sets $C \subset \mathbf{X}$, $\limsup_{k \to \infty} \nu_k\{C\} \leq \nu\{C\}$

(iv) for every equicontinuous family of functions $\mathcal{C} \subset \mathbf{C}$,

$\lim_{k \to \infty} \sup_{f \in \mathcal{C}} |\langle \nu_k - \nu, f \rangle| = 0.$ □

Let $\mathbf{C}_0 \subset \mathbf{C}$ denote the set of continuous functions on $\mathbf{X}$ which converge to zero on the "boundary" of $\mathbf{X}$. That is, $f \in \mathbf{C}_0$ if for some (and hence any) sequence $\{C_k : k \in \mathbb{Z}_+\}$ of compact sets which satisfy

$$C_k \subset C_{k+1}, \qquad \text{and} \bigcup_{k=0}^{\infty} C_k = \mathbf{X},$$

we have

$$\lim_{k \to \infty} \sup_{x \in C_k^c} |f(x)| = 0.$$

A sequence of sub-probabilities $\{\nu_k\}_{k=1}^{\infty}$ is said to converge *vaguely* to a sub-probability ν if for all $f \in \mathbf{C}_0$

$$\lim_{k\to\infty} \langle \nu_k, f\rangle = \langle \nu, f\rangle,$$

and in this case we will write

$$\nu_k \overset{\text{vaguely}}{\longrightarrow} \nu \qquad \text{as } k \to \infty$$

Obviously weak convergence implies vague convergence. On the other hand, it is easy to verify that a sequence of probabilities $\{\nu_k\}_{k=1}^{\infty} \overset{\text{weakly}}{\longrightarrow} \nu$ if and only if $\{\nu_k\}_{k=1}^{\infty} \overset{\text{vaguely}}{\longrightarrow} \nu$, and $\{\nu_k\}_{k=1}^{\infty}$ is tight.

We say the function $f: \mathbf{X} \to \mathbb{R}$ is *uniformly integrable* with respect to the probabilities $\{\nu_k : k \in \mathbb{Z}_+\}$ if

$$\limsup_{N\to\infty} \sup_{k\in\mathbb{Z}_+} \int_{\{|f|>N\}} |f|\, d\nu_k = 0.$$

This condition is satisfied if

$$\sup_{k\in\mathbb{Z}_+} \int |f|^{1+\delta}\, d\nu_k \le M$$

for some $\delta, M > 0$. We conclude this section with the following sufficient condition to ensure the convergence of moments on $\mathbf{X}$ when $\nu_k \overset{\text{weakly}}{\longrightarrow} \nu$. This result is taken from Theorem 5.4 of [Billingsley, 1968].

Theorem 1.1.2. *Suppose the function $f: \mathbf{X} \to \mathbb{R}$ is continuous, uniformly integrable with respect to the probabilities $\{\nu_k : k \in \mathbb{Z}_+\}$, and $\nu_k \overset{\text{weakly}}{\longrightarrow} \nu$ as $k \to \infty$. Then,*

$$\lim_{k\to\infty} \int f\, d\nu_k = \int f\, d\nu.$$

□

A Markov chain $\mathbf{\Phi}$ is not really one stochastic process but a family of stochastic processes parameterized by the initial distribution; that is, the distribution μ_0 of Φ_0. The fundamental object which makes the definition of a Markov chain possible and facilitates it's analysis is the Markov transition function.

1.2 Markov Transition Functions

Here we give the standard definition of a Markov transition function as in [Doob, 1953], and then we will use it to define linear operators $\mathbf{T}$ and $\mathbf{U}$, defined on $\mathbf{B}$ and $\mathcal{M}$ respectively. The key property of $\mathbf{U}$ which is investigated in [Saperstone, 1981] is that if a technical condition known as the *Feller property* holds, then the pair $(\mathcal{M}, \mathbf{U})$ is a semidynamical system. Besides it's intuitive appeal, this property of Feller chains enables us to exploit many important results from the theory of semidynamical systems. In particular, important notions from this field such as positive limit sets, stationary points, periodic orbits and stability have new significance in the context of Markov chains and stochastic system theory.

A *Markov transition function* is a mapping $P: \mathbf{X} \times \mathcal{B}(\mathbf{X}) \to [0,1]$ such that for *each* $x \in \mathbf{X}$,

$$P(x, \cdot) \in \mathcal{M} \tag{1.1}$$

and for *each* $A \in \mathcal{B}(\mathbf{X})$,

$$P(\cdot, A) \in \mathbf{B}. \tag{1.2}$$

A Markov transition function together with an initial distribution μ_0 generates a Markov chain $\mathbf{\Phi} = \{\Phi_k\}_{k=0}^{\infty}$ on $(\mathbf{X}^{\mathbf{Z}_+}, \mathcal{B}(\mathbf{X}^{\mathbf{Z}_+}), \mathrm{P}_{\mu_0})$ where $\mathbf{X}^{\mathbf{Z}_+}$ is the set of sequences

$$\{(s_0, s_1, \ldots, s_k, \ldots) : s_i \in \mathbf{X}\},$$

and $\mathcal{B}(\mathbf{X}^{\mathbb{Z}_+})$ is the smallest σ-algebra on $\mathbf{X}^{\mathbb{Z}_+}$ containing the sets

$$\{A_0 \times \cdots \times A_k \times \mathbf{X} \times \cdots : A_i \in \mathcal{B}(\mathbf{X}), k \in \mathbb{Z}_+\}.$$

With the stochastic process $\mathbf{\Phi}$ so defined it follows that the distribution of each Φ_k is $\mathbf{U}^k\mu_0$ (denoted $\Phi_k \sim \mathbf{U}^k\mu_0$), and that $\mathbf{\Phi}$ satisfies the Markov property:

$$\begin{aligned} \mathrm{P}\{\Phi_k \in A|\Phi_0,\ldots,\Phi_{k-1}\} &= \mathrm{P}\{\Phi_k \in A|\Phi_{k-1}\} \\ &= P(\Phi_{k-1}, A) \qquad a.s. [\mathrm{P}_{\mu_0}] \text{ for } A \in \mathcal{B}(\mathbf{X}), \end{aligned} \tag{1.3}$$

where $\mathrm{P}\{\Phi_n \in A|\Phi_0,\ldots,\Phi_{k-1}\} \triangleq \mathrm{E}[\mathbf{1}_{\Phi_n \in A}|\sigma\{\Phi_0,\ldots,\Phi_{k-1}\}]$.

A *Markov transition operator* $\mathbf{T}: \mathbf{B} \to \mathbf{B}$ is defined for $f \in \mathbf{B}$ by

$$\mathbf{T}f(x) = \int P(x,dy)f(y). \tag{1.4}$$

It's adjoint $\mathbf{U}: \mathcal{M} \to \mathcal{M}$ is defined for $\mu \in \mathcal{M}$ and $A \in \mathcal{B}(\mathbf{X})$ by

$$\mathbf{U}\mu(A) = \int \mu(dx)P(x,A). \tag{1.5}$$

Note that the domain and range of $\mathbf{U}$ may be extended to include all σ-finite measures on $\mathcal{B}(\mathbf{X})$ and similarly, the domain and range of $\mathbf{T}$ may be extended to include all positive $\mathcal{B}(\mathbf{X})$-measurable functions. We have

$$\langle \mathbf{U}\mu, f\rangle = \langle \mu, \mathbf{T}f\rangle$$

for any signed measure μ and Borel function f which makes one of the expressions meaningful. Observe that for an initial distribution μ_0, and a function $f \in L^2(\mathbf{X}, \mathcal{B}(\mathbf{X}), \mu_0)$ the least squares estimate of $f(\Phi_{N+k})$ given $\{\Phi_0,\ldots,\Phi_N\}$ is

$$\mathrm{E}[f(\Phi_{N+k})|\Phi_0^N] = \mathbf{T}^k f(\Phi_N), \tag{1.6}$$

and the mean square error is

$$\mathrm{E}\left|f(\Phi_{N+k}) - \mathrm{E}[f(\Phi_{N+k})|\Phi_0^N]\right|^2 = \mathrm{E}\left|f(\Phi_{N+k}) - \mathbf{T}^k f(\Phi_N)\right|^2.$$

As an example, consider the stochastic process generated by the recursion

$$\Phi_k = F(\Phi_{k-1}, w_k), \qquad k \in \mathbb{Z}_+, \tag{1.7}$$

where $F : \mathbf{X} \times \mathbb{R}^p \to \mathbf{X}$ is Borel measurable. Suppose that Φ_0 and the disturbance process $\mathbf{w}$ are mutually independent Borel random variables on the probability space $(\Omega, \mathcal{F}, \mathrm{P}_{\Phi_0})$, and that $\mathbf{w}$ is an independent and identically distributed (i.i.d.) process. Then the stochastic process $\mathbf{\Phi}$ generated by (1.7) is a Markov chain with Markov transition operator

$$\mathbf{T}f(x) = \int f(F(x,w))\, \mu_w(dw)$$

where μ_w is the distribution of w_k.

1.3 Feller Processes and Invariant Probabilities

The majority of important results concerning the asymptotic behavior of Markov chains require the existence of an *invariant measure.* By this we mean a (positive σ-finite) measure π with the property that

$$\mathbf{U}\pi = \pi. \tag{1.8}$$

If π is a probability and if Φ_0 has distribution π then $\Phi_k \sim \pi$ for all $k > 0$, and in fact $\mathbf{\Phi}$ is a stationary stochastic process in this case. The first result below gives necessary and sufficient conditions for the existence of an invariant *probability.*

A function $f\colon \mathbf{X} \to \mathbb{R}_-$ is called a *moment* if there exists a sequence of compact sets, $K_n \subset \mathbf{X}$, $K_n \uparrow \mathbf{X}$ s.t.

$$\lim_{n\to\infty} \left(\inf_{x \epsilon K_n^c} f(x) \right) = \infty$$

where we adopt the convention that the infimum of a function over the empty set is infinity. If $\mathbf{X}$ is closed and unbounded it is evident that $f(x) = \|x\|^p$ is a moment for any $p > 0$. Furthermore, if $\mathbf{X}$ is *compact* then our convention implies that f is still a moment because we may set $K_n = \mathbf{X}$ for all $k \in \mathbb{Z}_+$.

A Markov transition function P is said to have the *Feller property* if

$$\int P(y, dx) h(x)$$

is a continuous function of $y \in \mathbf{X}$ for every $h \in$ C. Hence, P has the Feller property if and only if $\mathbf{T}\colon \mathbf{C} \to \mathbf{C}$ where $\mathbf{T}$ is the Markov transition operator corresponding to P. It follows that the map $\mathbf{U}\colon \mathcal{M} \to \mathcal{M}$ is continuous, and in particular $x \to \mathbf{U}\delta_x$ is a continuous mapping from $\mathbf{X}$ to $\mathcal{M}$. As an example, the Markov chain generated by the recursion in (1.7) has the Feller property if the function $F(\cdot, z)$ is continuous for *a.e* $[\mu_w]$ $z \in \mathbb{R}^p$.

Theorem 1.3.1. (Beneš, 1967 and Saperstone, 1981) *Suppose that the Markov transition function P satisfies the Feller property. Then an invariant measure π exists if and only if a moment f exists such that for some initial distribution μ_0 either*

$$\sup_{k \epsilon \mathbb{Z}_+} \mathrm{E}_{\mu_0}[f(\Phi_k)] < \infty, \tag{1.9}$$

or

$$\sup_{N \geq 0} \frac{1}{N} \sum_{k=1}^{N} \mathrm{E}_{\mu_0}[f(\Phi_k)] < \infty. \tag{1.10}$$

Furthermore, if one of these conditions holds we have for all $g \in \mathbf{C}$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \mathrm{E}_{\mu_0}[g(\Phi_k)] = \mathrm{E}_\pi[g]. \tag{1.11}$$

Proof.

The proof of the existence of an invariant probability is straight forward: If (1.9) or (1.10) holds, then the collection of probabilities

$$\left\{ \frac{1}{N+1} \sum_{k=0}^{N} \mathbf{U}^k \mu_0 : N \in \mathbb{Z}_+ \right\} \tag{1.12}$$

is tight, and hence is a precompact subset of $\mathcal{M}$. By the Feller property, any weak limit point must be an invariant probability.

To establish (1.11) it is sufficient to show that there is at most one invariant probability in the closed convex hull of the probabilities in (1.12). For a proof of this fact the reader is referred to [Saperstone, 1981]. ☐

Observe that equation (1.11) is equivalent to the statement that

$$\frac{1}{N} \sum_{k=1}^{N} \mu_k \xrightarrow{\text{weakly}} \pi$$

where for $k \in \mathbb{Z}_+$, $\mu_k \triangleq \mathbf{U}^k \mu_0 =$ the distribution of Φ_k.

If $\mathbf{X}$ is a closed subset of $\mathbb{R}^n$, and for some initial distribution there exists a γ^2 such that either

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \|\Phi_k\|^2 \leq \gamma^2 \tag{1.13}$$

or

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \mathrm{E}\left[\|\Phi_k\|^2 \mid \mathcal{F}_{k-1} \right] \leq \gamma^2, \tag{1.14}$$

then by Theorem 1.3.1 an invariant probability exists. In fact if (1.13) is satisfied then for every $L > 0$

$$\frac{1}{N}\sum_{k=1}^{N}\mathbf{1}_{\{\|\Phi_k\|^2\le L^2\}} = \frac{1}{N}\sum_{k=1}^{N}\left(1-\mathbf{1}_{\{\|\Phi_k\|^2> L^2\}}\right)$$
$$\ge \frac{1}{N}\sum_{k=1}^{N}\left(1-\frac{\|\Phi_k\|^2}{L^2}\right)$$

Taking expectations and applying Fatou's lemma gives

$$\liminf_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\mathrm{P}\left\{\|\Phi_k\|^2\le L^2\right\} \ge \mathrm{E}\left[\liminf_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\left(1-\frac{\|\Phi_k\|^2}{L^2}\right)\right]$$
$$\ge 1-\frac{\gamma^2}{L^2}.$$

Hence, denoting the distribution of Φ_k by μ_k, the probabilities $\left\{\frac{1}{N}\sum_{k=1}^{N}\mu_k\right\}_{N=1}^{\infty}$ are tight. This is equivalent to the existence of a moment satisfying (1.10) and hence an invariant probability exists.

We conclude this section with a new characterization of systems which possess invariant probabilities.

Theorem 1.3.2. *Suppose that an invariant probability does not exist for* $\mathbf{U}$. *Then for any* $f \in C_o$

$$\lim_{N\to\infty}\left\|\frac{1}{N+1}\sum_{j=0}^{N}\mathbf{T}^j f\right\|_\infty = 0. \tag{1.15}$$

That is, $\frac{1}{N+1}\sum_{j=0}^{N}\mathbf{T}^j f \to 0$ *uniformly as* $k\to\infty$.

Conversely, if an invariant probability π does exist then by Theorem 1.3.1 the limit in (1.15) is non-zero for any $f \in C_o$ for which $\int f\,d\pi \neq 0$.

Proof.

Fix $f \in C_0$ and $\delta > 0$. Define the open sets $\{A_k : k \in \mathbb{Z}_+\}$ by

$$A_N = \left\{ x \in \mathbf{X} : \frac{1}{N+1} \sum_{j=0}^{N} \mathbf{T}^j f > \delta \right\}. \tag{1.16}$$

If (1.15) does not hold then there exists $\delta > 0$ and a subsequence $\{N_i : i \in \mathbb{Z}_+\}$ of $\mathbb{Z}_+$ with $A_{N_i} \neq \phi$ for all i. Let $\{\mu_i : i \in \mathbb{Z}_+\} \subset \mathcal{M}$ be probabilities for which $\mu_i\{A_{N_i}\} = 1$, and define

$$\lambda_i \stackrel{\Delta}{=} \frac{1}{N_i} \sum_{j=1}^{N_i} \mathbf{U}^j \mu_i .$$

The set of sub-probabilities is sequentially compact with respect to vague convergence (see [Chung, 1974].) Let λ_∞ be any vague limit point: $\lambda_{n_i} \xrightarrow{\text{vaguely}} \lambda_\infty$ for some subsequence $\{n_i : i \in \mathbb{Z}_+\}$ of $\mathbb{Z}_+$. The sub-probability $\lambda_\infty \neq 0$ because by the definition of vague convergence and the definition of A_n,

$$\begin{aligned} \int f \, d\lambda_\infty &\geq \liminf_{i\to\infty} \frac{1}{N_i} \sum_{j=1}^{N_i} \int f \, d\left(\mathbf{U}^j \mu_i\right) \\ &= \liminf_{i\to\infty} \int \left(\frac{1}{N_i} \sum_{j=1}^{N_i} \mathbf{T}^j f \right) d\mu_i \\ &\geq \delta \liminf_{i\to\infty} \mu_i \left\{ A_{N_i} \right\} = \delta > 0. \end{aligned} \tag{1.17}$$

We will now show that λ_∞ is invariant. Letting $g \in C_0$ satisfy $g \geq 0$, and $\eta \in C_0$ satisfy $0 \leq \eta(x) \leq 1$ for all $x \in \mathbf{X}$, we have

$$\begin{aligned} \int g \, d\lambda_\infty &= \lim_{i\to\infty} \int g \, d\lambda_{n_i} \\ &= \lim_{i\to\infty} \frac{1}{N_{n_i}} \sum_{j=1}^{N_{n_i}} \int \mathbf{T}^j g \, d\mu_i \\ &= \lim_{i\to\infty} \frac{1}{N_{n_i}} \sum_{j=1}^{N_{n_i}} \int \mathbf{T}^{j+1} g \, d\mu_i \end{aligned}$$

$$\geq \lim_{i\to\infty} \frac{1}{N_{n_i}} \sum_{j=1}^{N_{n_i}} \int \mathbf{T}^j(\eta \mathbf{T}g)\, d\mu_i$$

$$= \lim_{i\to\infty} \int \left(\eta(x) \int P(x,dy)g(y)\right) \lambda_{n_i}(dx)$$

$$= \int \left(\eta(x) \int P(x,dy)g(y)\right) \lambda_\infty(dx). \tag{1.18}$$

Letting the function $\eta \uparrow 1$ it follows that

$$\int g\, d\lambda_\infty \geq \int g\, d(\mathbf{U}\lambda_\infty),$$

and this implies that for all $A \in \mathcal{B}(\mathbf{X})$

$$\lambda_\infty\{A\} \geq \mathbf{U}\lambda_\infty\{A\}.$$

This is only possible if

$$\lambda_\infty\{A\} = \mathbf{U}\lambda_\infty\{A\},$$

and hence λ_∞ is an invariant sub-probability. Since we have assumed that no invariant probability exists it follows that $\lambda_\infty = 0$, which contradicts (1.17). So, $A_N = \phi$ for sufficiently large N and this completes the proof. □

1.4 Irreducible Markov Chains

In this section we present some results from the theory of irreducible Markov chains. Most of this material comes from [Nummelin, 1984]. Irreducible Markov chains exhibit many of the properties of Markov chains evolving on a finite set. In particular, in the first part of this section it is shown that there exists a (unique) cycle of disjoint Borel sets $\{\mathbf{X}_i : 1 \leq i \leq \lambda\}$ for which $\mathbf{1}_{\mathbf{X}_i}$ is taken to $\mathbf{1}_{\mathbf{X}_{i-1}}$ (mod λ) by the Markov transition operator $\mathbf{T}$.

1.4.1 Periodic Behavior in Markov Chains

We say a set $A \in \mathcal{B}(\mathbf{X})$ is *attainable* from $x \in \mathbf{X}$, and write $x \rightharpoonup A$ if for some $k \in \mathbb{Z}_+$, $P^k(x, A) > 0$. It is called *absorbing* if $P(x, A) = 1$ for all $x \in A$, and $\mathbf{X}$ is called *indecomposable* if it does not contain two disjoint absorbing sets.

The *potential kernel* G is defined for $x \in \mathbf{X}$ and $A \in \mathcal{B}(\mathbf{X})$ by

$$G(x, A) = \sum_{k=0}^{\infty} P^k(x, A).$$

For a set $A \in \mathcal{B}(\mathbf{X})$ the set $A_0 \in \mathcal{B}(\mathbf{X})$ is defined as the set of points in $\mathbf{X}$ from which A is not attainable. Hence,

$$A_0 = \{x \in \mathbf{X} : G(x, A) = 0\}.$$

The set A_0 is either absorbing or empty, so if A is absorbing and $\mathbf{X}$ is indecomposible then A_0 must be empty.

Let φ be a measure. The Markov chain $\mathbf{\Phi}$ is said to be *irreducible* if $x \rightharpoonup A$ for every $A \in \mathcal{B}(\mathbf{X})$ for which $\varphi\{A\} > 0$. We say that the measure μ is *absolutely continuous* with respect to the measure ν if for $A \in \mathcal{B}(\mathbf{X})$, $\nu\{A\} = 0 \Longrightarrow \mu\{A\} = 0$; μ and ν are said to be *equivalent* (denoted $\mu \approx \nu$) if $\mu \prec \nu$, and $\nu \prec \mu$. Suppose that $\mathbf{m}$ is a measure which satisfies

$$\mathbf{Um} \prec \mathbf{m}. \tag{1.19}$$

Such a measure always exists since if $\mu \in \mathcal{M}$ then

$$\mathbf{m} \doteq \sum_{k=0}^{\infty} 2^{-(k+1)} \mathbf{U}^k \mu$$

is such a measure. Furthermore, any *invariant* measure trivially satisfies equation (1.19). If $\mathbf{m}$ is an irreducibility measure for $\mathbf{\Phi}$ satisfying (1.19) then $\mathbf{m}$ is called a *maximal* irreducibility measure. It is a remarkable fact that in this case $\mathbf{m}$ is indeed maximal in the sense that if φ is any other irreducibility measure then $\varphi \prec \mathbf{m}$, and in particular, if $\mathbf{m}$ and $\mathbf{n}$ are maximal irreducibility measures then $\mathbf{n} \approx \mathbf{m}$ (see [Nummelin, 1984].)

The following lemma shows that it is possible to restrict the Markov chain $\mathbf{\Phi}$ to a set of full $\mathbf{m}$-measure when $\mathbf{m}$ is a maximal irreducibility measure.

Lemma 1.4.1. *Let* $\mathbf{m}$ *be a maximal irreducibility measure. Then:*

(i) *If* A *is an absorbing set then* $\mathbf{m}\{A^c\} = 0$;

(ii) *If* $F \in \mathcal{B}(\mathbf{X})$ *and* $\mathbf{m}\{F^c\} = 0$ *then there exists an absorbing set* $A \subset F$.

□

Assume now that $\mathbf{\Phi}$ is irreducible. We will show that for some $\lambda \in \mathbb{Z}_+$ the state space $\mathbf{X}$ may be written as a disjoint union

$$\mathbf{X} = \left\{ \bigcup_{k=0}^{\lambda-1} \mathbf{X}_k \right\} \bigcup N$$

where N is a set of $\mathbf{m}$-measure zero, and the sets $\{\mathbf{X}_i : 0 \leq i \leq \lambda - 1\}$ form a *cycle:* That is,

$$P(x,\mathbf{X}_j) = 1 \quad \text{for } x \in \mathbf{X}_{j-1} \quad (\text{mod } \lambda).$$

Observe that if $\{\mathbf{X}_i : 0 \leq i \leq \lambda - 1\}$ is a cycle then their union is an absorbing set. A positive Borel function s and non-zero finite positive measure ν are called *small* if for some $N \in \mathbb{Z}_+$, and all $x \in \mathbf{X}$, and $A \in \mathcal{B}(\mathbf{X})$

$$P^N(x, A) \geq s(x)\nu\{A\}. \tag{1.20}$$

Surprisingly, for irreducible chains there always exist small functions and measures:

Theorem 1.4.1. *Suppose* $\mathbf{\Phi}$ *is irreducible. Then there exists a small function* s, *and a small measure* ν *for which* $\int s\, d\nu > 0$.

□

It turns out that a small function s vanishes on all but one set $\mathbf{X}_i$ of a cycle: The proof of the following lemma follows directly from the definitions.

Lemma 1.4.2. *If $\{\mathbf{X}_i : 0 \le i \le \lambda - 1\}$ is a cycle and s is a small function then for some i*

$$s(x) = 0 \qquad \text{for all } x \in \mathbf{X}_j, \quad \text{and all } j \neq i \pmod{\lambda}.$$

□

We now present the existence theorem for cycles. Let λ be the greatest common divisor of the set

$$I \triangleq \{m \ge 1 : P^m(\cdot, \cdot) \ge \beta_m s(\cdot)\nu\{\cdot\} \quad \text{for some } \beta_m > 0\}.$$

The set I is closed under addition and hence contains all sufficiently large multiples of λ (see [Orey, 1971].)

Theorem 1.4.2. *Suppose that Φ is irreducible with maximal irreduciblity measure* m. *Let $\lambda \in \mathbb{Z}_+$, s and ν be as above. Then:*

(*i*) *There is a λ-cycle $\{\mathbf{X}_i : 0 \le i \le \lambda - 1\}$.*

(*ii*) *If $\{\mathbf{X}_i' : 0 \le i \le \lambda' - 1\}$ is another cycle then λ' divides λ, and any $\mathbf{X}_i'$ is the union a.e.* [m] *of sets from the collection $\{\mathbf{X}_i : 0 \le i \le \lambda - 1\}$.*

Proof.

For $j = 0, \ldots, \lambda - 1$ set

$$\tilde{\mathbf{X}}_j \triangleq \left\{ x : \sum_{k=0}^{\infty} \int P^{k\lambda - j}(x, dy) s(y) > 0 \right\} \tag{1.21}$$

It is easy to show that by irreducibility

$$\sum_{k=0}^{\infty} \int P^k(x, dy)s(y) = \int G(x, dy)s(y) > 0 \qquad \text{everywhere,} \tag{1.22}$$

and hence $\bigcup_{i=0}^{\lambda-1} \widetilde{\mathbf{X}}_i = \mathbf{X}$. Furthermore, by irreducibility and the definition of λ a simple argument (see [Nummelin, 1984]) shows that these sets are $\mathbf{m} - a.e.$ disjoint:

$$\mathbf{m}\{\widetilde{\mathbf{X}}_i \cap \widetilde{\mathbf{X}}_j\} = 0 \qquad \text{for } i \neq j \pmod{\lambda}.$$

By Lemma 1.4.1 there is an absorbing set F with $\mathbf{m}\{F^c\} = 0$ such that the sets $\mathbf{X}_i = \widetilde{\mathbf{X}}_i \cap F$ are disjoint. By (1.21) if $P(x, \mathbf{X}_i) > 0$ then x must belong to $\mathbf{X}_{i-1}$ (mod λ), and hence $\{\mathbf{X}_i : 0 \leq i \leq \lambda - 1\}$ is a cycle. The uniqueness assertion (ii) follows easily. □

1.4.1 Recurrence and Convergence Of The Underlying Distributions

In this section we present two standard recurrence conditions for Markov chains, and a variety of limit theorems for recurrent Markov chains.

Suppose that the Markov Chain $\mathbf{\Phi}$ is irreducible with maximal irreducible measure $\mathbf{m}$. It is called:

(i) *recurrent* if

$$\mathbf{m}\{A\} > 0 \Rightarrow \mathrm{P}_x\{\Phi_k \in A \quad i.o.\} \begin{cases} > 0 & \text{for all } x \in \mathbf{X}; \\ = 1 & \text{for } a.a.\ x \in A\ [\mathbf{m}]. \end{cases}$$

(i) ***Harris recurrent*** if

$$\mathbf{m}\{A\} > 0 \Rightarrow \mathrm{P}_x\{\Phi_k \in A \quad i.o.\} = 1 \qquad \text{for all } x \in \mathbf{X},$$

where $\{\Phi_k \in A \quad i.o.\} = \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} \{\Phi_k \in A\}$.

We call a Markov chain φ-recurrent if for every $A \in \mathcal{B}(\mathbf{X})$.

$$\varphi\{A\} > 0 \Longrightarrow \mathrm{P}_x\{\Phi_k \in A \quad \text{for some } k \in \mathbb{Z}_+\} = 1. \tag{1.23}$$

It follows from Proposition 3.12 of [Nummelin, 1984] that $\mathbf{\Phi}$ is φ-recurrent if and only if it is Harris recurrent.

We see in the following theorem that there is not a great deal of difference between recurrent and Harris recurrent Markov chains:

Theorem 1.4.3. *Suppose that $\mathbf{\Phi}$ is recurrent Then there exists an absorbing set $H \in \mathcal{B}(\mathbf{X})$ such that the restriction of $\mathbf{\Phi}$ to H is Harris.*

□

It turns out that for Harris recurrent Markov chains the existence of an invariant measure is guaranteed:

Theorem 1.4.4. *Suppose that $\mathbf{\Phi}$ is Harris recurrent. Then there exists a positive σ-finite invariant measure $\mathbf{m}$. If $\mathbf{n}$ is any other σ-finite invariant measure then $\mathbf{n} = c\mathbf{m}$ for some $c \in \mathbb{R}$.*

We summarize here some extremely important limit theorems for Harris recurrent Markov chains. The Markov chain $\mathbf{\Phi}$ is called *aperiodic* if $\lambda = 1$ where λ is the integer defined above Proposition 1.4.2, otherwise it is called *periodic*.

Theorem 1.4.5. *Suppose that $\mathbf{\Phi}$ is an aperiodic Harris recurrent Markov chain with invariant measure $\mathbf{m}$. Let μ and ν be any two initial distributions, and let $f, g \in L^1(\mathbf{X}, \mathcal{B}(\mathbf{X}), \mathbf{m})$ be such that $\int f\, d\mathbf{m} = \int g\, d\mathbf{m}$. Then:*

$$\lim_{k\to\infty} \sup_{B \in \mathcal{B}(\mathbf{X})} |\mathbf{U}^k\mu\{B\} - \mathbf{U}^k\nu\{B\}| = 0,$$

$$\lim_{k\to\infty} \int |\mathbf{T}^k f - \mathbf{T}^k g|\, d\mathbf{m} = 0,$$

and

$$\lim_{k\to\infty} \mathbf{T}^k f = \frac{\int f\,d\mathbf{m}}{\mathbf{m}\{\mathbf{X}\}} \qquad \text{for a.a. } [\mathbf{m}]\ x \in \mathbf{X}.$$

□

We have the following corollary in case $\mathbf{\Phi}$ possesses an invariant *probability:*

Corollary 1.4.5. *If $\mathbf{\Phi}$ possesses an invariant probability π then under the conditions of Theorem 1.4.5 we have*

$$\lim_{k\to\infty} \sup_{B\in\mathcal{B}(\mathbf{X})} |\mathrm{P}_\mu\{\Phi_k \in B\} - \pi\{B\}| = 0,$$

$$\lim_{k\to\infty} \int |\mathrm{E}_x[f(\Phi_k)] - \int f\,d\pi|\,\pi(dx) = 0,$$

and

$$\lim_{k\to\infty} \mathrm{E}_x[f(\Phi_k)] = \int f\,d\pi \qquad \text{for a.a. } [\pi]\ x \in \mathbf{X}.$$

□

For a probability $\mu_0 \in \mathcal{M}$ the sequence $\{\mu_k \triangleq \mathbf{U}^k\mu_0 : k \in \mathbb{Z}_+\}$ is called the *trajectory starting at* μ_0. Call a trajectory $\nu \triangleq \{\nu_k,\ k \in \mathbb{Z}_+\}$ a *periodic orbit* if there exists $\lambda \in \mathbb{Z}_+$ such that

$$\nu_{k+\lambda} \triangleq \mathbf{U}^\lambda \nu_k = \nu_k \quad \text{for each } k \in \mathbb{Z}_+. \tag{1.24}$$

The smallest $\lambda \geq 1$ for which (1.24) holds will be called the *period* of ν.

If $\mathbf{\Phi}$ is positive Harris recurrent (that is, $\mathbf{\Phi}$ is Harris recurrent and its invariant measure is finite), and if $\mathbf{\Phi}$ is periodic with period $\lambda > 1$ then Proposition 1.4.2 implies that non-trivial periodic orbits always exist. The Corollary to Proposition 1.4.5 implies that for a positive Harris recurrent Markov chain every trajectory converges to a periodic orbit in total variation norm.

1.5 Sample Path Properties of Markov Chains

In this section we describe the sample path properties of Markov chains which possess invariant probabilities. As remarked before, if π is an invariant measure then the Markov chain $\mathbf{\Phi}$ evolving on $\mathbf{X}^{\mathbb{Z}}$ generated by π is strictly stationary. Let $\Sigma_I \subset \mathcal{B}(\mathbf{X}^{\mathbb{Z}})$ denote the σ-algebra of *invariant sets* of the stationary stochastic process $\mathbf{\Phi}$. For the special case of a stationary *Markov* process every $\mathcal{A} \in \Sigma_I$ is of the form

$$\mathcal{A} = \{\cdots\cdots \times A \times A \times A \times \cdots\cdots\} \tag{1.25}$$

for some $A \in \mathcal{B}(\mathbf{X})$ where A has the invariance property

$$P(x, A) = \mathbf{1}_A(x), \quad a.e.\ [\pi]. \tag{1.26}$$

The set of all $A \in \mathcal{B}(\mathbf{X})$ satisfying (1.26) is a sub σ-algebra of $\mathcal{B}(\mathbf{X})$ and any such A will also be called *π-invariant*. Similarly, let $\mathcal{P}_\infty \subset \mathcal{B}(\mathbf{X}^{\mathbb{Z}})$ denote the *negative tail σ-algebra* of the stationary stochastic process $\mathbf{\Phi}$. That is,

$$\mathcal{P}_\infty \triangleq \bigcap_{k \le 0} \sigma\{\cdots \Phi_{k-1}, \Phi_k\}.$$

Let $\Sigma_D \subset \mathcal{P}_\infty$ denote the σ-algebra

$$\Sigma_D \triangleq \sigma\{\Phi_0\} \cap \mathcal{P}_\infty.$$

Every set $\mathcal{A} \in \Sigma_D$ is of the form

$$\mathcal{A} = \mathbf{1}_{\Phi_0 \in A}$$

for a set $A \in \mathcal{B}(\mathbf{X})$, and since $\mathcal{A} \in \mathcal{P}_\infty$ it follows by the Markov property that there exists a sequence of sets $\{A_k : k \in \mathbb{Z}_+\}$ such that

$$P^k(x, A) = \mathbf{T}^k \mathbf{1}_A(x) = \mathbf{1}_{A_k}(x) \quad a.s.\ [\pi]. \tag{1.27}$$

Conversely, if $A \in \mathcal{B}(\mathbf{X})$ satisfies (1.27) then $\mathcal{A} \triangleq \mathbf{1}_{\Phi_0 \in A} \in \mathcal{P}_\infty$. Hence we shall not distinguish between the sets $\mathcal{A}$ and A and we will use Σ_D to denote the sub-σ-algebra of sets $A \in \mathcal{B}(\mathbf{X})$ which satisfy (1.27).

The ergodic theorem for stationary processes applies:

For any $\mathcal{Y} \in L^1(\mathbf{X}^{\mathbb{Z}_+}, \mathcal{B}(\mathbf{X}^{\mathbb{Z}_+}), \mathrm{P}_\pi)$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \mathcal{Y}(\Phi_k, \Phi_{k-1}, \cdots) = \mathrm{E}_\pi[\mathcal{Y} | \Sigma_I] \quad a.s. \ [\mathrm{P}_\pi]. \tag{1.28}$$

Hence,

$$\begin{aligned} 1 &= \mathrm{P}_\pi\Big\{ \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \mathcal{Y}(\Phi_k, \Phi_{k-1}, \cdots) = \mathrm{E}_\pi[\mathcal{Y} | \Sigma_I] \Big\} \\ &= \int \pi(dx) \mathrm{P}_x \Big\{ \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \mathcal{Y}(\Phi_k, \Phi_{k-1}, \cdots) = \mathrm{E}_\pi[\mathcal{Y} | \Sigma_I] \Big\}, \end{aligned} \tag{1.29}$$

and it follows that

$$1 = \mathrm{P}_x\Big\{ \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \mathcal{Y}(\Phi_k, \Phi_{k-1}, \cdots) = \mathrm{E}_\pi[\mathcal{Y} | \Sigma_I] \Big\} \tag{1.30}$$

for *a.a.* $x \in \mathbf{X}$ $[\pi]$. This proves:

Theorem 1.5.1. [Doob, 1953] *Let* $\mathcal{Y} \in L^1(\mathbf{X}^{\mathbb{Z}_+}, \mathcal{B}(\mathbf{X}^{\mathbb{Z}_+}), \mathrm{P}_\pi)$. *Then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \mathcal{Y}(\Phi_k, \Phi_{k-1}, \cdots) = \mathrm{E}_\pi[\mathcal{Y} | \Sigma_I](\Phi_0, \Phi_1, \cdots) \tag{1.31}$$

almost surely for a.e. (π) initial condition $\Phi_0 = x \in \mathbf{X}$, or almost surely when Φ_0 has initial distributions μ_0 which is absolutely continuous with respect to π.

□

We have the following Corollary to Proposition 1.5.1 which relates positive Harris recurrence to the existence of the limit (1.31) for all initial condition distributions. Although this is a simple result, it appears to be new.

Corollary 1.5.1. *$\mathbf{\Phi}$ is positive Harris recurrent if and only if there exists a unique invariant probability π, and for every function $\mathcal{Y}$ satisfying the conditions of Proposition 1.5.1*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\mathcal{Y}(\Phi_k,\Phi_{k-1},\cdots)=\mathrm{E}_\pi[\mathcal{Y}]\qquad a.s.\ [\mathrm{P}_{\mu_0}] \tag{1.32}$$

for every initial distribution $\mu_0 \in \mathcal{M}$.

Proof.

If $\mathbf{\Phi}$ is positive Harris recurrent with invariant probability π then by Proposition 1.5.1 there exists a set $G \in \mathcal{B}(\mathbf{X})$ of full π-measure such that (1.32) holds whenever the distribution of Φ_0 is supported on G. Since $\mathbf{\Phi}$ is positive Harris recurrent and $\pi\{G\} = 1 > 0$, for an arbitrary initial condition distribution $\mu_0 \in \mathcal{M}$, $\mathrm{P}_{\mu_0}\{\mathbf{\Phi}$ enters $G\} = 1$, and it follows by a standard argument (see the proof of Theorem 6.2 of Chapter V of [Doob, 1953]) that (1.32) holds for arbitrary initial distributions μ_0.

Conversely, if (1.32) holds for every initial condition distribution then in particular for every $x \in \mathbf{X}$ and $A \in \mathcal{B}(\mathbf{X})$,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\mathbf{1}_{\{\Phi_k\in A\}}=\pi\{A\}\qquad a.s.\ [\mathrm{P}_x].$$

Hence if $\pi\{A\} > 0$ then $\mathrm{P}_x\{\mathbf{\Phi}$ enters A *i.o.*$\} = 1$. This shows that $\mathbf{\Phi}$ is positive Harris recurrent and the corollary is proved. □

We state here the following theorem of Wiener which will be useful later in the thesis. The function $\log^+ : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is defined by $\log^+(x) = \max(0, \log(x))$.

Theorem 1.5.2. (Wiener, 1939) *Let* $\mathcal{Y} \in L^p(\mathbf{X}^{\mathbb{Z}_+}, \mathcal{B}(\mathbf{X}^{\mathbb{Z}_+}), \mathrm{P}_\pi)$ *for some* $p > 1$, *or more generally suppose that*

$$\int_{\mathbf{X}^{\mathbb{Z}_+}} |\mathcal{Y}| \log^+ |\mathcal{Y}| \, d\mathrm{P}_\pi < \infty.$$

Then,

$$\sup_N \left\{ \frac{1}{N+1} \sum_{k=0}^{N} \mathcal{Y}(\Phi_k, \Phi_{k+1}, \cdots) \right\} \in L^1(\mathbf{X}^{\mathbb{Z}_+}, \mathcal{B}(\mathbf{X}^{\mathbb{Z}_+}), \mathrm{P}_\pi).$$

□

PART II
SYSTEM THEORY

In it's most general formulation, a (discrete time input-output) stochastic system is a causal random mapping $\varphi: \mathcal{U} \to \mathcal{Y}$ where $\mathcal{U}$ and $\mathcal{Y}$ are sets of discrete time stochastic processes on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ taking values in Euclidean space. $\mathcal{U}$ is called the set of *output processes*, and $\mathcal{Y}$ is called the set of *input processes*. By *causal* we mean that if $k \in \mathbb{Z}_+$, $\mathbf{u}, \mathbf{v} \in \mathcal{U}$ and $u_n = v_n$ for $n \leq k$ then $(\varphi \mathbf{u})_n = (\varphi \mathbf{v})_n$ for all $n \leq k$. A precise definition is difficult, but for an interesting discussion on stochastic systems the reader is referred to [Caines, 1987].

An example which we will often be referring to is the ARMAX system model of the form

$$
\begin{aligned}
y_k + A_1^{(k)} y_{k-1} + \dots + A_{n_1}^{(k)} y_{k-n_1} \\
= B_d^{(k)} u_{k-d} + \dots + B_{n_2}^{(k)} u_{k-n_2} \\
+ w_k + C_1^{(k)} w_{k-1} + \dots + C_{n_3}^{(k)} w_{k-n_3} \\
d \geq 1, k \geq 1
\end{aligned}
\tag{1.33}
$$

where the processes $\mathbf{y}$ and $\mathbf{w}$ are $\mathbb{R}^p$-valued, $\mathbf{u}$ is $\mathbb{R}^m$-valued, and initial conditions are assigned at $k = 0$. The process $\mathbf{u} = \{u_k\}_{k=0}^{\infty}$ is such that u_k is $\mathcal{F}_k$ measurable where $\mathcal{F}_k \triangleq \sigma\{y_0, \dots, y_k\}$. Furthermore, the parameter process $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is independent of the disturbance process $\mathbf{w}$. It is readily seen that these equations generate a random mapping from input processes $\mathbf{u}$ to output processes $\mathbf{y}$. However, if a non-random mapping is desired then the input space must be enlarged. In this case the new input process $\mathbf{v}$ takes on the form

$$
v_k = \begin{cases} \left\{(A^{(0)}, B^{(0)}, C^{(0)}), u_0, w_0, \mathbf{x}\right\} & \text{for } k = 0; \\ \left\{(A^{(k)}, B^{(k)}, C^{(k)}), u_k, w_k\right\} & \text{otherwise.} \end{cases}
$$

It is not appealing to be forced to consider the parameters and disturbances as *inputs* and this is the main reason for defining stochastic systems as *random* mappings.

1.6 Stochastic State Space Systems

In this section we introduce a definition of a stochastic state space system by generalizing the notion of a countable state controlled Markov chain as described in [Kushner, 1971]. A stochastic state-space system is just an input-output system of a special form. A state process $\mathbf{x}$ is assumed to evolve along with the input process $\mathbf{u}$, and the output process

$\mathbf{y}$. The state process summarizes the past behavior of the input and state processes in the sense that y_{N+k} is independent of x_0^{N-1} and u_0^{N-1} given observations on x_N and u_N^{N+k}. Already we see a close connection between Markov chains and stochastic state space systems.

We introduce here a formal definition of a state space system, but first we make the following generalization of a Markov transition function: Let $\mathbf{Y}$, $\mathbf{X}$, and $\mathbf{U}$ denote Borel subsets of $\mathbb{R}^{n_1}$, $\mathbb{R}^{n_2}$, and $\mathbb{R}^{n_3}$ respectively. A *controlled Markov transition function* is a mapping $P: \mathbf{X} \times \mathbf{U} \times \mathcal{B}(\mathbf{X}) \to [0,1]$ such that for each $u \in \mathbf{U}$,

$$P(\cdot\,; u, \cdot) \quad \text{is a Markov transition function,} \tag{1.34}$$

and for each $A \in \mathcal{B}(\mathbf{X})$,

$$P(\cdot\,; \cdot, A) \in \mathcal{B}(\mathbf{X}) \otimes \mathcal{B}(\mathbf{U}) \tag{1.35}$$

where for two σ-algebras $\mathcal{F}$ and $\mathcal{G}$ on $\mathbf{X}_1$, and $\mathbf{X}_2$ respectively, we let $\mathcal{F} \otimes \mathcal{G}$ denote the smallest σ-algebra on $\mathbf{X}_1 \times \mathbf{X}_2$ containing the sets $A \times B$ for $A \in \mathcal{F}$ and $B \in \mathcal{G}$.

Let Ψ be a Borel function on $\mathbf{X} \times \mathbf{U}$, and P be a controlled Markov transition function. We call the pair (P, Ψ) a *(time invariant) stochastic state space system*. The transition function P is used to define probabilistically the dynamics of the map $\mathbf{u} \to \mathbf{x}$, and the *read out map* Ψ takes (x_k, u_k) to y_k without memory or dynamics.

To define the input-state-output process $(\mathbf{u}, \mathbf{x}, \mathbf{y})$ we need a procedure for determining the input u_k given past observations on $\mathbf{x}$ and $\mathbf{y}$. There are many situations where one wishes to use feedback which is not only a function of the state, but also of variables which are independent of the system. For example, when P or Ψ contains unknown parameters a "dither signal" is added to the control for the purpose of identification in some estimation schemes (see [Caines and Lafortune, 1984]), or to ensure "noise controllability" (see Chapters II and III) In order to incorporate this into our model we assume that there is a probability space $(\Omega, \mathcal{F}, \mathrm{P})$, and a sequence of *feedback laws* $\{f_n : n \in \mathbb{Z}_+\}$ such that each $f_n \in \mathcal{B}(\mathbf{X}^n) \otimes \mathcal{F}$. Two special cases are when $f_n(x_0, \ldots, x_n, \omega) = g_n(\omega)$, which is a typical feedback law if the only goal is to identify unknown parameters, and $f_n(x_0, \ldots, x_n, \omega) = g_n(x_0, \ldots, x_n)$, which is typical in optimal control. Finally, to "start up the system", we need a probability μ_0 on $\mathcal{B}(\mathbf{X})$ which will serve as an initial condition distribution for x_0.

To summarize, we have defined the following objects:

(P, Ψ)	a state space system;
μ_0	an initial condition distribution;
$\{f_n : n \in \mathbb{Z}_+\}$	a set of feedback control laws;
$(\Omega, \mathcal{F}, \mathrm{P})$	a probability space.

Using these we now construct the input-state-output process on the probability space $(\Omega_{\mathrm{x}}, \mathcal{F}_{\mathrm{x}}, \mathrm{P}_{\mu_0})$ where

$$\Omega_{\mathrm{x}} \triangleq \mathbf{X}^{\mathbb{Z}_+} \times \Omega, \qquad \text{and} \qquad \mathcal{F}_{\mathrm{x}} \triangleq \mathcal{B}(\mathbf{X}^{\mathbb{Z}_+}) \otimes \mathcal{F}.$$

The stochastic processes of interest will be defined by specifying the probability P_{μ_0} on $\mathcal{F}_{\mathrm{x}}$. For a set of the form $A_0 \times B \in \mathcal{B}(\mathbf{X}) \otimes \Omega$ we define

$$\mathrm{P}_{\mu_0}\{\omega \in B,\ x_0 \in A_0\} \triangleq \mathrm{P}\{B\}\mu_0\{A_0\}.$$

For $k \geq 1$, a sequence of sets $\{A_i : A_i \in \mathcal{B}(\mathbf{X}), 0 \leq i \leq k\}$ and B as above we define

$$u_{k-1} = f_{k-1}(x_{k-1}, \ldots, x_0, \omega)$$

and the probability $\mathrm{P}_{\mu_0}\{x_k \in A_k, \ldots, x_0 \in A_0, \omega \in B\}$ is defined by

$$\int_{\omega \in B} \int_{x_0 \in A_0} \cdots \int_{x_{k-1} \in A_{k-1}} \mathrm{P}(d\omega)\mu_0(dx_0)P(x_0; f_0(x_0, \omega), dx_1) \cdots P(x_{k-1}; f_{k-1}(x_{k-1}, \ldots, x_0, \omega), A_k)$$

These equations define a consistent set of finite dimensional distributions on $\mathcal{F}_{\mathrm{x}}$, and hence define a probability P_{μ_0} on $\mathcal{F}_{\mathrm{x}}$ (see [Doob, 1953].) Finally, we may define

$$y_k \triangleq \Psi(x_k, u_k) \qquad \text{for all } k \in \mathbb{Z}_+.$$

Observe that if the feedback laws $\{f_k : k \in \mathbb{Z}_+\}$ are independent of k, and depend only on the present state so that $u_k = f(x_k)$, $k \in \mathbb{Z}_+$, then the state process $\mathbf{x}$ becomes a Markov process with Markov transition function Q given by

$$Q(x, A) = P(x; f(x), A) \qquad \text{for } x \in \mathbf{X} \text{ and } A \in \mathcal{B}(\mathbf{X}).$$

This definition of a state space system is general enough to model almost any time invariant stochastic system in which the disturbance processes are assumed to be i.i.d.* As an example, consider the linear state space system

$$x_{k+1} = Ax_k + Bu_k + Gw_{k+1}, \tag{1.36)(i}$$

$$y_k = Cx_k + Du_k + Hv_{k+1}, \qquad k \in \mathbb{Z}_+, \tag{ii}$$

where A, B, C, D, G, and H are real matrices, and the joint process $\binom{\mathbf{w}}{\mathbf{v}}$ is i.i.d.. Letting $X_k \triangleq \binom{x_k}{y_k}$ we have

$$X_{k+1} = F(X_k, u_k, v_{k+1}, w_{k+1})$$

for some continuous function F, and hence letting

$$P(X; u, A) \triangleq \int\int \mathbf{1}_{\{F(X,u,v,w) \in A\}}\, \nu(dv)\mu(dw), \tag{1.37}$$

---

* The time varying case may be modeled in an analogous way using a sequence $\{P_k : k \in \mathbb{Z}_+\}$ of controlled Markov transition functions, and defining $P_{k-1}(x_{k-1}; u_{k-1}, \cdot)$ to be the distribution of x_k given x_{k-1} and u_{k-1}.

where ν and μ are the distributions of v_k and w_k respectively for $k \in \mathbb{Z}_+$, it is easy to verify that P is a controlled Markov transition function which is equivalent to the system description (1.36). Similarly, the ARMAX system (1.33) may be modeled using a controlled Markov transition function with state process

$$x_{k-1}^{\top} \triangleq (y_{k-1}, \cdots, y_{k-n_1}, u_{k-1}, \cdots, u_{k-n_2}, w_{k-1}, \cdots, w_{k-n_3}), \qquad k \geq 1.$$

We now show how the results of Part I may be applied to the stability analysis of stochastic state space systems operating under feedback.

1.7 Stability and Optimal Control

A number of stability criteria are available for stochastic state space systems operating under feedback of the general form described above. Below we summarize a few reasonable choices. In this thesis we will be concerned exclusively with the *infinite horizon* control problem. Hence, the stability criteria presented below evaluate the long run performance of the closed loop system.

(*i*) We say a control law is *mean square stabilizing* if for each initial condition $x \in \mathbf{X}$, and for some bound $\gamma_x^2 > 0$ the familiar performance criterion J_k satisfies

$$\begin{aligned} J_\infty = \limsup_{k\to\infty} J_k &\triangleq \limsup_{k\to\infty} \mathrm{E}_x \left[\|y_k\|^2 + \rho \|u_k\|^2 \right] \\ &\leq \gamma_x^2. \end{aligned} \tag{1.38}$$

(*ii*) A natural sample path analogue of the above is the following: A control law is called *sample mean square* (s.m.s) *stabilizing* if the sample path criterion

L_∞ satisfies for each initial condition $x \in \mathbf{X}$,

$$L_\infty = \limsup_{N\to\infty} L_N$$
$$\triangleq \limsup_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \|y_k\|^2 + \rho\|u_k\|^2 \in L^1(\mathbf{X}^{\mathbf{z}_+}, \mathcal{B}(\mathbf{X}^{\mathbf{z}_+}), \mathrm{P}_x). \quad (1.39)$$

That is, $\mathrm{E}_x[L_\infty] < \infty$.

We will also say that a feedback law is *L^p-stabilizing* for $p > 1$ if (1.38) holds with 2 replaced by p.

All of the stochastic systems treated in this thesis will assumed to be *Markovianizable* under feedback. That is, for some Markov chain $\mathbf{\Phi}$ evolving on a state space $\mathbf{X} \subset \mathbb{R}^M$, and continuous functions $u: \mathbf{X} \to \mathbf{U}$, and $y: \mathbf{X} \to \mathbf{Y}$ the output processes $\mathbf{u}$ and $\mathbf{y}$ have the form

$$u_k = u(\Phi_k), \qquad \text{and} \qquad y_k = y(\Phi_k), \qquad k \in \mathbb{Z}_+, \qquad (1.40)$$

and in this case $\mathbf{\Phi}$ will be called the *Markov state process*. For example, if (P, Ψ) is a state space system, and the control u_k is chosen to be a continuous function of the present state; $u_k = u(x_k)$ for all $k \in \mathbb{Z}_+$, then the closed loop system is Markovianizable with Markov state process $\mathbf{x}$.

In many cases, the mean square or s.m.s. stability of a Markovianizable stochastic system depends on the stability of the Markov state process $\mathbf{\Phi}$. Here we introduce two useful stability criteria for Markov chains:

(*iii*) A Markov chain $\mathbf{\Phi}$ is called *stable in probability* if for each initial condition $\Phi_0 = x \in \mathbf{X}$ and $\varepsilon > 0$, there exists a compact set $C \subset \mathbf{X}$ such that

$$\limsup_{k\to\infty} \mathrm{P}_x\{\Phi_k \in C^c\} < \varepsilon; \qquad (1.41)$$

(*iv*) The Markov chain Φ is called *stable in probability on average* if for each initial condition $\Phi_0 \in \mathbf{X}$ and $\varepsilon > 0$, there exists a compact set $C \subset \mathbf{X}$ such that

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \mathrm{P}_x \{\Phi_k \in C^c\} < \varepsilon. \tag{1.42}$$

These four stability criteria are closely related for a Markovianizable system with Markov state process Φ. In particular, (*iii*) implies (*iv*), and in many instances, (*i*) and (*ii*) each imply (*iv*). In Chapter II we will find that if Φ satisfies a condition known as *local stochastic controllability* then (*iii*) and (*iv*) are equivalent.

Here we state a necessary and sufficient condition for the last form of stability.

Proposition 1.7.1. *The Markov chain Φ is stable in probability on average if and only if for each deterministic initial condition $\Phi_0 = x \in \mathbf{X}$ there exists an invariant probability π_x such that for all $f \in \mathbf{C}$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \int P^k(x,dy) f(y) = \int f(y)\, \pi_x(dy) \tag{1.43}$$

or, equivalently,

$$\frac{1}{N} \sum_{k=1}^{N} \mathrm{U}^k \delta_x \xrightarrow{\text{weakly}} \pi_x.$$

Proof.

This follows from Theorem 1.3.1 because the Markov process Φ has the Feller property and (1.42) is equivalent to the existence of a moment. □

In the following example we show how the ergodic theory of Markov chains described in Part I above may be applied to the optimal control of stochastic state space systems, and exhibit the close connection between the four stability criteria presented above.

Consider the stochastic state space system defined by the recursion

$$x_{k+1} = \alpha x_k + u_k + w_{k+1}, \qquad k \in \mathbb{Z}_+, \tag{1.44}$$

where the output process $\mathbf{y} = \mathbf{x}$. We assume that $-1 < \alpha < 1$, the initial condition x_0 is independent of $\mathbf{w}$, $\mathbf{w}$ is an i.i.d. stochastic process on $\mathbb{R}$, and for all $k \in \mathbb{Z}_+$, $w_k \sim \mu_w$ where μ_w is the uniform distribution on $[-1,1]$. Hence μ_w possess the density $p_w(x) \triangleq 1/2\mathbf{1}_{[-1,1]}(x)$. The control u_k will be a continuous function of x_k, and hence each feedback law in this class generates a Feller Markov chain $\mathbf{x}$.

Our objective is to minimize the sample mean square criterion function

$$L_\infty \triangleq \limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} y_k^2 + \rho u_k^2. \tag{1.45}$$

When $\rho = 0$ the unique solution to this problem is to set $u_k = -\alpha x_k$, and in this case the Markov chain $\mathbf{x}$ becomes an i.i.d. stochastic process (for $k \geq 1$) with invariant probability μ_w.

However, if $\rho \neq 0$, or because of saturation considerations or imperfect knowledge of the parameter α, the feedback control might be chosen to be of the form $u_k = g(x_k)$ where g is continuous. Suppose now that the resulting Markov chain $\mathbf{x}$ is stable in probability. Then Proposition 1.7.1 implies that for each $x \in \mathbf{X}$ there exists an invariant probability π_x such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \mathrm{E}_x[f(x_k)] = \int f\, d\pi_x, \tag{1.46}$$

for every $f \in C$. Hence for any continuous positive function $f: \mathbf{X} \to \mathbb{R}_+$ we have for every $m > 0$,

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \mathrm{E}_x[f(x_k)] \geq \liminf_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \mathrm{E}_x[f \wedge m(x_k)]$$
$$= \int f \wedge m \, d\pi_x,$$

and by the monotone convergence theorem this implies that

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \mathrm{E}_x[f(x_k)] \geq \int f \, d\pi_x, \tag{1.47}$$

which in particular implies that the mean square stability criterion J_∞ may be bounded from below using the invariant probability π_x.

Furthermore, it may be verified that by the assumptions made on μ_w, and since the control law u is continuous, the Markov state process $\mathbf{x}$ generated by the system (1.44) is locally stochastically controllable (see Chapter II.) By Proposition 2.2.4 of Chapter II, for each $x \in \mathbf{X}$ the limit

$$L_\infty = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} y_k^2 + \rho u_k^2$$

exists, and in fact the expectation of L_∞ may be computed using the invariant probability π_x:

$$\mathrm{E}_x[L_\infty] = \int (t^2 + \rho u(t)^2)\, \pi_x(dt). \tag{1.48}$$

Hence the control g is s.m.s. stabilizing if and only if (1.48) is finite for every $x \in \mathbf{X}$.

This example illustrates how feedback laws may be evaluated by an analysis of the invariant probabilities that they generate. For example, if $\rho = 0$ and the optimal feedback law $u_k = -\alpha x_k$ which makes $x_k \sim \mu_w$ for $k \geq 1$ is not available, then a reasonable alternative is to attempt to find an admissible control which makes the resulting invariant probability π_x *close* to μ_w for all $x \in \mathbf{X}$.

Let us now restrict our attention to linear control laws of the form $u_n = -\mathrm{k}x_n$. We will find that the stability criterion functions J_∞ and L_∞ take on a simple form in this special case. If $\delta \stackrel{\Delta}{=} |\alpha - \mathrm{k}| < 1$ then the closed loop system is stable in probability, and using the results to be presented in Chapters II and III we may show that in this case $\mathbf{x}$ is an aperiodic, positive Harris recurrent Markov chain. This example is studied in [Athreya and Pantula, 1986] where it is shown that the Markov process $\mathbf{x}$ is also uniformly mixing.

Hence, there exists a unique invariant probability π such that for every initial distribution μ_0 for x_0, the resulting trajectory $\{\mu_k : k \in \mathbb{Z}_+\}$ of probabilities governing the state process converge to π in total variation norm:

$$\lim_{k\to\infty} \sup_{B\in\mathcal{B}(\mathbf{X})} |\mathrm{P}_{\mu_0}\{x_k \in B\} - \pi\{B\}| = 0,$$

and for every $x \in \mathbf{X}$ the costs J_∞ and L_∞ may be computed using the invariant probability π;

$$J_\infty = L_\infty = \int (t^2 + \rho u(t)^2)\, \pi(dt). \tag{1.49}$$

We will now compute L_∞ for each $\mathrm{k} \in [0,\alpha]$ to find the optimal control law in this class. It is easily verified that with

$$\xi \stackrel{\Delta}{=} \sum_{n=0}^{\infty} (\alpha - \mathrm{k})^n w_n,$$

the distribution of the random variable ξ is the invariant probability π. Hence by (1.49), for each $k \in [0,\alpha]$ the s.m.s. criterion function becomes

$$L_\infty = \mathrm{E}[\xi^2(1 + \rho\mathrm{k}^2)] = \sigma_w^2 \left[\frac{(1 + \rho\mathrm{k}^2)}{1 - (\alpha - \mathrm{k})^2}\right], \tag{1.50}$$

where $\sigma_w^2 = \mathrm{E}[w_0^2]$. The optimal linear control law k^* may therefore be computed by setting the derivative of L_∞ with respect to k equal to zero, and solving a cubic polynomial equation.

For general continuous control laws we do not have an explicit description of the invariant probability π_x, and in this general case computing the s.m.s. cost L_∞ for a given control may not be possible. However in many cases useful bounds are available as is illustrated in the examples studied in Chapters IV and V.

1.8 Stochastic Adaptive Control

Over the past ten years there has been extensive study of the stochastic adaptive control of ARMAX systems of the form (1.33) where the parameter process takes on different forms in different papers. In [Goodwin, Ramadge and Caines, 1981], [Goodwin, Sin, 1982], and [Kumar, Praly, 1985] the parameters $A^k(z), B^{(k)}(z)$ and $C^{(k)}(z)$ are not time dependent, while in [Chen, Caines, 1985] these parameters are the sum of a bounded martingale difference process and an unknown constant value. In each of these papers the objective is to s.m.s. stabilize the system, and minimize the s.m.s. performance criterion L_∞.

As an example, consider the control algorithm in [Goodwin, Ramadge and Caines, 1981]; this treats the system model given in equation (1.33) with all of the parameters taken to be constant. Further, for simplicity, we take the delay $d = 1$, all processes to be scalar, and the reference signal $\mathbf{y}^*$ to be zero. Then the regulation algorithm of (Goodwin, Ramadge, Caines, 1981) is given by

$$\widehat{\theta}_k = \widehat{\theta}_{k-1} + r_{k-1}^{-1}\varphi_{k-1}y_k$$

$$r_k^{-1} = \frac{r_{k-1}^{-1}}{1 + r_{k-1}^{-1}\|\varphi_k\|^2}, \qquad k \in \mathbb{Z}_+$$

where

$$\varphi_{k-1} \triangleq (y_{k-1}, \cdots, y_{k-n_1}, u_{k-1}, \cdots, u_{k-n_2}),$$

φ_0 is given as initial condition, and u_k is computed by setting

$$\varphi_k^\top \hat{\theta}_k = 0.$$

Observe that with $\Phi_k \in \mathbb{R}_+ \times \mathbb{R}^{2(n_1+n_2)+n_3}$ defined as

$$\Phi_k \triangleq \begin{pmatrix} r_k^{-1} \\ \varphi_k \\ \hat{\theta}_k \\ w_k \\ \vdots \\ w_{k-n_3+1} \end{pmatrix} \tag{1.51}$$

Φ is of the form

$$\Phi_{k+1} = S(\Phi_k, w_k)$$

where S is a Borel measurable function on $\mathbb{R}_+ \times \mathbb{R}^{2(n_1+n_2)+n_3}$ (because it is continuous there) and hence the closed loop system is Markovianizable.

In each of the papers cited above a method based on Neveu's version of the martingale convergence Theorem (see [Neveu, 1975]) is applied to establish the s.m.s. stability of (1.33) under the appropriate hypotheses. A $\mathcal{F}_k$-adapted *stochastic Lyapunov function* V_k is introduced which has the super martingale property

$$\mathbf{E}[V_{k+1} \mid \mathcal{F}_k] \leq V_k + \alpha_k - \beta_k; \qquad k \in \mathbb{Z}_+,$$

where the random variables α_k and β_k are functionals of the sequence φ and are almost surely positive, and $\sum_0^\infty \alpha_k < \infty$. It follows that $V_k \longrightarrow V_\infty$ *a.s.* as $k \to \infty$, and that $\sum_0^\infty \beta_k < \infty$.

Using these facts it is shown in [Goodwin, Ramadge, Caines, 1981] and [Goodwin, Sin, 1982] that the system is s.m.s. stable, and that for some $C > 0$ and all initial conditions $\Phi_0 = x \in \mathbf{X}$,

$$L_\infty = \limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} y_k^2 + \rho u_k^2 < C,$$

and

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} y_k^2 = \sigma_w^2$$

where $\sigma_w^2 \triangleq \mathrm{E}[w_k^2]$.

Until recently, the super-martingale technique has been the principle tool available for the stability analysis of adaptive control laws. It is limited because the job of finding a suitable super-martingale becomes extremely difficult if the dynamics of the process $\theta = \{\theta_n, n \geq 1\}$ become more complicated than, for instance, those treated in [Chen, Caines, 1985].

In the remainder of this thesis we investigate a new approach to the stability analysis of stochastic systems based upon the ergodic theory of Markov chains.

Chapter 2 NOISE CONTROLLABILITY

2.1 Introduction

Recently, there has been considerable interest in applying the ergodic theory of Markov chains to the analysis of stochastic control systems (see for example [Meyn, Caines, 1987], [Kumar, 1983], and [Kushner, 1971].) In Chapter 1 we showed that if a Markov state may be constructed for the controlled output process then subject to technical conditions which include stability of the Markovian state process one may deduce (amongst other facts) (*i*) the existence of an invariant probability π for the process and (*ii*) the convergence almost surely of the sample averages of a function of the state process (and of its expectation) to its conditional expectation $[\pi]$ with respect to a sub-σ-field of invariant sets Σ_I. One of the drawbacks to this approach is that many of the desired ergodicity properties hold only when the initial condition lies in a set of full measure with respect to the invariant probability π. The goal of generalizing these results to arbitrary initial conditions is one of the major objectives of this thesis.

One solution to this problem is to search for an irreducibility measure for the state process Φ and apply the theory of irreducible Markov chains (see [Revuz, 1975] and [Nummelin, 1984].) The major drawbacks to this approach are that finding an irreducibility measure can be a formidable task, to obtain useful results a proof that the Markov chain satisfies a recurrence condition is needed and furthermore, stochastic systems do not possess irreducible Markov state processes in general. The approach which we introduce in this chapter is based on the concept of *controllability* from linear system theory. The task of finding an irreducibility measure and verifying a recurrence condition will be replaced by a computation of the rank of a controllability matrix, and a proof that the Markov state process is stable in probability on average.

To motivate the discussion and definitions that follow, consider the Gaussian Markov process $\mathbf{\Phi}$ generated by the recursion

$$\Phi_{k+1} = A\Phi_k + Bw_{k+1} \tag{2.1}$$

where A and B are respectively $n \times n$ and $n \times p$ matrices, $\mathbf{w} = \{w_k : k \geq 1\}$ is an i.i.d. Gaussian stochastic process on $\mathbb{R}^p$ with $w_k \sim N(0, I)$ for all k, and the deterministic initial condition $\Phi_0 \in \mathbb{R}^n$ is given.

Suppose that the eigenvalues of A fall strictly within the unit circle in $\mathbb{C}$. Then many of the asymptotic properties of (2.1) are determined by its unique invariant probability π. The probability π is Gaussian with zero mean and covariance matrix F where F is the unique solution to the Lyapunov equation

$$F = AFA^T + BB^T.$$

If the pair (A, B) is *controllable* then an analysis of the asymptotic properties of $\mathbf{\Phi}$ is straight forward. It may be verified that in this case the matrix F is positive definite and it follows that $\mathbf{\Phi}$ is a positive Harris recurrent Markov chain. Hence for example, if f is any positive Borel function on $\mathbb{R}^n$, then for every initial condition $\Phi_0 = x \in \mathbb{R}^n$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(\Phi_k) = \int f\, d\pi \quad a.s. \ [\mathrm{P}_x], \tag{2.2(i)}$$

and by a simple computation,

$$\lim_{k\to\infty} \mathrm{E}_x[f(\Phi_k)] = \int f\, d\pi. \tag{ii}$$

Hence if (2.1) describes a stochastic system operating under feedback, and f is a loss function on the state process Φ, then by (2.2) the infinite horizon performance is determined by the invariant probability π.

On the other hand if (A,B) is not controllable then Φ is *not* Harris in general, and this and other technicalities prevent (2.2) from holding for such a general class of functions. Because the covariance matrix F is not full rank in this case, the invariant probability π is supported on a hyperplane $L \subset \mathbb{R}^n$ whose dimension is strictly less than n. Hence (2.2) may not hold unless f is continuous on L. To establish (2.2) even for continuous functions requires extensive exploitation of the linear structure of (2.1). Our objective in this chapter is to generalize the notion of controllability to analyse nonlinear stochastic systems operating under feedback.

In Section 2 the concept of local stochastic controllability is introduced. It is shown that for locally stochastically controllable systems the concepts of stability in probability, and stability in probability on average are equivalent, and that such systems exhibit very regular asymptotic behavior. For example, averages of functions of the state process converge for *every* initial condition for locally stochastically controllable systems which are stable in probability. For locally stochastically controllable systems it is shown that if the closed loop system generating Φ is stable in probability, then for every initial distribution $\mu_0 \in \mathcal{M}$, the resulting trajectory $\{\mu_k : k \in \mathbb{Z}_+\}$ converges in total variation norm to a convex combination of periodic orbits. Furthermore, if the system is stable in probability and there is exactly one invariant probability π, then the Markov chain $\mathbf{\Phi}$ is positive Harris recurrent. Hence, the probabilities $\{\mu_k : k \in \mathbb{Z}_+\}$ governing the state process either converge to π, or to a periodic orbit consisting of weighted averages of restrictions of the invariant probability to cyclical sets.

2.2 Locally Stochastically Controllable Systems

In this chapter we consider input-output stochastic systems possessing Markovian realizations of the form

$$\Phi_{k+1} = F(\Phi_k, w_{k+1}), \qquad k \in \mathbb{Z}_+, \tag{2.3}$$

where for all k, $\Phi_k \in \mathbf{X}$ = an open subset of $\mathbb{R}^n$, $w_k \in \mathbb{R}^p$, and $F : \mathbf{X} \times \mathbb{R}^p \to \mathbf{X}$ is continuous.

To complete the description of the state process $\mathbf{\Phi}$ we assume that the initial condition Φ_0 and the disturbance process $\mathbf{w}$ satisfy

A1 $(\Phi_0, \mathbf{w})$ are Borel random variables on the probability space $(\Omega, \mathcal{F}, \mathrm{P}_{\Phi_0})$;

A2 Φ_0 is independent of $\mathbf{w}$;

A3 $\mathbf{w}$ is an independent and identically distributed (i.i.d) process;

and we will occasionally assume:

A4 There exists an open set $\mathcal{O}_w \subset \mathbb{R}^p$ such that the distribution μ_w of w_k, $k \in \mathbb{Z}_+$, is equivalent to Lebesgue measure on $\mathcal{O}_w$. (We say two measures μ and

ν are *equivalent* if for all $N \in \mathcal{B}(\mathbf{X})$, $\mu\{N\} = 0 \iff \nu\{N\} = 0$ and this shall be written $\mu \approx \nu$.)

Assumption A4 is satisfied when the distribution of w_k possesses a continuous density. Markovian systems of this form will be obtained from stochastic state space systems of the form introduced in Chapter I by the choice of time invariant feedback control laws. For example, the Markov state process Φ defined in equation (1.51) is of this general form. In order to obtain the ergodic properties of interest for (2.3) it will, of course, be necessary to verify that each particular feedback law generates a system satisfying the appropriate hypotheses.

The *state readout map* $S_x^k : \mathbb{R}^{kp} \to \mathbf{X}$ of the system (2.3) is defined inductively for $k \in \mathbb{Z}_+$ and $\mathbf{z} \doteq (z_1, \ldots, z_k)^\top \in \mathbb{R}^{pk}$ by

$$\begin{aligned} S_x^k &= F\left(S_x^{k-1}(z_1, \cdots, z_{k-1}), z_k\right), \quad k \geq 1, \\ S_x^0 &\doteq x. \end{aligned}$$

The state readout map is so named because for all $k \geq 1$, $\Phi_k = S_x^k(w_1, \ldots, w_k)$ when $\Phi_0 = x$.

We now introduce a notion of stochastic controllability:

Definition. The system (2.2) is called *locally stochastically controllable* if there exists $T \in \mathbb{Z}_+$ such that for each initial condition $x \in \mathbf{X}$ there exists an open set $O_x \subset \mathbb{R}^p$ for which the distribution of the random variable $\Phi_T \doteq S_x^T(w_1, \ldots, w_T)$ is equivalent to Lebesgue measure on O_x.

□

One consequence of this definition may be roughly described as follows: If (2.3) is locally stochastically controllable, and if starting at a point $x \in \mathbf{X}$ it is possible to reach a point $y \in \mathbf{X}$ at some time $k \geq T$, then at time k all points in some neighborhood of y are reachable from x. The terminology may also be motivated by the fact that if $F : \mathbf{X} \times \mathbb{R}^p \to \mathbf{X}$ is linear then the notions of local stochastic controllability, and controllability in the usual sense are equivalent.

Figure 2.1 below illustrates the evolution of the underlying distributions governing a locally stochastically controllable state process $\mathbf{\Phi}$.

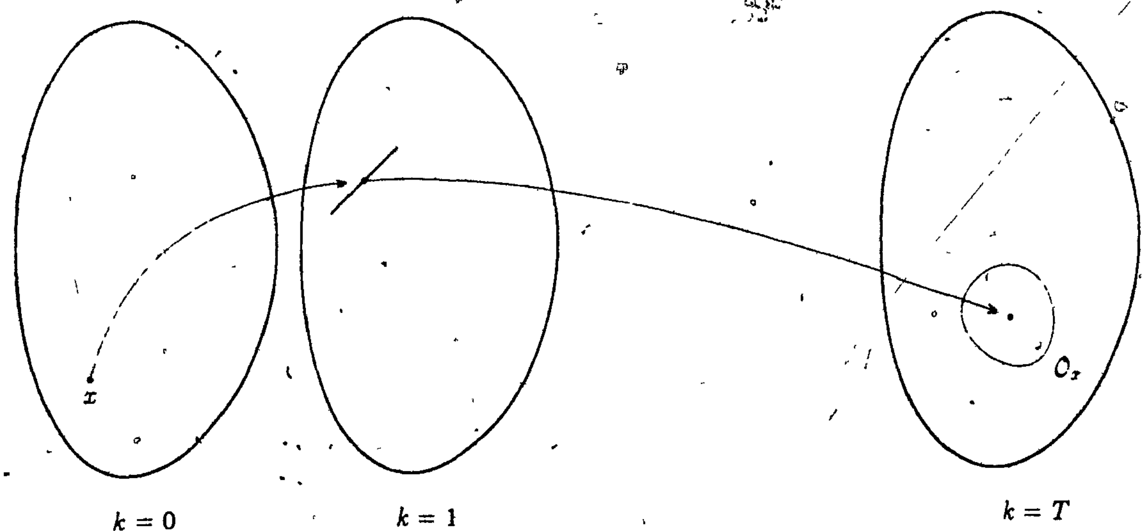


Figure 2.1 Local Stochastic Controllability

Here we give a sufficient condition for local stochastic controllability. For $y \in \mathbf{X}$ and a sequence $\{z_k : z_k \in \mathbb{R}^p,\ k \in \mathbb{Z}_+\}$ let $\{\mathrm{A}_k,\ \mathrm{B}_k,\ k \in \mathbb{Z}_+\}$ denote the matrices

$$\mathrm{A}_k \triangleq \left[\frac{\partial F}{\partial x}\right]_{(S_y^k, z_{k+1})} \quad \text{and} \quad \mathrm{B}_k \triangleq \left[\frac{\partial F}{\partial z}\right]_{(S_y^k, z_{k+1})}$$

Proposition 2.2.1. *The system (2.3) is locally stochastically controllable if $F: \mathbf{X} \times \mathbb{R}^p \to \mathbf{X}$ is a continuously differentiable (C^1) function, and for all initial conditions $x \in \mathbf{X}$ there exists some $T \geq 1$ such that the generalized controllability matrix*

$$C_T = C_T(x, z_1, \ldots, z_T) \triangleq [A_{T-1} \cdots A_1 B_0 | A_{T-1} \cdots A_2 B_1 | \cdots\cdots | A_{T-1} B_{T-2} | B_{T-1}] \tag{2.4}$$

is full rank for all sequences $(z_1, \ldots, z_T) \in \mathcal{O}_w^T \setminus Z$ where $Z \subset \mathbb{R}^{pT}$ has zero Lebesgue measure.

We remark that if F is of the form

$$F(x, z) = Ax + Bz$$

then the generalized controllability matrix becomes the familiar controllability matrix

$$\left[A^{T-1}B | A^{T-2}B | \cdots | AB | B\right].$$

Note that all quantities in the matrix (2.4) are deterministic.

Proof.

To prove Proposition 2.2.1 we will need the following lemma:

Lemma 2.2.1. *Let $\mathcal{U}_1 \subset \mathbb{R}^m$ and $\mathcal{V}_1 \subset \mathbb{R}^n$ be open and suppose $G: \mathcal{U}_1 \times \mathcal{V}_1 \to \mathbb{R}^n$, $(x, y) \to z$, is C^1, and that the matrix $\frac{\partial G}{\partial y}$ is full rank at some $(x_0, y_0) \in \mathcal{U}_1 \times \mathcal{V}_1$. Then there exists an open set $\mathcal{U} \times \mathcal{V} \subset \mathcal{U}_1 \times \mathcal{V}_1$ containing (x_0, y_0) and an open set $\mathcal{O} \subset \mathbb{R}^n$ such that for any strictly positive Borel function $p: \mathcal{U} \times \mathcal{V} \to (0, \infty)$ the measure ν defined for $A \in \mathcal{B}(\mathbb{R}^n)$ by*

$$\nu\{A\} = \int_{\mathcal{U}} \int_{\mathcal{V}} \mathbf{1}_{G(x,y) \in A}\, p(x, y)\, dx\, dy \tag{2.5}$$

is equivalent to Lebesgue measure on $\mathcal{O}$.

Proof.

Consider the function $G^\star : \mathcal{U}_1 \times \mathcal{V}_1 \to \mathbb{R}^m \times \mathbb{R}^n$ defined for $(x,y) \in \mathcal{U}_1 \times \mathcal{V}_1$ by

$$G^\star(x,y) \triangleq \begin{pmatrix} x \\ G(x,y) \end{pmatrix}.$$

Under the conditions of Lemma 2.2.1 the function $G^\star$ is C^1 and its derivative is full rank at (x_0, y_0). By the implicit function theorem there exist open sets $\mathcal{W} \subset \mathbb{R}^m \times \mathbb{R}^n$, $\mathcal{U} \times \mathcal{V} \subset \mathcal{U}_1 \times \mathcal{V}_1$ with $(x_0, y_0) \in \mathcal{U} \times \mathcal{V}$ and a C^1 function $H^\star : \mathcal{W} \to \mathcal{U} \times \mathcal{V}$ such that

$$\mathcal{W} = \left\{ \begin{pmatrix} x \\ G(x,y) \end{pmatrix} : x \in \mathcal{U},\ y \in \mathcal{V} \right\} \qquad \text{and} \quad H^\star(G^\star(x,y)) = (x,y)$$

for $(x,y) \in \mathcal{U} \times \mathcal{V}$. Applying a projection to the function $H^\star$ we may find a C^1 function $H : \mathcal{W} \to \mathcal{V}$ for which

$$H(x, G(x,y)) = H(G^\star(x,y)) = y$$

for $(x,y) \in \mathcal{U} \times \mathcal{V}$.

We now construct a density for the measure ν by a change of variables and Fubini's Theorem:

$$\begin{aligned} \nu\{A\} &= \int_{\mathcal{U}} \left\{ \int_{G(x,\mathcal{V})} \mathbf{1}_{\{z \in A\}}\, p(x, H(x,z))\, \left|\det \frac{\partial H}{\partial z}\right| dz \right\} dx \\ &= \int_{\mathbb{R}^n} \mathbf{1}_{\{z \in A\}} \left\{ \int_{\mathbb{R}^m} \mathbf{1}_{\{(x,z) \in \mathcal{W}\}}\, p(x, H(x,z))\, \left|\det \frac{\partial H}{\partial z}\right| dx \right\} dz \end{aligned}$$

Let $p_1 : \mathbb{R}^n \to \mathbb{R}_+$ be the density defined for $z \in \mathbb{R}^n$ by

$$p_1(z) \triangleq \int_{\mathbb{R}^m} q(x,z)\, dx$$

where $q(x,z) \triangleq \mathbf{1}_{\{(x,z) \in \mathcal{W}\}}\, p(x, H(x,z))\, |\det \frac{\partial H}{\partial z}|$. Observe that $\{(x,z) : q(x,z) > 0\} = \mathcal{W}$ and hence defining the open set $\mathcal{O} \subset \mathbb{R}^n$ by

$$\mathcal{O} \triangleq \{z : q(x,z) > 0 \quad \text{for some} \quad x \in \mathcal{U}\} = \{G(x,y) : x \in \mathcal{U},\ y \in \mathcal{V}\}$$

it easily follows that $\{z : p_1(z) > 0\} = \mathcal{O}$. Hence ν is equivalent to Lebesgue measure on the open set $\mathcal{O}$ and this completes the proof.

□

We now prove Proposition 2.2.1. For a set $A \in \mathcal{B}(\mathbf{X})$, and a measure μ we let $\mathbf{1}_A\mu$ denote the measure defined for $B \in \mathcal{B}(\mathbf{X})$ by

$$(\mathbf{1}_A\mu)\{B\} = \mu\{A \cap B\}.$$

Fix a point $y \in \mathbf{X}$. By Lemma 2.2.1 we can cover $\mathcal{O}_w \times \cdots \times \mathcal{O}_w \setminus Z$ by a countable union of open rectangles,

$$B^i \stackrel{\Delta}{=} B_1^i \times \cdots \times B_T^i, \qquad i \in \mathbb{Z}_+,$$

where for each $i \in \mathbb{Z}_+$, the distribution ν_i defined by

$$\nu_i\{A\} \stackrel{\Delta}{=} \int_{B_1^i} \cdots \int_{B_T^i} \mathbf{1}_{S_y^T(\lambda_1 \ldots \lambda_T) \in A}\, \mu_w(d\lambda_1) \cdots \mu_w(d\lambda_T)$$

is equivalent to Lebesgue measure on an open set $\mathcal{O}_y^i \subset \mathbf{X}$. Set $\mathcal{O}_y \stackrel{\Delta}{=} \bigcup_{i=0}^{\infty} \mathcal{O}_y^i$. Then $\sum_{i=0}^{\infty} \nu_i$ is equivalent to Lebesgue measure on $\mathcal{O}_y$. Furthermore, since

$$\sum_{i=0}^{\infty} \nu_i\{\,\cdot\,\} \geq P^T(y, \cdot), \qquad \text{and } P^T(y, \cdot) \geq \nu_i \qquad \text{for all } \quad i \in \mathbb{Z}_+,$$

it follows that $P^T(y, \cdot) \approx \sum_{i=0}^{\infty} \nu_i$. Hence, $P^T(y, \cdot) \approx \mathbf{1}_{\mathcal{O}_y}\mu^{Leb}$ and this proves the proposition.

□

2.2.1 Invariant Probabilities

We now investigate the invariant probabilities of locally stochastically controllable systems. In the lemma below we establish an important property of the invariant probabilities of such systems which will be used to establish Propositions 2.2.2 and 2.2.3 below.

Lemma 2.2.2. *If (2.3) is locally stochastically controllable then for any invariant measure π there exists an open set $\mathcal{W}$ such that*

$$\pi \approx \mathbf{1}_{\mathcal{W}}\mu^{Leb},$$

that is, π is equivalent to Lebesgue measure on the open set $\mathcal{W}$.

Proof.

Let S denote the support of the invariant measure π (this is often denoted $\mathrm{supp}\,\pi$). That is, $y \in S$ if for every open set $\mathcal{U} \subset \mathbf{X}$ containing y, $\pi\{\mathcal{U}\} > 0$. To prove the lemma we first show that $\mathcal{O}_y \subset S$ for every $y \in S$ where $\mathcal{O}_y$ is the open set used in the definition of local stochastic controllability.

Let $y \in S$, $z \in \mathcal{O}_y$, and let $\mathcal{U} \subset \mathbf{X}$ be any open set containing z. By the Feller property $P^T(\cdot, \mathcal{U})$ is lower semi-continuous (see [Billingsley, 1968]), and by the definition of $\mathcal{O}_y$, $P^T(y, \mathcal{U}) > 0$. Hence, $P^T(x, \mathcal{U}) > 0$ for all x in an open set $\mathcal{O}_+$ containing y. But since $y \in S$, $\pi\{\mathcal{O}_+\} > 0$, and these facts imply that $\pi\{\mathcal{U}\} > 0$. Since $\mathcal{U}$ is an arbitrary open set containing z, we must have $z \in S$, and since z is an arbitrary element of $\mathcal{O}_y$ this shows that $\mathcal{O}_y \subset S$.

Let $\mathcal{W} \triangleq \bigcup_{x \in S} \mathcal{O}_x$. We have just shown that $\bar{\mathcal{W}} \subset S$ and on the other hand, because

$$\pi\{\cdot\} = \int_S \pi\{dx\} P^T(x, \cdot),$$

we must also have $S \subset \bar{\mathcal{W}}$ and this shows that $S = \bar{\mathcal{W}}$.

We will now show that $\pi \approx \mathbf{1}_{\mathcal{W}} \mu^{Leb}$. To do this we need the following fact: Suppose G is a dense subset of $\mathcal{W}$. Then with $\mathcal{W}_G \triangleq \bigcup_{x \in G} \mathcal{O}_x$,

$$\mu^{Leb}\{\mathcal{W} \setminus \mathcal{W}_G\} = 0. \tag{2.6}$$

To establish this fact observe that if (2.6) does not hold then the open set $\mathcal{W}_0 \triangleq \mathcal{W} \setminus \bar{\mathcal{W}}_G$ is non-empty (this follows because the boundary of an open set has Lebesgue measure zero.) Since $\mathcal{W}_0$ is open, we may use the Feller property to show that $P^T(\cdot, \mathcal{W}_0)$ is positive on an open subset of $\mathcal{W}$, but this contradicts the hypothesis that G is dense in $\mathcal{W}$.

We may now complete the proof of the lemma.

First of all, since $\pi = \mathbf{1}_S \pi$, and $\pi \prec \mu^{Leb}$ we have

$$\pi \prec \mathbf{1}_S \mu^{Leb} = \mathbf{1}_{\mathcal{W}} \mu^{Leb}, \tag{2.7}$$

where the equality in (2.7) follows from the fact that the boundary of an open set in $\mathbf{X}$ has Lebesgue measure zero. To show that $\pi \approx \mathbf{1}_{\mathcal{W}} \mu^{Leb}$ and complete the proof of Lemma 2.2.2 we will now show that $\mathbf{1}_{\mathcal{W}} \mu^{Leb} \prec \pi$. Let $A \subset \mathbf{X}$ be a Borel set for which $\pi\{A\} = 0$. Then since $\bar{\mathcal{W}} = S$ and

$$\int_{\mathcal{W}} \pi(dx) P^T(x, A) = \pi\{A\} = 0,$$

there exists a dense subset $G \subset \mathcal{W}$ with $P^T(x, A) = 0$ for $x \in G$, and hence by local stochastic controllability,

$$\mu^{Leb}\{\mathcal{O}_x \cap A\} = 0, \qquad \text{for } x \in G. \tag{2.8}$$

Using (2.8) and the fact that $\mathcal{W} \setminus \bigcup_{x \in G} \mathcal{O}_x$ has Lebesgue measure zero, it follows that $\mu^{Leb}\{\mathcal{W} \cap A\} = 0$, and this proves the lemma.

□

One very important fact established in the proof of Lemma 2.2.2 is that if $\mathbf{\Phi}$ is locally stochastically controllable and π is an invariant measure which is equivalent to Lebesgue measure on the open set $\mathcal{W}$, then $\mathcal{O}_x \subset \bar{\mathcal{W}}$ for every $x \in \bar{\mathcal{W}}$. This implies that $P^k(x, \cdot) \prec \pi$ for every $x \in \bar{\mathcal{W}}$ and $k \geq T$. Furthermore, we may prove that $\bar{\mathcal{W}}$ is absorbing, and hence the Markov chain $\mathbf{\Phi}$ may be restricted to $\bar{\mathcal{W}}$. This follows from the Feller property: Let $f \in \mathbf{C}$ be any continuous function which vanishes on $\bar{\mathcal{W}}$. Then,

$$0 = \int f \, d\pi = \int \int \pi(dy) P(y, dx) f(x),$$

which shows that $\int P(y,dx)f(x) = 0$ for *a.e.* $[\mu^{Leb}]$ $y \in \bar{W}$, and by continuity it follows that $\int P(y,dx)f(x) = 0$ for every $y \in \bar{W}$. This shows that $P(y,\bar{W}) = 1$ for every $y \in \bar{W}$, and hence $\mathbf{\Phi}$ may be restricted to $\bar{W}$.

Let $\mathcal{I} \subset \mathcal{M}$ dentote the set of all invariant probabilities. $\mathcal{I}$ and $\mathcal{M}$ are obviously convex sets. Using the terminology of [Rosenblatt, 1967] we call an invariant probability $\pi \in \mathcal{I}$ *ergodic* if

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \mathbf{T}^k f = \int f\, d\pi \qquad a.e.\ [\pi]$$

for every $f \in L^1(\mathbf{X}, \mathcal{B}(\mathbf{X}), \pi)$. This is equivalent to the condition that the invariant σ-algebra corresponding to π is trivial. In [Rosenblatt, 1967] it is shown that the set of ergodic probabilities is precisely the set of extreme points of the convex set $\mathcal{I}$: That is, if $\pi \in \mathcal{I}$ is ergodic and

$$\pi = \lambda\pi^1 + (1-\lambda)\pi^2$$

for $\lambda \in [0,1]$ and $\pi^1, \pi^2 \in \mathcal{I}$ then $\lambda = 1$ or $\lambda = 0$, and conversely, if $\pi \in \mathcal{I}$ is an extreme point in $\mathcal{I}$ then it is ergodic.

In Proposition 2.2.2 below we show that when (2.3) is locally stochastically controllable there exists an at most countable collection $\{\pi^k : k \in \mathbb{Z}_+\} \subset \mathcal{I}$ of ergodic probabilities for (2.3). Furthermore, the set of ergodic probabilities in $\mathcal{I}$ (the extreme points) actually generate $\mathcal{I}$, just as the set of extreme points in a compact convex subset $C \subset \mathbb{R}^n$ generate the set C (see the Krein Milman Theorem in [Dunford, Schwarz, 1957].) Using Lemma 2.2.2 we now prove this important result.

Proposition 2.2.2. *Suppose that (2.3) is locally stochastically controllable. Then there exists an at most countable (possibly empty) collection of ergodic probabilities* $\{\pi^i : i \in \mathbb{Z}_+\}$ *and open sets* $\{O^i : i \in \mathbb{Z}_+\}$ *such that:*

(i) $\pi^i \approx \mathbf{1}_{O^i}\,dx$, for $i \in \mathbb{Z}_+$,

(ii) *Any finite invariant measure* π *has the form* $\pi = \sum q_i \pi^i$ *for a summable sequence* $\{q_i : i \in \mathbb{Z}_+\} \subset \mathbb{R}_+$.

Hence, letting

$$O \triangleq \bigcup_{i=0}^{\infty} O^i, \tag{2.9}$$

every finite invariant measure is weaker than Lebesgue measure on O.

Proof.

The proof is similar to the proof of Theorem D of Chapter V of [Foguel, 1969]. We first show that the invariant sigma field of any invariant probability is atomic. If this is not the case then there exists an invariant probability π which is equivalent to Lebesgue measure on an open set $\mathcal{U}$ and contains no atoms.

Fix $B \subset \mathsf{X}$ open with compact closure such that $\mathcal{U} \cap B \neq \phi$. Since π has no atoms we may construct a decreasing sequence of invariant sets $\{A_k : k \in \mathbb{Z}_+\}$ such that $A_k \cap B \neq \phi$ and $0 < \pi\{A_k\} < 2^{-k}$ for all $k \in \mathbb{Z}_+$. By local stochastic controllability we may assume that for each k, A_k is open. Let

$$x \in \bigcap_{k=1}^{\infty} \overline{A_k \cap B},$$

such a point exists by compactness, and since A_k is invariant we may show that $O_x \subset \bar{A}_k$ for each $k \in \mathbb{Z}_+$. However, $\bigcap_{k\geq 0} \bar{A}_k \subset \bar{\mathcal{U}}$ has π-measure zero and hence Lebesgue measure zero. We conclude that $O_x = \phi$ which is impossible.

This implies that every invariant probability is a weighted average of ergodic invariant probabilities supported on disjoint open sets. Since there can be no more than a countable number of disjoint open subsets in $\mathbf{X}$ the proposition is proved.

□

2.2.2 Asymptotic Behavior

We now present some important properties of the ergodic invariant probabilities $\{\pi^i : i \in \mathbb{Z}_+\}$. For each $i \in \mathbb{Z}_+$ the stationary Markov process $\mathbf{\Phi}$ with initial distribution π^i is ergodic. Hence by Proposition 1.5.1, if $\mu_0 \prec \pi^i$ then for every $A \in \mathcal{B}(\mathbf{X})$,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \mathbf{1}_{\Phi_k \in A} = \pi\{A\} \qquad a.s.\ [\mathrm{P}_{\mu_0}],$$

and so whenever $\mu_0 \prec \pi^i$ and $\pi^i\{A\} > 0$,

$$\mathrm{P}_{\mu_0}\{\mathbf{\Phi} \in A \quad i.o.\} = 1. \qquad (2.10)$$

For any $x \in \bar{O}^i$ we have shown that $P^T(x, \cdot) \prec \pi^i$ and using (2.10),

$$\mathrm{P}_x\{\mathbf{\Phi} \in A \quad i.o.\} = 1.$$

This and the remarks below the proof of Lemma 2.2.2 proves:

Lemma 2.2.3. *If (2.3) is locally stochastically controllable then for any $i \in \mathbb{Z}_+$ the Markov process $\mathbf{\Phi}$ may be restricted to $\bar{O}^i$, and the restricted process is positive Harris recurrent.*

□

The following useful result describes the asymptotic behavior of averages of functions of $\mathbf{\Phi}$ which vanish on the open set O defined in (2.9), and will be used to establish Proposition 2.2.3 below.

Lemma 2.2.4. *Let $f \in \mathrm{C}_0$ and suppose $f \equiv 0$ on O. Then*

$$\lim_{N\to\infty} \Big\| \frac{1}{N+1} \sum_{j=0}^{N} \mathbf{T}^j f \Big\|_\infty = 0. \qquad (2.11)$$

That is, $\frac{1}{N+1}\sum_{j=0}^{N} \mathbf{T}^j f \to 0$ uniformly as $N \to \infty$.

Proof.

Fix $f \not\equiv 0$ satisfying the conditions of the Proposition, and $\delta > 0$. Define the open sets $\{A_N : N \in \mathbb{Z}_+\}$ by

$$A_N = \left\{x \in \mathbf{X} : \frac{1}{N+1} \sum_{j=0}^{N} \mathbf{T}^j f(x) > \delta\right\}, \tag{2.12}$$

and observe that since $\bar{O}$ is absorbing, $A_k \subset O^c$ for each k. If (2.11) does not hold then there exists $\delta > 0$ and a subsequence $\{N_i : i \in \mathbb{Z}_+\}$ of $\mathbb{Z}_+$ with $A_{N_i} \neq \phi$ for all i. Let $\{\mu_i : i \in \mathbb{Z}_+\} \subset \mathcal{M}$ be probabilities for which $\mu_i\{A_{N_i}\} = 1$, and define

$$\lambda_i \triangleq \frac{1}{N_i} \sum_{j=1}^{N_i} \mathbf{U}^j \mu_i.$$

As in the proof of Theorem 1.3.2 we may find a sub-probability λ_∞ for which $\lambda_{n_i} \xrightarrow{\text{vaguely}} \lambda_\infty$ for some subsequence $\{n_i : i \in \mathbb{Z}_+\}$ of $\mathbb{Z}_+$. The sub-probability $\lambda_\infty \neq 0$ because

$$\begin{aligned} \int f\, d\lambda_\infty &\geq \liminf_{i\to\infty} \frac{1}{N_i} \sum_{j=1}^{N_i} \int f\, d\left(\mathbf{U}^j \mu_i\right) \\ &= \liminf_{i\to\infty} \int \left(\frac{1}{N_i} \sum_{j=1}^{N_i} \mathbf{T}^j f\right) d\mu_i \\ &\geq \delta \liminf_{i\to\infty} \mu_i \left\{A_{N_i}\right\} = \delta > 0. \end{aligned} \tag{2.13}$$

By the same argument used in the proof of Proposition 1.3.2, it may be shown that λ_∞ is an invariant sub-probability. We have $\lambda_\infty\{O\} = 0$ since by Theorem 2.1 of [Billingsly, 1969]

$$\lambda_\infty\{O\} \leq \liminf_{k\to\infty} \lambda_k\{O\}$$

and $\lambda_k\{O\} = 0$ for all $k \in \mathbb{Z}_+$. Furthermore, $\lambda_\infty\{O^c\} = 0$ since we have already shown that $\Upsilon\{O^c\} = 0$ for any finite invariant measure Υ. It follows that $\lambda_\infty = 0$ which contradicts (2.13). So $A_N = \phi$ for sufficiently large N, and this completes the proof. □

If (2.3) is locally stochastically controllable, and π^k is one of the invariant probabilities introduced above, then because Φ restricted to $\mathcal{O}^k$ is positive Harris recurrent it may be decomposed uniquely as an average of probabilities $\{d_i : 1 \le i \le \lambda\}$;

$$\pi^k = \frac{1}{\lambda}\sum_{i=1}^{\lambda} d_i. \tag{2.14}$$

where d_i and d_j are mutually singular (denoted $d_i \perp d_j$) for $i \neq j$. The probabilities $\{d_i : 1 \le i \le \lambda\}$ satisfy

$$\mathbf{U}d_i = d_{i+1},\ 1 \le i \le \lambda - 1, \quad \text{and} \quad \mathbf{U}d_\lambda = d_1. \tag{2.15}$$

Hence the trajectory starting at d_1 is a periodic orbit. Because the probability d_i is invariant under $\mathbf{U}^\lambda$ for each i, it follows by Lemma 2.2.3 that the probability d_i is equivalent to Lebesgue measure on an open set $\mathcal{D}^i \subset \mathbf{X}$. Hence $\mathcal{O}^k$ may be written as the *disjoint* union

$$\mathcal{O}^k \triangleq \bigcup_{i=1}^{\lambda} \mathcal{D}^i.$$

The following Proposition demonstrates how the underlying distributions of locally stochastically controllable systems exhibit asymptotically periodic behavior.

Proposition 2.2.3. *If (2.3) is locally stochastically controllable then for each initial condition $x \in \mathbf{X}$, the resulting trajectory $\{\mu_k \triangleq \mathbf{U}^k\delta_x : k \in \mathbb{Z}_+\}$ may be written*

$$\mu_k = n_k + \sum_{i=0}^{\infty} \alpha_i \mu_k^i \tag{2.16}$$

where $\{n_k : k \in \mathbb{Z}_+\}$ is a sequence of sub-probabilities for which

$$\frac{1}{N}\sum_{k=1}^{N} n_k \xrightarrow{\text{vaguely}} 0 \quad \text{as} \quad N \to \infty, \tag{2.17}$$

and for each $i \in \mathbb{Z}_+$, the sequence of sub-probabilities $\{\mu_k^i : k \in \mathbb{Z}_+\}$ converges in total variation norm to a periodic orbit: That is, there exists a periodic orbit $\{\gamma_k^i : k \in \mathbb{Z}_+\}$ such that

$$\lim_{k\to\infty}\left(\sup_{A \in \mathcal{B}(\mathbf{X})} |\mu_k^i\{A\} - \gamma_k^i\{A\}|\right) = 0. \tag{2.18}$$

Proof.

In the remarks after the proof of Lemma 2.2.2 we showed that for each $i \in \mathbb{Z}_+$, the open set $\mathcal{O}^i$ has the invariance property

$$P(x, \bar{\mathcal{O}}^i) = 1 \quad \text{for each} \quad x \in \bar{\mathcal{O}}^i.$$

This implies that $\alpha_i \triangleq \lim_{k\to\infty} P^k(x, \bar{\mathcal{O}}^i)$ exists and is in fact equal to $\mathrm{P}_x\{\Phi_k \in \mathcal{O}^i$ for some $k \in \mathbb{Z}_+\}$.

The proof of Proposition 2.2.3 will be completed in two steps. Step 1: We show that for each $i \in \mathbb{Z}_+$, the sequence of sub-probabilities $\{\mu_k^i \triangleq (1/\alpha_i)\mathbf{1}_{\mathcal{O}^i}\mu_k : k \in \mathbb{Z}_+\}$ converges to a periodic orbit whenever $\alpha_i \neq 0$.

For each $1 \leq k \leq \lambda$, the set $\bar{\mathcal{D}}^k$ defined below equation (2.15) has the invariance property

$$P^\lambda(x, \bar{\mathcal{D}}^k) = 1 \quad \text{for every} \quad x \in \bar{\mathcal{D}}^k.$$

So, defining $\beta_k \in \mathbb{R}_+$ by $\beta_k \triangleq \lim_{n\to\infty} P^{n\lambda}(x, \bar{\mathcal{D}}^k)$ we have $\alpha_i = \sum_{k=1}^{\lambda} \beta_k$. If we let

$$\gamma_0 = \frac{1}{\alpha_i}\sum_{k=1}^{\lambda} \beta_k d_k, \quad \text{and} \quad \gamma_k = \mathbf{U}^k \gamma_0 \tag{2.19}$$

then, since (2.15) is satisfied and $\gamma_k \in \mathcal{M}$ for each $k \in \mathbb{Z}_+$, $\gamma \triangleq \{\gamma_k : k \in \mathbb{Z}_+\}$ is a periodic orbit.

By Corollary 1.4.5,

$$\lim_{k\to\infty} \sup_{B\in\mathcal{B}(\mathbf{X})} |\mathbf{1}_{\mathcal{D}^j}\mu_{k\lambda}\{B\} - \beta_j d_j\{B\}| = 0.$$

It follows that

$$\lim_{k\to\infty} \sup_{B\in\mathcal{B}(\mathbf{X})} |\frac{1}{\alpha_i}\mathbf{1}_{\mathcal{O}^i}\mu_{k\lambda}\{B\} - \gamma_0\{B\}| = 0,$$

and further that for each $j \in \mathbb{Z}_+$

$$\lim_{k\to\infty} \sup_{B\in\mathcal{B}(\mathbf{X})} |\frac{1}{\alpha_i}\mathbf{1}_{\mathcal{O}^i}\mu_{k\lambda+j}\{B\} - \gamma_j\{B\}| = 0.$$

Hence $\{\mu_k^i : k \in \mathbb{Z}_+\}$ converges in total variation norm to the periodic orbit γ.

Step 2: We are left to show that with $\mathcal{O} = \bigcup_{i=0}^{\infty} \mathcal{O}^i$, and $n_k \triangleq \mathbf{1}_{\mathcal{O}^c}\mu_k$,

$$\frac{1}{N}\sum_{k=1}^{N} n_k \xrightarrow{\text{vaguely}} 0. \tag{2.20}$$

Using the same methods used in the proof of Theorem 1.3.2 we may show that any vague limit point n_∞ of the probabilities $\{\frac{1}{N}\sum_{k=1}^{N} n_k : N \geq 1\}$ must be invariant. Since $n_\infty\{\mathcal{O}\} = 0$, and every invariant sub-probability vanishes on $\mathcal{O}^c$ it follows that $n_\infty = 0$, and this completes the proof of Proposition 2.2.3.

□

Observe that by Proposition 2.2.3, if (2.3) is locally stochastically controllable then for each $x \in \mathbf{X}$

$$\frac{1}{N}\sum_{k=1}^{N} \mu_k \xrightarrow{\text{vaguely}} \alpha\pi_x \tag{2.21}$$

where $\alpha = \sum \alpha_i$, and for $\alpha > 0$ the invariant probability π_x is defined by

$$\pi_x = \frac{1}{\alpha}\sum_{k=0}^{\infty} \alpha_i \pi^i. \tag{2.22}$$

If $\alpha = 0$ then (2.21) still holds with $\pi_x = 0$.

The statement of Proposition 2.2.3 takes on a simpler form when (2.3) is stable in probability. As already shown, by the invariance of $\bar{\mathcal{O}}$, $P^k(x, \bar{\mathcal{O}}^c)$ is decreasing in k for every $x \in \mathbf{X}$, and hence for each fixed x it's limit as $k \to \infty$ exists. If (2.3) is stable in probability, or just stable in probability on average then we will show in Proposition 2.2.4 that $\lim_{k\to\infty} P^k(x, \mathcal{O}^c) = 0$, and hence $\lim_{k\to\infty} n_k\{\mathbf{X}\} = 0$. In other words, $\{n_k : k \in \mathbb{Z}_+\}$ converges to zero in total variation norm if $\mathbf{\Phi}$ is stable in probability. We will see that this implies that the law of large numbers holds for every initial condition distribution for locally stochastically controllable systems which are stable in probability.

The following lemma illustrates the sample path properties of locally stochastically controllable systems.

Lemma 2.2.5. *Suppose that (2.3) is locally stochastically controllable. Let the invariant probability π_x and $T \in \mathbb{Z}_+$ be as above, and recall that*

$$\mathcal{O} = \bigcup_{i=0}^{\infty} \mathcal{O}^i.$$

Then for any $x \in \mathbf{X}$, and $f \in L^1(\mathbf{X}^{\mathbb{Z}_+}, \mathcal{B}(\mathbf{X}^{\mathbb{Z}_+}), \mathrm{P}_{\pi_x})$ there exists a function $f_\infty \in L^1(\mathbf{X}^{\mathbb{Z}_+}, \mathcal{B}(\mathbf{X}^{\mathbb{Z}_+}), \mathrm{P}_x)$ such that

$$\mathrm{P}_x\Big\{\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f(\Phi_k, \Phi_{k+1}, \ldots) = f_\infty(\Phi_0, \Phi_1, \ldots)\Big\} \geq \alpha_x \tag{2.23}$$

where $\alpha_x \in [0,1]$ is defined by

$$\alpha_x \triangleq \mathrm{P}_x\Big\{\bigcup_{k=0}^{\infty} \{\Phi_k \in \mathcal{O}\}\Big\} = \lim_{k\to\infty} P^k(x, \mathcal{O}). \tag{2.24}$$

In fact,

$$f_\infty = \mathbf{1}_{\tau<\infty} \sum_{k=0}^{\infty} \mathbf{1}_{\{\Phi_\tau \in \mathcal{O}^k\}} \mathrm{E}_{\pi^k}[f(\Phi_0, \Phi_1, \ldots)] \tag{2.25}$$

where τ is the first entrance time into the set $\mathcal{O}$:

$$\tau = \min\{k \in \mathbb{Z}_+ : \Phi_k \in \mathcal{O}\}.$$

Proof.

By the corollary to Theorem 1.5.1, for each $n \in \mathbb{Z}_+$, and $f \in L^1(\mathbf{X}^{\mathbb{Z}_+}, \mathcal{B}(\mathbf{X}^{\mathbb{Z}_+}), \mathrm{P}_{\pi^n})$,

$$\mathrm{P}_{\mu_0}\Big\{\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f(\Phi_k, \Phi_{k+1}, \ldots) = \mathrm{E}_{\pi^n}[f(\Phi_0, \Phi_1, \ldots)]\Big\} = 1$$

for any initial condition distribution μ_0 for which $\mu_0\{\bar{O}^n\} = 1$. From this it follows that for any $x \in \mathbf{X}$,

$$\begin{aligned}
&\mathrm{E}_x\Big[\mathrm{P}\Big\{\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} f(\Phi_k,\Phi_{k+1},\ldots) = f_\infty(\Phi_0,\Phi_1,\ldots) \mid \Phi_\tau\Big\}\mathbf{1}_{\tau<\infty}\Big] \\
&= \mathrm{E}_x\Big[\mathrm{P}\Big\{\lim_{N\to\infty}\frac{1}{N}\sum_{k=\tau+1}^{\tau+N} f(\Phi_k,\Phi_{k+1},\ldots) = f_\infty(\Phi_0,\Phi_1,\ldots) \mid \Phi_\tau\Big\}\mathbf{1}_{\tau<\infty}\Big] \\
&= \mathrm{E}_x\Big[\mathrm{P}_{\Phi_\tau}\Big\{\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} f(\Phi_k,\Phi_{k+1},\ldots) = f_\infty(\Phi_0,\Phi_1,\ldots)\Big\}\mathbf{1}_{\tau<\infty}\Big] \\
&= \mathrm{P}_x\{\tau<\infty\} = \alpha_x.
\end{aligned}$$

Hence,

$$\mathrm{P}_x\Big\{\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} f(\Phi_k,\Phi_{k+1},\ldots) = f_\infty(\Phi_0,\Phi_1,\ldots)\Big\} \geq \alpha_x,$$

and this establishes Lemma 2.2.5.

□

In Proposition 2.2.4 below local stochastic controllability is used to establish the equivalence of the two forms of stability for Markov chains introduced in Chapter I, and Theorem 1.5.1 is generalized to hold for arbitrary initial conditions. These results hold even in the case where $\mathbf{\Phi}$ possesses more than one invariant probability and hence is not Harris recurrent or even irreducible.

Proposition 2.2.4. *If (2.3) is locally stochastically controllable then the following are equivalent:*

(i) (2.3) is stable in probability

(ii) (2.3) is stable in probability on average

(*iii*) $\alpha_x = \mathrm{P}_x\{\Phi_k \in \bar{O}$ *for some* $k \in \mathbb{Z}_+\} = 1$ *for all* $x \in \mathbf{X}$.

Hence if any of the above hold then by Lemma 2.2.5,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(\Phi_k, \Phi_{k+1}, \ldots) = f_\infty(\Phi_0, \Phi_1, \ldots) \quad a.s. \ [\mathrm{P}_x] \tag{2.26}$$

for any $f \in L^1(\mathbf{X}^{\mathbb{Z}_+}, \mathcal{B}(\mathbf{X}^{\mathbb{Z}_+}), \mathrm{P}_{\pi_x})$ where f_∞ is defined in equation (2.25), and π_x is defined in equation (2.22).

Proof.

We will proceed by establishing that (*i*) is equivalent to (*ii*), and that (*ii*) is equivalent to (*iii*).

(*ii*) $\Rightarrow$ (*iii*). For $x \in \mathbf{X}$ we must show that $\mathrm{P}_x\{\tau < \infty\} = 1$. Since (*ii*) and Proposition 1.7.1 imply that $\frac{1}{N}\sum_{k=1}^{N} P^k(x, \cdot) \overset{\text{weakly}}{\longrightarrow} \pi_x$ where π_x is an invariant probability, Theorem 1.1.1 (*ii*) applies to give

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \mathrm{P}_x\{\Phi_k \in O\} \geq \pi_x\{O\} = 1. \tag{2.27}$$

This implies that $\boldsymbol{\Phi}$ enters $\bar{O}$ at some time $k \in \mathbb{Z}_+$ *a.s.* $[\mathrm{P}_x]$ since for any $i \in \mathbb{Z}_+$

$$\mathrm{P}_x\Big\{\bigcup_{k=0}^{\infty} \{\Phi_k \in \bar{O}\}\Big\} \geq \mathrm{P}_x\{\Phi_i \in \bar{O}\}.$$

Consequently,

$$\begin{aligned}\mathrm{P}_x\Big\{\bigcup_{k=0}^{\infty} \{\Phi_k \in \bar{O}\}\Big\} &\geq \sup_{i\geq 0} \mathrm{P}_x\{\Phi_i \in \bar{O}\} \\ &= 1\end{aligned} \tag{2.28}$$

by (2.27), and this establishes (*iii*).

$(ii) \Rightarrow (i)$. Let $\varepsilon > 0$. If (2.3) is stable in probability on average then we have already seen in equation (2.28) that

$$\mathrm{P}_x\Big\{\Phi_k \in \bar{O}_\varepsilon \quad \text{for some } k \in \mathbb{Z}_+\Big\} = 1. \tag{2.29}$$

Hence,

$$\mu_k = \sum_{i=0}^{\infty} \alpha_i \mu_k^i + n_k, \quad \text{where } \sum_{i=0}^{\infty} \alpha_i = 1,$$

and $n_k \xrightarrow{\text{vaguely}} 0$ as $k \to \infty$. Choose $M \in \mathbb{Z}_+$ so large that $\sum_{i=0}^{M} \alpha_i > 1 - \varepsilon$, and choose a compact set $C \subset \mathbf{X}$ such that

$$\gamma_k^i\{C\} > 1 - \varepsilon, \quad \text{for } 0 \le i \le M, \text{ and } 1 \le k \le \lambda^i.$$

Then,

$$\begin{aligned}
\liminf_{k\to\infty} \mu_k\{C\} &\ge \liminf_{k\to\infty} \sum_{i=1}^{M} \alpha_i \mu_k^i\{C\} + n_k\{C\} \\
&\ge \sum_{i=1}^{M} \alpha_i \min_k \gamma_k^i\{C\} \\
&\ge (1-\varepsilon)^2.
\end{aligned} \tag{2.30}$$

Hence (2.3) is stable in probability.

It is obvious that $(i) \Rightarrow (ii)$, so we are left to show that $(iii) \Rightarrow (ii)$. If (iii) holds then by Lemma 2.2.5 for every $x \in \mathbf{X}$, and every $f \in \mathbf{C}$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(\Phi_k) = f_\infty \quad a.s. \ [\mathrm{P}_x] \tag{2.31}$$

Taking expectations of both sides of (2.31), and using the dominated convergence theorem and (2.25) shows that as $N \to \infty$,

$$\frac{1}{N} \sum_{k=1}^{N} \mathbf{U}^k \delta_x \xrightarrow{\text{weakly}} \sum_{i=0}^{\infty} \alpha_i \pi^i. \tag{2.32}$$

It follows from this and Proposition 1.7.1 that (2.3) is stable in probability on average, and this establishes the proposition. □

Proposition 2.2.4 together with Corollary 1.5.1 makes the following connection between stability in probability and positive Harris recurrence for a locally stochastically controllable Markovian system:

Corollary 2.2.4. *Suppose that (2.3) is locally stochastically controllable. Then* $\mathbf{\Phi}$ *is positive Harris recurrent if and only if it is stable in probability and possesses exactly one invariant probability.* □

2.3 Irreducible Stochastic Systems

In this section we will continue our investigation of stochastic systems of the form (2.3). Our goal is to find sufficient conditions to ensure that the Markov state processes for such systems are irreducible and more generally, to find conditions which ensure that there is at most one invariant probability. This is an important question in stochastic system theory because it is a necessary condition for the Markov state process to be Harris recurrent which, as we saw in Chapter I, facilitates the computation of the performance criteria J_∞ and L_∞ introduced in that chapter. Moreover, if certain technical conditions are met, then the values of these performance criteria do not depend on the initial conditions of the system.

2.3.1 Recurrence and Stability

We saw in the previous section that when a Markov state process is locally stochastically controllable, stability and Harris recurrence are strongly related. In fact, if the state process $\mathbf{\Phi}$ is locally stochastically controllable and irreducible then Harris recurrence and stability in probability are equivalent concepts. It would be very useful if this remained true without the stochastic controllability assumption. However, this is not the case as can be seen from the following simple example: Let $\mathbf{X} = \mathbb{R}$, and consider the Markov transition function P defined by

$$P(x, \{1/2x\}) = 2^{-|x|}$$
$$P(x, \{0\}) = 1 - 2^{-|x|}.$$

The corresponding Markov chain $\mathbf{\Phi}$ has the Feller property, is stable in probability, and is irreducible with maximal irreducibility measure δ_0. However, $\mathbf{\Phi}$ is *not* Harris recurrent since for any $x \in \mathbf{X}$, $x \neq 0$,

$$\mathrm{P}_x\{\Phi_k \neq 0 \quad \text{for all } k\} = 2^{-|x|}2^{-|x/2|}2^{-|x/4|}\dots$$
$$= 2^{-2|x|} > 0.$$

The following result relates the notions of stability and positive recurrence:

Proposition 2.3.1. *If $\mathbf{\Phi}$ is irreducible and possesses an invariant probability then it is positive recurrent.*

Proof.

First we show that if $\mathbf{\Phi}$ is irreducible with irreducibility measure φ, and π is an invariant probability then

$$\varphi \prec \pi. \tag{2.33}$$

If $\varphi\{A\} > 0$ then $G(x, A) > 0$ everywhere. Hence, there exists an $N \in \mathbb{Z}_+$ such that $\frac{1}{N}\sum_{k=1}^{N} P^k(x, A) > 0$ on a set of positive π measure. It follows that,

$$0 < \frac{1}{N}\sum_{k=1}^{N}\int \pi(dx)P^k(x, A) = \pi\{A\},$$

and hence (2.33) holds. It follows that there exists only one invariant probability since (2.33) cannot hold for two mutually singular invariant probabilities.

To complete the proof we will now show that whenever $\pi\{A\} > 0$,

$$P_x\{\Phi \quad \text{enters} \quad A \ i.o.\} > 0 \qquad \text{for every } x \in \mathbf{X}. \tag{2.34}$$

By Theorem 1.5.1 and since π is ergodic, for each $A \in \mathcal{B}(\mathbf{X})$ for which $\pi\{A\} > 0$ there exists a set $F \in \mathcal{B}(\mathbf{X})$ of full π measure such that

$$P_x\{\Phi \quad \text{enters} \quad A \ i.o.\} = 1 \qquad \text{for all } x \in F. \tag{2.35}$$

By (2.33) it follows that $\varphi\{F\} = \varphi\{\mathbf{X}\} > 0$. Hence by irreducibility we have for all $x \in \mathbf{X}$,

$$P_x\{\Phi \quad \text{enters} \quad F\} > 0,$$

and by a standard argument (2.34) follows.

From (2.34) and (2.33) it follows that π is a maximal irreducibility measure, and by (2.34) and (2.35) the Markov chain Φ is positive recurrent. □

Hence by Proposition 2.3.1, if Φ is irreducible and stable in probability on average then it is positive recurrent. We now turn to the problem of finding general conditions under which a Markov chain of the form (2.3) is irreducible.

2.3.2 Irreducibility

In most cases, stochastic systems of the form (2.3) which are stable in probability exhibit the following related property: Given a system of the form (2.3) we will call the deterministic system

$$d_{k+1} = F(d_k, 0), \quad k \in \mathbb{Z}_+ \tag{2.36}$$

with initial condition $d_0 \in \mathbf{X}$ the *freely evolving system.* We say the system (2.3) *satisfies condition* **GA** if some $x^* \in \mathbf{X}$ is globally attracting for the freely evolving system. That is, for each initial condition $d_0 \in \mathbf{X}$,

Condition **GA** $$\lim_{k\to\infty} d_k = \lim_{k\to\infty} S^k_{d_0}(0,\cdots,0) = x^*.$$

Hence, if the disturbance sequence $\mathbf{w}$ is replaced by $(0,\cdots,0,\cdots)$ in (2.3) then $\Phi_k \to x^*$ as $k \to \infty$ for all initial condition distributions. To simplify the statements of the results that follow and their corresponding proofs, we shall henceforth assume that $x^* = 0$. This does not lead to any loss of generality since we may always replace the Markov chain $\mathbf{\Phi}$ by $\{\Phi_k - x^* : k \in \mathbb{Z}_+\}$ when $x^* \neq 0$. For example, the controlled random parameter $AR(p)$ system to be examined in Chapter V satisfies condition **GA** when $\sigma_e^2 < 1$, and the linear system (1.44) under linear control satisfies condition **GA** if and only it is stable in probability.

For a system of the form (2.3) satisfying condition **GA** suppose that the support of the distribution μ_w of w_0 contains the origin. In this case $\mathbf{\Phi}$ enters every neighborhood of the origin with positive probability, and hence the support of every invariant probability contains the origin. This is reminiscent of the definition of irreducibility and might suggest that stochastic systems satisfying these assumptions possess no more than one invariant probability. However, this is not the case as can be seen by the following example: First consider the Markov chain $\mathbf{\Psi}$ on $(0,\infty)$ generated by the recursion

$$\Psi_{k+1} = \Psi_k^p w_k \tag{2.37}$$

where $\mathbf{w}$ is i.i.d. with μ_w equivalent to Lebesgue measure or $(0,\infty)$, $\mathrm{E}[|\log(w_0)|] < \infty$, and $1 > p > 0$.

The system (2.37) is stable in probability because $|\log(\cdot)|$ is a moment on $(0,\infty)$, and

$$E_x[|\log(\Psi_{k+1})|] \leq \rho E_x[|\log(\Psi_k)|] + E[|\log(w_k)|].$$

Hence,

$$\limsup_{k\to\infty} E_x[|\log(\Psi_k)|] \leq \frac{E[|\log(w_k)|]}{1-\rho} < \infty,$$

and this shows that $\mathbf{\Psi}$ is (uniformly) stable in probability. Furthermore, the first order generalized controllability matrix for (2.37) is full rank for every $x \in (0,\infty)$ which shows that $\mathbf{\Psi}$ is locally stochastically controllable. Since $\mathcal{O}_x = (0,\infty)$ for every $x \in (0,\infty)$, there is a unique invariant probability π which is equivalent to Lebesgue measure on $(0,\infty)$.

We now use $\mathbf{\Psi}$ to define a system of the form (2.3) which is stable in probability and satisfies condition **GA** yet possesses three invariant probabilities. Let $\mathbf{\Phi}$ be generated by the recursion

$$\Phi_{k+1} = \operatorname{sign}(\Phi_k)|\Phi_k|^\rho w_k \tag{2.38}$$

where $\mathbf{w}$ and ρ have the properties given below equation (2.37) and $\operatorname{sign}(\cdot)\colon \mathbb{R} \to \mathbb{R}$ is defined by

$$\operatorname{sign}(t) = \begin{cases} 1 & \text{if } t > 0; \\ -1 & \text{if } t < 0; \\ 0 & \text{if } t = 0. \end{cases}$$

Then (2.38) is of the form (2.3) with $\mathbf{X} = \mathbb{R}$. The system (2.38) is stable in probability and satisfies condition **GA**.

If π denotes the invariant probability for (2.37), then (2.38) possesses the invariant probabilities π_+, π_-, and δ_0 where $\pi_+ = \pi$, $\pi_-\{(a,b)\} = \pi\{(-b,-a)\}$ for $(a,b) \subset (-\infty,0)$, and δ_0 is the point mass at the origin. We see in the following proposition that the reason $\mathbf{\Phi}$ can possess more than one invariant probability is that

$P^T(0, \cdot) = \delta_0\{\cdot\}$ for all T, where P is the Markov transition function for $\mathbf{\Phi}$, and hence (2.38) is not locally stochastically controllable.

Proposition 2.3.2. *Suppose that the Markov chain* $\mathbf{\Phi}$ *is of the form (2.3) where: (i) (2.3) is locally stochastically controllable and satisfies condition* **GA**; *(ii)* $0 \in \operatorname{supp} \mu_w = \bar{O}_w$. *Then there is at most one invariant probability and there are no non-trivial cyclical sets.*

Hence by Corollary 2.2.4, if (2.3) satisfies the conditions of Proposition 2.3.2 and is stable in probability then the state process $\mathbf{\Phi}$ is positive Harris recurrent and aperiodic. Before proving this proposition we must establish the following lemma:

Lemma 2.3.1. *Suppose that* $\mathbf{\Phi}$ *is generated by the system (2.3) satisfying conditions* **A1-A3**, *and* **GA**, *and that* $0 \in \operatorname{supp} \mu_w$. *Then for each* $x \in \mathbf{X}$, *and every open set* $\mathcal{U} \subset \mathbf{X}$ *containing the origin,*

$$\sup_{k \geq 0} P^k(x, \mathcal{U}) > 0.$$

Proof.

Fix $x \in \mathbf{X}$, and let $\mathcal{U}$ satisfy the hypotheses of the lemma. By condition **GA** we may choose $k \in \mathbb{Z}_+$ so large that

$$S_x^k(0, \cdots, 0) \in \mathcal{U},$$

and by continuity there exists a $\delta > 0$ such that

$$S_x^k(z_1, \cdots, z_k) \in \mathcal{U},$$

for all $(z_1, \cdots, z_k) \in \{B_\delta(0)\}^k$ where $B_\delta(0)$ is the open rectangle of width δ centered at the origin. It follows that

$$\begin{aligned} P^k(x, \mathcal{U}) &> \mathrm{E}[\mathbf{1}_{\|w_1\| < \delta} \cdots \mathbf{1}_{\|w_k\| < \delta}] \\ &= (\mu_w\{B_\delta(0)\})^k > 0, \end{aligned}$$

and this proves the lemma. □

We may now present the proof of Proposition 2.3.2.

Proof.

It follows from Lemma 2.3.1 that there can be only one invariant probability because if π^1 and π^2 are ergodic invariant probabilities then there are disjoint open sets $\mathcal{O}^1$ and $\mathcal{O}^2$ for which

$$\pi^j \approx \mathbf{1}_{\mathcal{O}^j}\mu^{Leb}, \qquad j = 1,\ 2, \tag{2.39}$$

and

$$0 \in \bar{\mathcal{O}}^1 \cap \bar{\mathcal{O}}^2. \tag{2.40}$$

By invariance $P^T(0, \bar{\mathcal{O}}^1) = P^T(0, \bar{\mathcal{O}}^2) = 1$. Furthermore, since $P^T(0, \cdot) \prec \mu^{Leb}$ and $\mathcal{O}^1 \cap \mathcal{O}^2 = \phi$, this implies that $P^T(0, \mathcal{O}^1 \cup \mathcal{O}^2) = 2$. This contradiction shows that there is at most one invariant probability.

We now show that there are no cyclical sets. Suppose that an invariant probability π exists for (2.3) and that D_i, $1 \le i \le \lambda$, is a cycle with period λ. Letting Q denote the Markov transition function P^λ we may show that the system corresponding to Q is locally stochastically controllable, and that for each i, $\lambda \mathbf{1}_{D_i}\pi$ is an invariant probability for Q. But the system corresponding to Q satisfies the conditions of this proposition and hence Q has at most one invariant probability. This shows that $\lambda = 1$, completing the proof. □

Let us now turn to the problem of finding general conditions under which a system satisfying condition **GA** is irreducible. A great deal of work in this direction has been carried out on the random walk $\boldsymbol{\Phi}$ on $\mathbb{R}$ where

$$\Phi_{k+1} = \Phi_k + w_{k+1}, \tag{2.41}$$

w is i.i.d. with $w_k \sim \mu_w$ for $k \in \mathbb{Z}_+$, and Φ_0 is independent of **w** (see [Nummelin, 1984].) If for example, $\mu_w = \frac{1}{2}(\delta_{-1} + \delta_1)$ then $\boldsymbol{\Phi}$ cannot be irreducible because for each deterministic initial condition $\Phi_0 \in \mathbb{R}$, the distribution of Φ_k is supported on $\mathbb{Z} + \Phi_0$ for all $k \in \mathbb{Z}_+$. We say that the probability μ_w is *spread out* if for some open set $O \subseteq \mathbb{R}$, Lebesgue measure on O is absolutely continuous with respect to μ_w; that is. $\mathbf{1}_O \mu^{Leb} \prec \mu_w$. This condition is equivalent to the condition that μ_w is non-singular with respect to Lebesgue measure. It is shown in [Nummelin, 1984] that if μ_∞ is spread out then (2.41) is irreducible, and if in addition $\int x\,\mu_w(dx) = 0$ then it is Harris recurrent. In [Athreya and Pantula, 1986] the spread out condition (among other assumptions) is used to show that an ARMA stochastic process is strongly mixing.

Motivated by these results, we call the Markov system (2.3) *spread out* if for some $T \in \mathbb{Z}_+$ and all $x \in \mathrm{X}$ the probability $P^T(x, \cdot)$ is spread out. Observe that this is a much weaker condition than local stochastic controllability because we only require $P^T(x, \cdot)$ to be *non-singular* with respect to Lebesgue measure.

We have the following sufficient condition for (2.3) to be spread out:

Proposition 2.3.3. *Suppose that the function F defined in (2.3) is C^1, and that the generalized controllability matrix C_T satisfies the rank condition (2.4). Then (2.3) is spread out if μ_w is spread out.*

Proof.

The proof is omitted since it is identical to the proof of Proposition 2.2.1.

□

In the remainder of this chapter the assumption that μ_w is equivalent to Lebesgue measure on an open set is unnecessary. Furthermore, we will restrict our attention to Markovian systems of the form (2.3) satisfying condition **GA** where F is C^1, and whose generalized controllability matrix C_T is full rank. Hence, replacing assumption **A4** by

> **A4′** There exists an open set $O_w \subset \mathbb{R}^p$ such that the distribution μ_w of w_k, $k \in \mathbb{Z}_+$, satisfies $\mathbf{1}_{O_w} \prec \mu_w$, and $0 \in \operatorname{supp} \mu_w$,

we see by Proposition 2.3.3 that such systems are spread out.

The following result is extremely useful in practice:

Proposition 2.3.4. *Suppose that the Markov chain Φ is of the form (2.3) where F is C^1, and that assumptions* **A1-A3** *and* **A4′** *are satisfied. Then Φ is irreducible and aperiodic under the assumptions:*

(i) The system (2.3) satisfies condition **GA**;

(ii) The generalized controllability matrix (2.4) satisfies the conditions of Proposition 2.2.1.

For a point $x \in \mathbb{R}^m$ and $\varepsilon > 0$, we let $B_\varepsilon(x)$ denote the open rectangle

$$B_\varepsilon(x) \triangleq \{y \in \mathbb{R}^m : |x_i - y_i| < \varepsilon \text{ for } 1 \leq i \leq m\}.$$

Proof.

To prove the proposition we will construct an open set O_0 containing the origin and a measure φ on $\mathcal{B}(\mathbf{X})$ such that for all $x \in \mathbf{X}$ and $A \in \mathcal{B}(\mathbf{X})$,

$$P^T(x, A) \geq \mathbf{1}_{O_0}(x)\varphi\{A\}. \tag{2.42}$$

That is, O_0 is a small set, and φ is a small measure. It will follow from Lemma 2.3.1 that φ is an irreducibility measure.

Let p_w denote the Radon-Nikodym derivative of μ_w, and $p: \mathbb{R}^{Tp} \to \mathbb{R}_+$ the density defined for $(z_1, \ldots, z_T) \in \mathbb{R}^{Tp}$ by

$$p(z_1, \ldots, z_T) \triangleq p_w(z_1) \cdots p_w(z_T).$$

Since μ_w is spread out we may find $\lambda_0 \in \mathbb{R}^{Tp}$, $\delta_p > 0$, and an open rectangle $B_{\delta_p}(\lambda_0) \subset \mathbb{R}^{Tp}$ such that

$$p > \delta_p \mathbf{1}_{B_{\delta_p}(\lambda_0)} \qquad a.e.\ [\mu^{Leb}] \tag{2.43}$$

Using the rank condition on the derivative of F we may assume without loss of generality that for some integers $\{i_1, \ldots i_n\}$,

$$\det \left[\frac{\partial S_0^T}{\partial \lambda_{i_1}} | \cdots | \frac{\partial S_0^T}{\partial \lambda_{i_n}} \right]_{\lambda^0} \neq 0. \tag{2.44}$$

For any $\Phi_0 \in \mathbf{X}$, $A \in \mathcal{B}(\mathbf{X})$,

$$P^T(\Phi_0, A) \geq \delta_p \int_{B_{\delta_p}(\lambda_0)} \mathbf{1}_{S_{\Phi_0}^T(\lambda) \in A}\, d\lambda, \tag{2.45}$$

and the term on the right may be written

$$\delta_p \int_{B_{\delta_p}(\lambda'^0)} \left\{ \int_{B_{\delta_p}(\lambda_{i_1}^0, \ldots, \lambda_{i_n}^0)} \mathbf{1}_{S_{\Phi_0}^T(\lambda) \in A}\, d\lambda_{i_1} \cdots d\lambda_{i_n} \right\} d\lambda' \tag{2.46}$$

where

$$\lambda' \triangleq (\lambda_1, \ldots, \lambda_{i_1-1}, \lambda_{i_1+1}, \ldots, \lambda_{i_n-1}, \lambda_{i_n+1}, \ldots, \lambda_{Tp}),$$

and

$$\lambda'^0 \triangleq (\lambda_1^0, \ldots, \lambda_{i_1-1}^0, \lambda_{i_1+1}^0, \ldots, \lambda_{i_n-1}^0, \lambda_{i_n+1}^0, \ldots, \lambda_{Tp}^0).$$

The main task in this proof is to estimate the term in brackets in (2.46).

Let $x \in \mathbf{X} \times \mathbb{R}^{Tp-n}$ and $y \in \mathbb{R}^n$ denote the generic variables

$$x \triangleq (\Phi_0, \lambda_1, \ldots, \lambda_{i_1-1}, \lambda_{i_1+1}, \ldots, \lambda_{i_n-1}, \lambda_{i_n+1}, \ldots, \lambda_{Tp})$$

$$y \triangleq (\lambda_{i_1}, \ldots, \lambda_{i_n})$$

and let $x_0 \in \mathbf{X} \times \mathbb{R}^{Tp-n}$ and $y_0 \in \mathbb{R}^n$ denote the fixed variables

$$x_0 \triangleq (0, \lambda_1^0, \ldots, \lambda_{i_1-1}^0, \lambda_{i_1+1}^0, \ldots, \lambda_{i_n-1}^0, \lambda_{i_n+1}^0, \ldots, \lambda_{Tp}^0)$$

$$y_0 \triangleq (\lambda_{i_1}^0, \ldots, \lambda_{i_n}^0).$$

Define the function $G(\cdot,\cdot): \{\mathbf{X} \times \mathbb{R}^{Tp-n}\} \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$G(x,y) \triangleq S_{\Phi_0}^T(\lambda_1, \ldots, \lambda_{Tp}),$$

so that by (2.44),

$$\det \left[\frac{\partial G}{\partial y}\right]_{(x_0,y_0)} \neq 0. \tag{2.47}$$

Consider the function $G^*: \{\mathbf{X} \times \mathbb{R}^{Tp-n}\} \times \mathbb{R}^n \to \{\mathbf{X} \times \mathbb{R}^{Tp-n}\} \times \mathbb{R}^n$ given by

$$G^*(x,y) \triangleq \begin{pmatrix} x \\ G(x,y) \end{pmatrix}.$$

By (2.47) and the inverse function theorem there exist open sets $\mathcal{O}$, $\mathcal{U} \times \mathcal{V} \subset \{\mathbf{X} \times \mathbb{R}^{Tp-n}\} \times \mathbb{R}^n$, and a C^1 function $H^* : \mathcal{O} \to \mathcal{U} \times \mathcal{V}$ such that $(x_0, y_0) \in \mathcal{U} \times \mathcal{V}$, $\mathcal{O} = \{\binom{x}{G(x,y)} : x \in \mathcal{U}, y \in \mathcal{V}\}$, and

$$H^*(G^*(x,y)) = (x,y)$$

for all $(x, y) \in \mathcal{U} \times \mathcal{V}$. Using the function H^* we may define a C^1 function $h\colon \mathcal{O} \to \mathcal{V}$ for which

$$h(x, G(x, y)) = y \quad \text{for all} \quad (x, y) \in \mathcal{U} \times \mathcal{V},$$

and we may assume that with $z_0 \triangleq G(x_0, y_0)$

$$\left|\det \left[\frac{\partial h}{\partial z}\right]_{(x_0, z_0)}\right| > 0.$$

Hence by reducing the size of $\mathcal{U} \times \mathcal{V}$ we may assume that for some $\delta_n > 0$,

$$\left|\det \left[\frac{\partial h}{\partial z}\right]_{(x, z)}\right| > \delta_n$$

for all $(x, z) \in \mathcal{O}$.

We may now estimate the term in brackets in equation (2.46):

$$\begin{aligned}
\int_{B_{\delta_p}(\lambda^0_{i_1}, \ldots, \lambda^0_{i_n})} \mathbf{1}_{S^T_{\Phi_0}(\lambda) \in A}\, d\lambda_{i_1} \ldots d\lambda_{i_n} &= \int_{B_{\delta_p}(y_0)} \mathbf{1}_{G(x,y) \in A}\, dy \\
&\geq \mathbf{1}_{\mathcal{U}}(x) \int_{\mathcal{V} \cap B_{\delta_p}(y_0)} \mathbf{1}_{G(x,y) \in A}\, dy \\
&= \mathbf{1}_{\mathcal{U}}(x) \int_{G(x, \mathcal{V} \cap B_{\delta_p}(y_0))} \mathbf{1}_{z \in A} \left|\det \left[\frac{\partial h}{\partial z}\right]_{(x,z)}\right| dz \\
&\geq \delta_n \mathbf{1}_{\mathcal{U}}(x) \int_{G(x, \mathcal{V} \cap B_{\delta_p}(y_0))} \mathbf{1}_{z \in A}\, dz
\end{aligned} \tag{2.48}$$

Observe that the set

$$\left\{ \begin{pmatrix} x \\ G(x, \mathcal{V} \cap B_{\delta_p}(y_0)) \end{pmatrix} : x \in \mathcal{U} \right\} \subset \left\{ \mathbf{X} \times \mathbb{R}^{Tp-n} \right\} \times \mathbb{R}^n$$

contains (x_0, z_0) and is open. So we may find a $\delta_0 > 0$ such that $\delta_0 \leq \delta_p$ and for all $(x, z) \in \{\mathbf{X} \times \mathbb{R}^{Tp-n}\} \times \mathbb{R}^n$,

$$\mathbf{1}_{B_{\delta_0}(x_0, z_0)}(x, z) \leq \mathbf{1}_{\mathcal{U}}(x) \mathbf{1}_{G(x, \mathcal{V} \cap B_{\delta_p}(y_0))}(z).$$

Hence by equations (2.45), (2.46) and (2.48)

$$\begin{aligned} P^T(\Phi_0, A) &\geq \delta_p \int_{B_{\delta_p}(\lambda'^0)} \left\{ \int_{B_{\delta_p}(\lambda^0_{i_1}, \ldots, \lambda^0_{i_n})} \mathbf{1}_{S^T_{\Phi_0}(\lambda) \epsilon A} \, d\lambda_{i_1} \ldots d\lambda_{i_n} \right\} d\lambda' \\ &\geq \delta_p \delta_n \int_{B_{\delta_p}(\lambda'^0)} \left\{ \int \mathbf{1}_{z \epsilon B_{\delta_0}(z_0) \cap A} \, dz \right\} \mathbf{1}_{B_{\delta_0}(x_0)}(\lambda') \, d\lambda' \\ &\geq \delta_p \delta_n \mathbf{1}_{B_{\delta_0}(0)}(\Phi_0) \mu^{Leb}\{B_{\delta_0}(z_0) \cap A\} \int_{B_{\delta_0}(\lambda'^0)} d\lambda' \end{aligned}$$

and so, letting $\alpha \triangleq 2\delta_0$, we have

$$P^T(\Phi_0, A) \geq \delta_p \delta_n \alpha^{Tp-n} \mathbf{1}_{B_{\delta_0}(0)}(\Phi_0) \mu^{Leb}\{B_{\delta_0}(z_0) \cap A\}. \tag{2.49}$$

Letting

$$\varphi\{\cdot\} \triangleq \delta_p \delta_n \alpha^{Tp-n} \mu^{Leb}\{B_{\delta_0}(z_0) \cap \cdot\},$$

and $\mathcal{O}_0 \triangleq B_{\delta_0}(0)$ we see from (2.49) that equation (2.42) is satisfied, and the proof is almost complete. To show that ϕ is an irreducibility measure let $A \in \mathcal{B}(\mathbf{X})$ satisfy $\phi\{A\} > 0$, and let $y \in \mathbf{X}$. By Lemma 2.3.1 there is a $k \in \mathbb{Z}_+$ such that $P^k(y, \mathcal{O}_0) > 0$ and by (2.49), $P^T(x, A) \geq \varphi\{A\} > 0$ for every $x \in \mathcal{O}_0$. Combining these facts we have

$$P^{T+k}(y, A) \geq \int_{\mathcal{O}_0} P^k(y, dz) P^T(z, A) > 0.$$

This shows that φ is an irreducibility measure, and hence $\mathbf{\Phi}$ is irreducible.

In fact, $P^k(x, \mathcal{O}_0) > 0$ for *all* k sufficiently large by a simple modification of the proof of Lemma 2.3.1, and we conclude that if $\varphi\{A\} > 0$ then $P^N(x, A) > 0$ for all N sufficiently large. This eliminates the possibility of non-trivial cyclical sets and hence $\mathbf{\Phi}$ is aperiodic.

□

Remarks

- It is easy to see that the rank condition on (2.4) is much more than is needed. The conclusions of Proposition 2.3.4 remain valid if $C_T(0, \lambda_0)$ is full rank for some $T > 0$ where the probability $\mu_w^T \triangleq \mu_w \cdots \mu_w$ on $\{\mathbb{R}^p\}^T$ satisfies

$$\mathbf{1}_{B_\delta(\lambda_0)} \mu^{Leb} \prec \mu_w^T$$

for some $\delta > 0$.

- Since writing this section we have discovered a similar yet less general result in the dissertation [Chan, 1986].

2.4 Applications to Linear System Theory

Consider the ARMAX system

$$\begin{aligned} y_k + A_1 y_{k-1} + \ldots + A_{n_1} y_{k-n_1} &= B_d u_{k-d} + \ldots + B_{n_2} u_{k-n_2} \\ &\quad + w_k + C_1 w_{k-1} + \ldots + C_{n_3} w_{k-n_3} \end{aligned} \tag{2.50}$$

where k, $d \geq 1$, the processes **y**, **u**, and **w** evolve on $\mathbb{R}$, **w** is an i.i.d. processes with distribution μ^w satisfying assumption **A4** and initial conditions independent of **w** are given.

Suppose that a control law u is given of the form

$$u_k = u(y_k, \ldots, y_{k-n_4}), \quad k \in \mathbb{Z}_+,$$

where $u : \mathbb{R}^{(n_4+1)} \to \mathbb{R}$ is C^1. The closed loop system is of form

$$y_k = G(y_{k-1}, \ldots, y_{k-m}) + w_k + \cdots, C_{n_3} w_{k-n_3}, \tag{2.51}$$

for some $m \in \mathbb{Z}_+$, where $G : \mathbb{R}^m \to \mathbb{R}$ is C^1.

We now embed the closed loop system (2.51) in a Markovian system which is of the general form introduced in (2.3):

$$\Phi_{k+1} = \begin{bmatrix} y_k \\ \vdots \\ \vdots \\ y_{k-m+1} \\ w_{k+1} \\ \vdots \\ \vdots \\ w_{k-n_3+1} \end{bmatrix} = \begin{bmatrix} G(y_{k-1}, \ldots, y_{k-m}) + w_k + \cdots + C_{n_3} w_{k-n_3} \\ y_{k-1} \\ \vdots \\ y_{k-m+1} \\ 0 \\ w_k \\ \vdots \\ w_{k-n_3+1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ w_{k+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.52}$$

valid for $k \geq 0$ and $\Phi_0 \in \mathbf{X} \triangleq \mathbb{R}^n$ given $(n = m + n_3 + 1)$. This Markov chain is of the form

$$\Phi_{k+1} = F(\Phi_k) + Bw_{k+1} \tag{2.53}$$

where F is C^1 and, since we have already assumed that μ_w satisfies **A4**, this process has the properties required in Section 2. It would be very desirable to prove that (2.53) is locally stochastically controllable. Necessary and sufficient conditions for local stochastic controllability have not been established as yet, but we can apply Proposition 2.2.1 to provide sufficient conditions. In establishing these sufficient conditions we have assumed that $m \geq n_3$. This simplifies the discussion and the general case may be proved using the same techniques as those used below.

If we let $G_i^k \triangleq \left(\frac{\partial G}{\partial y_i}\right)_{\Phi_k}$, $1 \leq i \leq m$, $k \in \mathbb{Z}_+$, then the matrices A_k and B_k

used in Proposition 2.2.1 are given by

$$\mathbf{A}_k = \begin{bmatrix} G_1^k & \cdots & G_{m-1}^k & G_m^k & 1 & C_1 & \cdots & C_{n_3} \\ 1 & 0 & \cdots & & & & \cdots & 0 \\ 0 & \ddots & 0 & \cdots & & & \cdots & 0 \\ \vdots & & 1 & 0 & \cdots & & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & \cdots & & \cdots & 0 \\ \vdots & & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \ddots & & \vdots \\ 0 & \cdots & & & \cdots & 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where the $m+1^{\text{th}}$ row of A_k consists of zeros, and the $m+1^{\text{th}}$ entry of B_k is 1. Hence when $m \geq n_3$ the generalized controllability matrix $\mathbf{C}_T$ becomes

$$\mathbf{C}_T = \begin{bmatrix} 0 & 1 & \alpha_1^{T-1} & \alpha_2^{T-1} & \alpha_3^{T-1} & \cdots & & & \cdots & \alpha_{T-1}^{T-1} \\ 0 & 0 & 1 & \alpha_1^{T-1} & \alpha_2^{T-1} & \alpha_3^{T-1} & \cdots & & \cdots & \alpha_{T-2}^{T-1} \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ 0 & 0 & \cdots & \cdots & 1 & \alpha_1^{T-1} & \alpha_2^{T-1} & \alpha_3^{T-1} & \cdots & \alpha_{T-m}^{T-1} \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots & & & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ 0 & \cdots & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{2.54}$$

where the α_k^{T-1} are defined by

$$\begin{aligned} \alpha_0^{T-1} &= 1 \\ \alpha_1^{T-1} &= G_1^{T-1} + C_1 \\ \alpha_2^{T-1} &= G_1^{T-1}(G_1^{T-2} + C_1) + G_2^{T-2} + C_2 \\ &\vdots \end{aligned} \tag{2.55}$$

Letting $\alpha^k(z)$ denote the formal power series $\sum_{n=0}^{\infty} \alpha_n^k z^n$ it follows from (2.55) that $\alpha^k(z)$ may be computed inductively:

$$\alpha^k(z) = \begin{cases} 0 & \text{for } k = 0; \\ (G^k(z) - 1)\alpha^{k-1}(z) + C(z) & \text{for } k \geq 1, \end{cases}$$

where $G^k(z) \triangleq 1 + G_1^k z + \cdots + G_m^k z^m$ and $C(z) \triangleq 1 + C_1 z + \ldots + C_{n_3} z^{n_3}$.

Hence a sufficient condition for local stochastic controllability in this case is that for some $T \in \mathbb{Z}_+$ and each $\Phi_0 = x \in \mathbf{X}$ the matrix below is full rank:

$$\begin{bmatrix} \alpha_m & \alpha_{m+1} & \alpha_{m+2} & \alpha_{m+3} & \cdots & \alpha_{m+T} \\ \vdots & & & & & \vdots \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_{1+T} \\ \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_T \end{bmatrix}. \tag{2.56}$$

As an illustration suppose G is linear so that

$$\alpha_k = G_1\alpha_{k-1} + \ldots + G_m\alpha_{k-m} + C_k, \qquad k \in \mathbb{Z}_+, \tag{2.57}$$

where the $G_i, 1 \le i \le m$, do not depend on Φ. Equation (2.57) may be written in the more compact form

$$G(z)\alpha(z) = C(z)$$

where $G(z) \triangleq 1 + G_1 z + \ldots + G_m z^m$. We assume without loss of generality that $G_m \neq 0$ so that $G(z)$ is an m^{th} order polynomial.

If the matrix (2.56) is not full rank for any $T \in \mathbb{Z}_+$, then there exist polynomials $d(z)$ and $D(z)$ whose orders do not exceed $m-1$ and m respectively such that

$$d(z)\alpha(z) = D(z), \tag{2.58}$$

which is possible if and only if C and G have common factors. It follows that the system (2.53) is locally stochastically controllable whenever the polynomials G and C are co-prime.

To illustrate what can go wrong with local stochastic controllability, consider the ARMAX system (2.50) with $n_3 \neq 0$ under the mean square optimal control law

$$B(z)u(z) = [A(z) - C(z)]\, y(z),$$

and define

$$\Phi_{k+1} = \left(y_k, \cdots, y_{k-n_1}, u_{k-d}, \cdots, u_{k-n_2}, w_k, \cdots, w_{k-n_3}\right)^{\top}.$$

Suppose that the zeros of the polynomials B and C lie outside the unit disc in $\mathbb{C}$. It is easily shown that the closed loop system is stable in probability in this case and that

$$\lim_{k\to\infty} y_k - w_k = 0 \quad a.s. \ [\mathrm{P}_{\Phi_0}] \tag{2.59}$$

for all initial conditions Φ_0.

By (2.59) we have, $y_0 = w_0 \quad a.s. \ [\mathrm{P}_{\mu_\infty}]$ for the (unique) invariant probability π, and hence π is supported on a hyperplane in $\mathbb{R}^n$. Consequently, this system is *not* locally stochastically controllable because the support of π is not equivalent to an open set. Similarly, the stochastic gradient algorithm of [Goodwin, Ramadge, and Caines, 1981] does not give rise to a locally stochastically controllable system because the variable r_k converges to zero almost surely.

However, one very active research area in stochastic control theory today is the adaptive control of time varying systems (see [Meyn, Caines, 1987], [Chen, Caines, 1986].) It is in this area that the ideas introduced in this chapter will be very useful. For example, the Markovian system of [Meyn, Caines, 1987] is locally stochastically controllable, and this fact was crucial in establishing many of the results in that paper. Furthermore, the ARMAX system (2.50) controlled by a forgetting factor type algorithm gives rise to a locally stochastically controllable Markovian system under mild restrictions on (2.50).

Chapter 3 STRUCTURAL ROBUSTNESS

3.1 Objectives

In this chapter we examine the robustness properties of stochastic systems which are Markovianizable under feedback; we assume that a Markov state process $\mathbf{\Phi}$ evolving on a Borel subset $\mathbf{X} \subset \mathbb{R}^M$ exists, and that for some Borel function $\Psi : \mathbf{X} \to \mathbb{R}^m$, the output process $\mathbf{y}$ has the representation

$$y_k = \Psi(\Phi_k), \quad k \in \mathbb{Z}_+. \tag{3.1}$$

For stochastic systems of the form (3.1) we will be concerned with finding conditions under which invariant probabilities on the state process vary continuously under perturbations of the state process $\mathbf{\Phi}$. As we shall see, this is an important question in stochastic system theory. In particular, if this is the case then the value f_∞ of ergodic averages of the form

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(y_k) = f_\infty \quad a.s. \ [\mathrm{P}_x] \tag{3.2}$$

also vary continuously for a large class of functions f.

To illustrate the basic ideas, consider the controlled system (1.44) where the control $u_k = -k\Phi_k$, $k \in \mathbb{R}$, and $|k - \alpha| \ll 1$. Then the invariant probability of the closed loop system possesses the density p_k which is compared to the disturbance density p_w in figure 3.1. This example suggests a number of questions. In particular, it is easy to verify that as k approaches α, the corresponding density p_k approaches p_w, and figure 3.1 illustrates this fact. It would be very interesting to establish a similar result for *nonlinear* perturbations of the optimal control, i.e. if $g(x) \cong \alpha x$ does it follow that the resulting invariant probability $p_u \cong p_w$? A related question is this: Under what conditions does the controlled invariant density p_u vary continuously under a perturbation of the density p_w? These questions are of fundamental importance in stochastic system theory because we can never model a plant, or the statistics of a disturbance process perfectly.

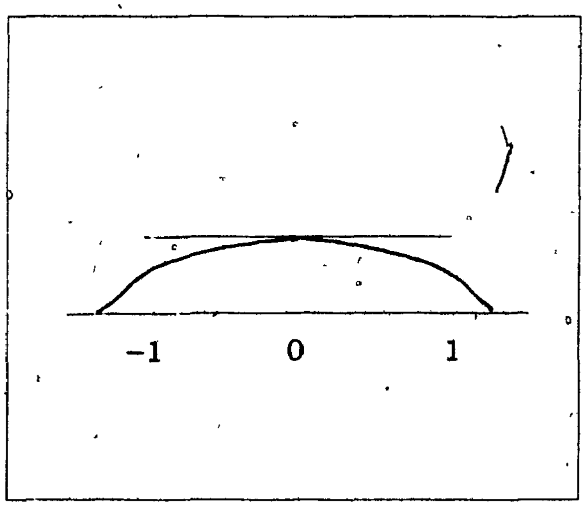


Figure 3.1 ———— p_k: the invariant density ———— p_w: the density of μ_w

A deterministic version of this problem has been carefully examined in [Mees and Chua, 1979]. In this work a dynamical system on $\mathbb{R}^n$ of the form

$$\mathbf{x}' = f(\mathbf{x}, \xi) \tag{3.3}$$

is studied for values of the parameter $\xi \in \mathbb{R}$ close to ξ_0. At $\xi = \xi_0$ it is assumed that there is a stable critical point x_0, and it is shown under very general conditions that in this case there exist periodic orbits near x_0 when ξ is near ξ_0, and that these orbits converge to the point x_0 as $\xi \to \xi_0$. It is shown that if a variable called the *curvature constant* is positive then each of these periodic orbits will be locally asymptotically stable. This is a special case of the problem under study in this chapter because the system described in 3.3 is a Markov process, x_0 is a critical point if and only if the probability δ_{x_0} is invariant, and if $\mathcal{L}$ is a closed curve in $\mathbb{R}^n$, then it is a periodic orbit if and only if the probability Υ† on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ given by $\Upsilon \triangleq C\frac{ds}{\|f(\mathbf{x})\|_2}$ is invariant where $C > 0$ is a normalizing constant, and ds is the increment of arc length on $\mathcal{L}$.

With this example in mind, consider the ε-parameterized system of the form (3.1) where the Markov chain $\mathbf{\Phi}^\varepsilon$ is generated by the recursion:

$$\Phi^\varepsilon_{k+1} = F(\Phi^\varepsilon_k, \xi^\varepsilon_{k+1}) \qquad \text{for } \varepsilon \in [0,1],\ k \in \mathbb{Z}_+, \tag{3.4}$$

and the output readout map Ψ is fixed. The function $F\colon \mathbb{R}^M \times \mathbb{R}^p \to \mathbb{R}^M$ is continuous, and ξ^ε is an $\mathbb{R}^p$-valued i.i.d. stochastic process with $\xi^\varepsilon_{k+1} \sim \mu^\varepsilon$ for $k \in \mathbb{Z}_+$, $\varepsilon \in [0,1]$. The probabilities $\{\mu^\varepsilon : \varepsilon \in [0,1]\}$ form a *curve* in $\mathcal{M}$, and to make the ε-parameterization in (3.4) continuous we assume that this curve is continuous in the topology of weak convergence in $\mathcal{M}$.

Suppose that at some $\varepsilon_0 \in \mathbb{R}$ the Markov chain $\mathbf{\Phi}^{\varepsilon_0}$ has a unique invariant probability π^{ε_0}. Can we then say that the Markovian system (3.4) has an invariant

† The invariant probability Υ for this example has an elegant interpretation. For two points $x(t_0), x(t_1) \in \mathcal{L}$ with $t_0 < t_1$ it may be verified that $\Upsilon\{x(t) : t_0 \le t \le t_1\} = (t_1 - t_0)/T$, where T is the amount of time required for $\mathbf{x}$ to complete one orbit. That is, the Υ-measure of the set of points between x_0 and x_1 on $\mathcal{L}$ is a constant times the length of time it takes to reach x_1, starting at x_0

probability π^ε for ε close to ε_0 in this case? And if so, will these invariant probabilities approach π^{ε_0} as $\varepsilon \to \varepsilon_0$? In the results that follow we show that this is indeed the case under certain conditions. Because of the extreme flexibility in the choice of the representation (3.1) we will find that these results have wide applicability to a variety of robustness issues in stochastic control and system theory such as:

- Finding the effect of a perturbation of the parameters, or the distribution of the disturbance process on the asymptotic properties of the output of a nonlinear stochastic system operating under feedback;

- Establishing convergence results for the underlying distributions of the output process of such systems;

- Estimating the performance criteria J^ε_∞ and L^ε_∞ of the perturbed systems for values of ε close to ε_0.

In section 3 we give a detailed example to show how these results may be applied to the stability analysis of stochastic time varying systems. In section 4 we discuss some open problems that could possibly be solved using these methods, and present a general result which will be applied in Chapter IV.

3.2 Approximation of Stochastic Systems

We begin our discussion on the approximation of Markovianizable stochastic systems by presenting a notion of convergence for stochastic processes on $\mathbf{X}^{\mathbb{Z}_+}$, and Markov transition operators on $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$. Let $\{\mathrm{P}^\varepsilon : 0 \leq \varepsilon \leq 1\}$ be a family of probabilities on

$(\mathbf{X}^{\mathbb{Z}_+}; \mathcal{B}(\mathbf{X}^{\mathbb{Z}_+}))$. Then when $\mathbf{X}^{\mathbb{Z}_+}$ is endowed with the product topology (see [Billingsley, 1968]) it follows that $\mathrm{P}^\varepsilon \xrightarrow{\text{weakly}} \mathrm{P}^0$ as $\varepsilon \to 0$ if and only if

$$\lim_{\varepsilon \to 0} \mathrm{P}^\varepsilon \{A_0 \times \cdots \times A_N \times \mathbf{X} \times \mathbf{X} \times \cdots\} = \mathrm{P}^0 \{A_0 \times \cdots \times A_N \times \mathbf{X} \times \mathbf{X} \times \cdots\} \quad (3.5)$$

for every $N \in \mathbb{Z}_+$, and every finite rectangle $(A_0 \times \cdots \times A_N \times \mathbf{X} \times \mathbf{X} \times \cdots) \in \mathcal{B}(\mathbf{X}^{\mathbb{Z}_+})$ whose boundary has P^0-measure zero.

We say that the Markov transition operators $\{\mathbf{T}_\varepsilon : 0 < \varepsilon \leq 1\}$ *converge* to the Markov transition operator $\mathbf{T}_0$ and write $\mathbf{T}_\varepsilon \to \mathbf{T}_0$ as $\varepsilon \to 0$, if for every $x \in \mathbf{X}$ and $f \in \mathbf{C}$,

$$\lim_{\varepsilon \to 0} \mathbf{T}_\varepsilon f(x) = \mathbf{T}_0 f(x).$$

Under very general conditions it will follow that

$$\mathrm{P}_x^\varepsilon \xrightarrow{\text{weakly}} \mathrm{P}_x^0 \quad \text{as} \quad \varepsilon \to 0 \quad (3.6)$$

for every $x \in \mathbf{X}$.

Unfortunately, the existence of the limit in (3.6) tells us little about the relationship between the asymptotic behavior of the Markov chain corresponding to $\mathbf{T}_\varepsilon$, and the asymptotic behavior of the limiting Markov chain generated by $\mathbf{T}_0$. Take for example, the deterministic Markov chain $\mathbf{\Phi}^\varepsilon$ on $\mathbf{X} \triangleq [-1, 1]$ given by

$$\Phi_{k+1}^\varepsilon = (1 - \varepsilon)\Phi_k^\varepsilon \qquad k \in \mathbb{Z}_+,\ 0 \leq \varepsilon \leq 1. \quad (3.7)$$

The Markov transition operator $\mathbf{T}_\varepsilon$ generating $\mathbf{\Phi}^\varepsilon$ is defined for $f \in \mathbf{B}$, $x \in \mathbf{X}$, and $0 \leq \varepsilon \leq 1$ by

$$\mathbf{T}_\varepsilon f(x) = f((1 - \varepsilon)x). \quad (3.8)$$

It is easily verified that for every $f \in \mathbf{C}$, $\mathbf{T}_\varepsilon f \to \mathbf{T}_0 f$ *in the uniform norm* as $\varepsilon \to 0$, and we conclude that (3.6) holds for this example. The Markovian system (3.7) is of the form (2.3) studied in Chapter II, and for each $\varepsilon \in [0,1]$ it is stable in probability. The invariant probabilities $\{\pi_{x,\varepsilon} : 0 \leq \varepsilon \leq 1\}$ for (3.7) are defined $f \in \mathbf{C}$, $x \in \mathbf{X}$, and $0 \leq \varepsilon \leq 1$ by

$$\int f \, d\pi_{x,\varepsilon} \triangleq \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \mathbf{T}_{\varepsilon}^{k} f(x), \tag{3.9}$$

and it follows that for all $x \in \mathbf{X}$ the invariant probability $\pi_{x,\varepsilon} = \delta_0$ when $\varepsilon > 0$, but the invariant probability $\pi_{x,0}$ of the limiting system is δ_x. So, even with this extremely strong form of convergence holding for $\{\mathbf{T}_\varepsilon : 0 \leq \varepsilon \leq 1\}$, and the *uniform* stability of the Markov chains $\{\mathbf{\Phi}^\varepsilon : 0 \leq \varepsilon \leq 1\}$ we still cannot infer that the invariant probabilities $\{\pi_{x,\varepsilon} : 0 < \varepsilon \leq 1\}$ converge to $\pi_{x,0}$ as $\varepsilon \to 0$.

In order to avoid such pathologies we will begin our robustness analysis of Markovianizable stochastic systems by considering problems in which the adjoint $\mathbf{U}_0$ of $\mathbf{T}_0$ possesses at most one invariant probability. Throughout this chapter $\{\mathbf{T}_\varepsilon : 0 \leq \varepsilon \leq 1\}$ will dentote a set of Markov transition operators, $\{\mathbf{U}_\varepsilon : 0 \leq \varepsilon \leq 1\}$ their respective adjoints, and $\{\pi_\varepsilon : 0 \leq \varepsilon \leq 1\}$ a set of probabilities such that π_ε is invariant under $\mathbf{U}_\varepsilon$ for each $\varepsilon \in [0,1]$. Finally, we let $\mathbf{C}_c$ denote the set of continuous real-valued functions on $\mathbf{X}$ which vanish outside of some compact set, and a family of real-valued functions $\{f_\alpha\}_{\alpha \in A}$ on $\mathbf{X}$ is said to be *equicontinuous on compacta* if for all $\gamma \in \mathbf{C}_c$, the family $\{\gamma f_\alpha\}_{\alpha \in A}$ is equicontinuous. We list here some of the assumptions that we will occasionally be using later in this chapter:

R0 $\mathbf{U}_0$ possesses at most one invariant probability;

R1 The set of probabilities $\{\pi_\varepsilon : 0 \leq \varepsilon \leq 1\}$ is tight;

R2 For each $f \in \mathbf{C}_c$ the collection of functions $\{\mathbf{T}_\varepsilon f : 0 \leq \varepsilon \leq 1\}$ is equicontinuous on compacta.

R3 Whenever the probabilities $\{\mu_\varepsilon : 0 \leq \varepsilon \leq 1\}$ satisfy

$$\mu_\varepsilon \xrightarrow{\text{weakly}} \mu_0 \quad \text{as} \quad \varepsilon \to 0,$$

it follows that

$$\mathbf{U}_\varepsilon \mu_\varepsilon \xrightarrow{\text{weakly}} \mathbf{U}_0 \mu_0 \quad \text{as} \quad \varepsilon \to 0.$$

We remark that conditions **R2** and $\mathbf{T}_\varepsilon \to \mathbf{T}_0$ as $\varepsilon \to 0$. are equivalent to the condition that the functions $\{\mathbf{T}_\varepsilon f : \varepsilon > 0\}$ converge uniformly on compact sets to the function $\mathbf{T}_0 f$ for every $f \in \mathbf{C}_c$.

The first result below concerns perturbations of the disturbance distribution μ_w. Suppose that the Markov chains $\{\mathbf{\Phi}^\varepsilon : 0 \leq \varepsilon \leq 1\}$, have the form

$$\Phi_k^\varepsilon = F(\Phi_{k-1}^\varepsilon, w_k^\varepsilon) \tag{3.10}$$

where $F : \mathbf{X} \times \mathbb{R}^p \to \mathbf{X}$ is Borel measurable, and for each $\varepsilon \in [0,1]$, $\mathbf{w}^\varepsilon \triangleq \{w_k^\varepsilon : k \in \mathbb{Z}_+\}$ is independent and identically distributed with $w_k^\varepsilon \sim \mu_w^\varepsilon$ for each $k \in \mathbb{Z}_+$, and $\varepsilon \in [0,1]$. Then the Markov transition operators $\mathbf{T}_\varepsilon$, $\varepsilon \in [0,1]$, are defined for $g \in \mathbf{B}$ by

$$\mathbf{T}_\varepsilon g\,(x) = \int_{\mathbb{R}^p} g(F(x,\lambda))\mu_w^\varepsilon(d\lambda). \tag{3.11}$$

Proposition 3.2.1. *For the Markov transition operators $\mathbf{T}_\varepsilon$, $\varepsilon \in [0,1]$, defined in (3.11) suppose that one of the following two conditions holds:*

(*i*) $\{\mu_w^\varepsilon : \varepsilon > 0\}$ *converges in total variation norm to* μ_w^0 as $\varepsilon \to 0$.

(*ii*) *The function F is continuous and* $\mu^\varepsilon \xrightarrow{\text{weakly}} \mu_w^0$ as $\varepsilon \to 0$.

Then $\mathbf{T}_\varepsilon \to \mathbf{T}_0$ and condition **R2** *holds.*

Proof.

To prove the proposition we will show that if either (i) or (ii) is satisfied then for each $g \in \mathbf{C}_c$, $\mathbf{T}_\varepsilon g \to \mathbf{T}_0 g$ uniformly on compact sets. Hence, result (i) follows by using the estimate

$$\|(\mathbf{T}_0 - \mathbf{T}_\varepsilon)g\|_\infty \leq \|g\|_\infty \, \|\mu_w^\varepsilon - \mu_w^0\|_{tv},$$

where $\|\mu\|_{tv}$ is the total variation norm of the finite signed measure μ, from which it follows that $\mathbf{T}_\varepsilon g$ converges uniformly to $\mathbf{T}_0 g$.

Suppose now that (ii) holds. Let $\delta > 0$, and $C_1 \subset \mathbf{X}$ be an arbitrary compact set. Choose $C_2 \subset \mathbb{R}^p$ so that $\mu_w^0\{\partial C_2\} = 0$ and $\mu_w^\varepsilon\{C_2\} \geq 1 - \delta$ for all $0 \leq \varepsilon \leq 1$. This is possible because the probabilities $\{\mu_w^\varepsilon : \varepsilon > 0\}$ are convergent and hence tight. The function F is uniformly continuous on $C_1 \times C_2$ and hence letting $\xi_x(\cdot) \triangleq F(x, \cdot)$ for $x \in C_1$ the family of functions $\{\xi_x : x \in C_1\}$ is equicontinuous on C_2.

Let $g \in \mathbf{C}_c$. Then because g is uniformly continuous, the family of functions $\{g(\xi_x) \, : x \in C_1\}$ is also equicontinuous on C_2. Since $\mu_w^0\{\partial C_2\} = 0$ it follows from assumption (ii) of the proposition and Theorem 2.1 of [Billingsley, 1968] that

$$\mu_w^\varepsilon \mathbf{1}_{C_2} \xrightarrow{\text{weakly}} \mu_w^0 \mathbf{1}_{C_2} \quad \text{as} \quad \varepsilon \to 0.$$

Hence Theorem 1.1.1 (iv) applies to give

$$\lim_{\varepsilon \to 0} \sup_{x \in C_1} \left| \int_{C_2} g(\xi_x(\lambda))\mu_w^\varepsilon(d\lambda) - \int_{C_2} g(\xi_x(\lambda))\mu_w^0(d\lambda) \right| = 0 \tag{3.12}$$

It now follows easily that $\mathbf{T}_\varepsilon g$ converges to $\mathbf{T}_0 g$ uniformly on C_1 as $\varepsilon \to 0$: By the hypotheses made on C_2 and the definition of $\mathbf{T}_\varepsilon$ we have

$$\begin{aligned}\sup_{x \in C_1} |\mathbf{T}_\varepsilon g(x) - \mathbf{T}_0 g(x)| &= \sup_{x \in C_1} \left| \int g(\xi_x(\lambda))\mu_w^\varepsilon(d\lambda) - \int g(\xi_x(\lambda))\mu_w^0(d\lambda) \right| \\ &\leq 2\delta \|g\|_\infty + \sup_{x \in C_1} \left| \int_{C_2} g(\xi_x(\lambda))\mu_w^\varepsilon(d\lambda) - \int_{C_2} g(\xi_x(\lambda))\mu_w^0(d\lambda) \right|\end{aligned}$$

Hence by equation (3.12),

$$\limsup_{\varepsilon \to 0} \sup_{x \epsilon C_1} \left| \mathbf{T}_\varepsilon g(x) - \mathbf{T}_0 g(x) \right| \leq 2\delta \|g\|_\infty$$

and since δ is an arbitrary constant $\mathbf{T}_\varepsilon g$ converges uniformly to $\mathbf{T}_0 g$ on C_1 and this completes the proof.

□

The following lemma is adapted from Theorem 6 of Chapter 6, section 4 of [Kushner, 1984]:

Lemma 3.2.1. *Suppose that assumptions* **R1** *and* **R3** *hold, and that* $\mathbf{U}_\varepsilon \pi_\varepsilon = \pi_\varepsilon$ *for each* $\varepsilon > 0$. *Then,*

$$\pi_\varepsilon \xrightarrow{\text{weakly}} \mathcal{I}_0 \quad \text{as} \quad \varepsilon \to 0, \tag{3.13}$$

where $\mathcal{I}_0 \subset \mathcal{M}$ *is the set of probabilities which are invariant under* $\mathbf{U}_0$.

Proof.

Let Υ be a limit point of $\{\pi_\varepsilon : 0 < \varepsilon \leq 1\}$. Then for some sequence $\{\varepsilon_i : i \in \mathbb{Z}_+\}$, converging to zero,

$$\pi_{\varepsilon_i} \xrightarrow{\text{weakly}} \Upsilon \quad \text{as} \quad i \to \infty.$$

Applying assumption **R3** we find that $\mathbf{U}_{\varepsilon_i} \pi_{\varepsilon_i} \xrightarrow{\text{weakly}} \mathbf{U}_0 \Upsilon$ as $i \to \infty$. Since $\mathbf{U}_{\varepsilon_i} \pi_{\varepsilon_i} = \pi_{\varepsilon_i}$ for all $i \in \mathbb{Z}_+$ we conclude that

$$\mathbf{U}_0 \Upsilon = \Upsilon,$$

and this proves the lemma.

□

In the following result we give a sufficient condition to ensure the convergence of the invariant probabilities corresponding to a convergent sequence of Markov transition operators. This result will be very useful in Chapter IV where we investigate the robustness and asymptotic behavior of linear systems operating under nonlinear feedback.

Proposition 3.2.2. *Suppose that* $\mathbf{T}_\varepsilon \to \mathbf{T}_0$ *as* $\varepsilon \to 0$, *where the Markov transition operator* $\mathbf{T}_0$ *has the Feller property, and that conditions* **R0**, **R1** *and* **R2** *hold. Then,*

$$\pi_\varepsilon \xrightarrow{\text{weakly}} \pi_0 \qquad \text{as} \quad \varepsilon \to 0$$

where π_0 *is invariant under* $\mathbf{U}_0$.

Observe that we do not require that the Markov transition operator $\mathbf{T}_\varepsilon$ be Feller for $\varepsilon > 0$.

Proof.

By Lemma 3.2.1 and **R0** it is enough to show that $\{\mathbf{T}_\varepsilon : 0 \leq \varepsilon \leq 1\}$ satisfies condition **R3**. Let $\mu_\varepsilon \xrightarrow{\text{weakly}} \mu$ as $\varepsilon \to 0$, and fix $f \in \mathbf{C}_c$. Then, letting $\langle \nu, f \rangle \triangleq \int f \, d\nu$,

$$\langle \mathbf{U}_\varepsilon \mu_\varepsilon, f \rangle = \langle \mu_\varepsilon, (\mathbf{T}_\varepsilon - \mathbf{T}_0) f \rangle + \langle \mu_\varepsilon, \mathbf{T}_0 f \rangle.$$

Since $\mathbf{T}_\varepsilon \to \mathbf{T}_0$ as $\varepsilon \to 0$ and condition **R2** is satisfied the first summand converges to zero as $\varepsilon \to 0$. Hence, because $\mathbf{T}_0 f \in \mathbf{C}$,

$$\lim_{\varepsilon \to 0} \langle \mathbf{U}_\varepsilon \mu_\varepsilon, f \rangle = \langle \mu_0, \mathbf{T}_0 f \rangle = \langle \mathbf{U}_0 \mu_0, f \rangle$$

which shows that $\mathbf{U}_\varepsilon \mu_\varepsilon \xrightarrow{\text{vaguely}} \mathbf{U}_0 \mu_0$ as $\varepsilon \to 0$. Since $\mathbf{U}_0 \mu_0$ is a probability it follows that $\mathbf{U}_\varepsilon \mu_\varepsilon \xrightarrow{\text{weakly}} \mathbf{U}_0 \mu_0$. Hence condition **R3** is satisfied, and this completes the proof. □

We remark that if assumption **R2** does not hold then the conclusions of Proposition 3.2.2 may not be valid. Take for example, the Markov chain $\mathbf{d}^\varepsilon$ evolving on $\mathbb{R}$ defined by

$$d_{k+1}^\varepsilon = f_\varepsilon(d_k), \quad k \in \mathbb{Z}_+, \quad 0 \le \varepsilon \le 1/2 \tag{3.14}$$

where $f_0 \equiv 0$, and the continuous function f_ε is defined for $\varepsilon > 0$ by

$$f_\varepsilon(t) = \begin{cases} 0 & \text{if } t \in [0, 1-2\varepsilon]; \\ 1-\varepsilon & \text{if } t = 1-\varepsilon; \\ \text{affine} & \text{on } [1-2\varepsilon, 1-\varepsilon] \text{ and on } [1-\varepsilon, 1]. \end{cases}$$

The graph of f_ε for $\varepsilon > 0$ is shown in figure 3.2.1.

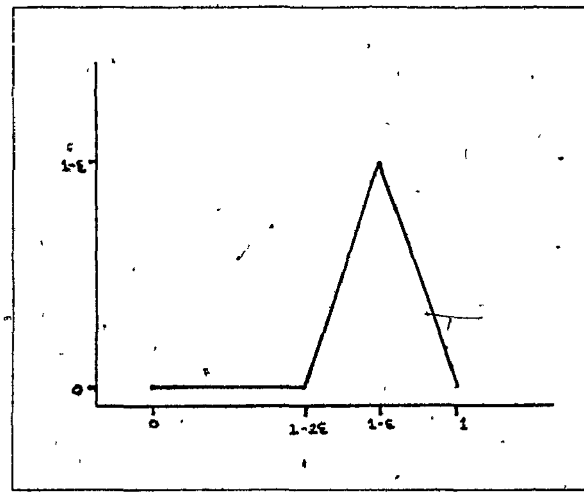


Figure 3.2 Graph of f_ε

The collection of Markov transition functions $\{\mathbf{T}_\varepsilon : 0 \le \varepsilon \le 1/2\}$ satisfy all of the conditions of Proposition 3.2.2 except for condition **R2**. Furthermore, for each $\varepsilon > 0$, $\delta_{1-\varepsilon}$ is an invariant probability for $\mathbf{d}^\varepsilon$, and does not converge to the invariant probability for $\mathbf{d}^0$ as $\varepsilon \to 0$.

The assumption **R0** is too strong in many applications. To illustrate the difficulty involved in relaxing this assumption we will assume in the next few paragraphs that the following condition is satisfied:

R4 Each of the Markov transition operators in the collection $\{\mathbf{T}_\varepsilon : 0 \leq \varepsilon \leq 1\}$ has the Feller property, and for each $x \in \mathbf{X}$ and $k \in \mathbb{Z}_+$ the set $\{\mathbf{U}_\varepsilon^k \delta_x : 0 \leq \varepsilon \leq 1\}$ is a continuous curve in $\mathcal{M}$.

Observe that $\{\mathbf{U}_\varepsilon^k \delta_x : 0 \leq \varepsilon \leq 1\}$ is a continuous curve in $\mathcal{M}$ if $\mathbf{T}_\varepsilon^k \to \mathbf{T}_{\varepsilon_0}^k$ as $\varepsilon \to \varepsilon_0$ for each $\varepsilon_0 \in [0,1]$.

Suppose that $\mathbf{\Phi}^\varepsilon$ is stable in probability for each $0 \leq \varepsilon \leq 1$ and let $\pi_{x,\varepsilon}$ denote the invariant probability defined as in (3.9). Our goal is to find general sufficient conditions to ensure that

$$\pi_{x,\varepsilon} \xrightarrow{\text{weakly}} \pi_{x,0} \quad \text{as} \quad \varepsilon \to 0 \tag{3.15}$$

for every $x \in \mathbf{X}$.

Fix $x \in \mathbf{X}$, and for $N \in \mathbb{Z}_+$ let $\mathrm{av}_N\mathbf{U}: [0,1] \to \mathcal{M}$ denote the continuous curve defined for $\varepsilon \in [0,1]$ by

$$\mathrm{av}_N\mathbf{U}(\varepsilon) \triangleq \frac{1}{N}\sum_{k=1}^{N} \mathbf{U}_\varepsilon^k \delta_x. \tag{3.16}$$

The functions $\{\mathrm{av}_N\mathbf{U} : N \in \mathbb{Z}_+\}$ are continuous whenever condition **R4** holds, and for each $0 \leq \varepsilon \leq 1$,

$$\mathrm{av}_N\mathbf{U}(\varepsilon) \xrightarrow{\text{weakly}} \pi_{x,\varepsilon} \qquad \text{as} \quad N \to \infty. \tag{3.17}$$

Hence (3.15) holds if and only if the function $\mathrm{av}_\infty\mathbf{U}: [0,1] \to \mathcal{M}$ defined for $\varepsilon \in [0,1]$ by $\mathrm{av}_\infty\mathbf{U}(\varepsilon) = \pi_{x,\varepsilon}$ is continuous at 0. This shows that finding conditions to ensure that the limit in (3.15) holds is equivalent to the solving the following problem:

Given a sequence of continuous functions $\{h_k : k \in \mathbb{Z}_-\}$ mapping $[0,1]$ into a metric space $\mathbf{Y}$, under what conditions does the existence of the pointwise limit

$$\lim_{k\to\infty} h_k = h_\infty$$

imply that h_∞ is continuous at zero?

Although this approach is illuminating, it does not yield any profound results. Another approach to this problem is to apply some of the ideas used in the proof of Proposition 1.3.1. In [Sapperstone, 1981] the existence of the limit

$$\frac{1}{N}\sum_{k=1}^{N} \mathbf{U}^k \delta_x \xrightarrow{\text{weakly}} \pi_x \quad \text{as} \quad N \to \infty$$

is established under the appropriate conditions by first showing that

$$\frac{1}{N}\sum_{k=1}^{N} \mathbf{U}^k \delta_x \xrightarrow{\text{weakly}} I \qquad \text{as} \quad N \to \infty,$$

where I is the set of probabilities which are invariant under $\mathbf{U}$. The proof is completed by establishing the following lemma:

Lemma 3.2.2. *Let $\overline{\text{co}}_x \subset \mathcal{M}$ denote the closed convex hull of the set of probabilities*

$$\left\{\frac{1}{N+1}\sum_{k=0}^{N} \mathbf{U}^k \delta_x : N \in \mathbb{Z}_+\right\} \tag{3.18}$$

where $\mathbf{U}$ is the adjoint of the Feller Markov transition operator $\mathbf{T}$, and assume that the set of probabilities in (3.18) is tight. Then $\overline{\text{co}}_x \cap I$ consists of exactly one probability π_x. □

The following result follows directly from Lemma 3.2.2. Let $\overline{\mathrm{co}}_{x,0}$ denote the closed convex hull of the set of probabilities (3.18) with $\mathbf{U}$ replaced by $\mathbf{U}_0$.

Proposition 3.2.3. *Suppose that the Markov transition operators $\mathbf{T}_\varepsilon$, $\varepsilon \in [0,1]$, have the Feller property and satisfy condition* **R2**. *Suppose that for some $x \in \mathbf{X}$ the corresponding invariant probabilities $\{\pi_{x,\varepsilon} : 0 \leq \varepsilon \leq 1\}$ satisfy condition* **R1**, *and that $\mathbf{T}_\varepsilon \to \mathbf{T}_0$ as $\varepsilon \to 0$. Then,*

$$\pi_{x,\varepsilon} \overset{\text{weakly}}{\longrightarrow} \pi_{x,0} \quad \text{as} \quad \varepsilon \to 0$$

if and only if

$$\pi_{x,\varepsilon} \overset{\text{weakly}}{\longrightarrow} \overline{\mathrm{co}}_{x,0} \quad \text{as} \quad \varepsilon \to 0.$$

Proof.

The only if part is trivial since $\pi_{x,0} \in \overline{\mathrm{co}}_{x,0}$. To establish the other direction we use the Feller property, assumptions **R1** and **R2**, and the technique used in the proof of Proposition 3.2.2 to show that

$$\pi_{x,\varepsilon} \overset{\text{weakly}}{\longrightarrow} \mathcal{I}_0 \quad \text{as} \quad \varepsilon \to 0$$

where $\mathcal{I}_0 \subset \mathcal{M}$ is the set of probabilities which are invariant under $\mathbf{U}_0$. Hence, applying Lemma 3.2.2 together with the assumptions of the proposition,

$$\pi_{x,\varepsilon} \overset{\text{weakly}}{\longrightarrow} \mathcal{I}_0 \cap \overline{\mathrm{co}}_{x,0} = \pi_{x,0} \quad \text{as} \quad \varepsilon \to 0,$$

and this completes the proof. □

We remark that the following condition is sufficient to ensure that $\pi_{x,\varepsilon} \xrightarrow{\text{weakly}} \overline{\text{co}}_{x,0}$ as $\varepsilon \to 0$: For every $f \in \mathbf{C}_c$,

$$\lim_{\varepsilon \to 0} \inf_{N \geq 0} \left\{ \left| \int f \, d\pi_{x,\varepsilon} - \frac{1}{N+1} \sum_{k=0}^{N} \mathbf{T}_0^k f(x) \right| \right\} = 0.$$

This concludes our presentation of the general robustness theory for stochastic systems. We now show how the results of this section may be applied to the stability analysis of a random parameter stochastic system operating under feedback, and in Chapter IV we will consider another application.

3.3 A Random Parameter Model

Consider the ARMAX system model:

$$\begin{aligned} y_k + a_1 y_{k-1} + \ldots + a_{n_1} y_{k-n_1} &= b_1 u_{k-1} + \ldots + b_{n_2} u_{k-n_2} \\ &\quad + w_k + c_1 w_{k-1} + \ldots + c_{n_3} w_{k-n_3} \qquad k \in \mathbb{Z}_+, \end{aligned} \tag{3.19}$$

which we will write in the form

$$y_{k+1} = \theta^{*\top} \varphi_k + b_1 u_k + w_{k+1} \tag{3.20(i)}$$

where,

$$\theta^{*\top} \triangleq (-a_1, \ldots, -a_{n_1}, b_2, \ldots, b_{n_2}, c_1, \ldots, c_{n_3}), \tag{ii}$$

$$\varphi_k^\top \triangleq (y_k, \ldots, y_{k-n_1+1}, u_{k-1}, \ldots, u_{k-n_2+1}, w_k, \ldots, w_{k-n_3+1}). \tag{iii}$$

We assume that the processes $\mathbf{y}$ and $\mathbf{u}$ are $\mathbb{R}$-valued, $\mathbf{w}$ is an $\mathbb{R}$-valued i.i.d. processes with $w_{k+1} \sim \mu_w$ for $k \in \mathbb{Z}_+$, and initial conditions independent of $\mathbf{w}$ are assigned at $k = 0$. Under these assumptions φ is a controlled Markov chain with state space $\mathbf{X} \triangleq \mathbb{R}^{n_1+n_2+n_3-1}$, and hence (3.20) defines a stochastic state space system as defined in Chapter I.

We also make the following technical assumptions. The polynomials $a(\cdot)$, $b(\cdot)$, and $c(\cdot)$ are defined by

$$a(z) \triangleq 1 + a_1 z + \cdots + a_{n_1} z^{n_1}, \quad zb(z) \triangleq b_1 z + \cdots + b_{n_2} z^{n_2}, \quad c(z) \triangleq 1 + c_1 z + \cdots + c_{n_3} z^{n_3}.$$

sys1 The zeros of the polynomials $c(\cdot)$ and $b(\cdot)$ fall outside the unit circle in $\mathbb{C}$;

sys2 The polynomials $b(\cdot)$ and $a(\cdot) - c(\cdot)$ are co-prime, b_1, b_{n_2}, $a_{n_1} \neq 0$, and for definiteness we assume that $a(\cdot) \not\equiv c(\cdot)$;

sys3 For some $p_0 > 2$ the probability μ_w satisfies

$$\int |t|^{p_0} \mu_w(dt) < \infty,$$

and hence, $\sigma_w^2 \triangleq \int t^2 \mu_w(dt) < \infty$.

sys4 The probability μ_w satisfies condition **A4** of Chapter II, and $0 \in \bar{\mathcal{O}}_w$.

Under these assumptions the s.m.s. and mean square optimal feedback control is given by solving the recursion

$$zb(z)u(z) = (a(z) - c(z))y(z). \tag{3.21}$$

With the control law so defined, the closed loop system is Markovianizable; $\mathbf{y}$, $\mathbf{u}$ and $\mathbf{w}$ are functions of the Feller Markov chain φ defined in equation (3.20) (*ii*). It is easy to show that corresponding to the Markov chain φ, there exists a unique invariant probability $P_{\pi}\varphi$ on the sequence space $(\mathbf{X}^{\mathbb{Z}_-}, \mathcal{B}(\mathbf{X}^{\mathbb{Z}_-}))$ under which $\mathbf{y} = \mathbf{w}$ *a.s.* $P_{\pi}\varphi$. Furthermore, for *any* initial condition $\varphi_0 = x \in \mathbf{X}$ we have

$$J_\infty \triangleq \lim_{k\to\infty} \mathrm{E}_x[y_k^2 + \rho u_k^2] = \sigma_w^2 + \rho\sigma_u^2, \tag{3.22(i)}$$

and

$$L_\infty \triangleq \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} y_k^2 + \rho u_k^2$$
$$= \sigma_w^2 + \rho\sigma_u^2 \qquad a.s.\ [\mathrm{P}_x], \tag{ii}$$

where $\sigma_w^2 \triangleq \mathrm{E}[w_0^2]$, and

$$\sigma_u^2 \triangleq \sigma_w^2 \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{a(e^{i\theta}) - c(e^{i\theta})}{b(e^{i\theta})}\right|^2 d\theta. \tag{3.23}$$

What can we say about the limit in (3.22) if the model (3.1) does not describe the true system *exactly*? For example, suppose that the parameters are in fact a stochastic process $\theta^\varepsilon \triangleq \{\theta_k^\varepsilon : k \in \mathbb{Z}_+\}$, for which $\|\theta_k^\varepsilon - \theta^*\|_2$ is small in a some statistical sense for all $k \in \mathbb{Z}_+$. We will show under very general conditions that the limits in (3.22) still exist in this case, and that the asymptotic and finite-time behavior of the time varying system is close to that of the time-invariant system. In fact, under the appropriate conditions we may show that the values of the criterion functions J_∞^ε and L_∞^ε of the perturbed system will be close to the criterion functions of the time invariant system. These results hold even for parameter processes with unbounded sample paths.

We henceforth suppose that the parameter process $\theta^\varepsilon \triangleq \{\theta_k^\varepsilon : k \in \mathbb{Z}_+\}$ is time varying and independent of the disturbance process $\mathbf{w}$, where $0 \le \varepsilon \le 1$, and thus (3.20) becomes

$$y_{k+1}^\varepsilon = \theta_k^{\varepsilon\top}\varphi_k^\varepsilon + b_1 u_k^\varepsilon + w_{k+1}. \tag{3.24}$$

Observe that we take the parameter b_1 to be time invariant. This is not necessary but is done to simplify the example. If the control law (3.21) is applied to this system and the parameter sequence θ^ε is viewed as the *input*, then (3.24) is a (nonlinear) stochastic state space system as defined in Chapter I. Let $P(\cdot;\cdot,\dots,\cdot,\cdot)$ denote the N-step controlled Markov transition function for (3.24). Then for all $A \in \mathcal{B}(\mathbf{X})$,

$$\mathrm{P}\{\varphi_N^\varepsilon \in A \mid \varphi_0^\varepsilon, \theta_0^\varepsilon, \dots, \theta_{N-1}^\varepsilon\} = P(\varphi_0^\varepsilon; (\theta^\varepsilon)_0^{N-1}, A). \tag{3.25}$$

The N-step controlled Markov transition function may be computed by noting that for each $N \in \mathbb{Z}_+$, the random variable φ_N^ε has the form

$$\varphi_N^\varepsilon = S_{\varphi_0^\varepsilon}^N(\theta_0^\varepsilon, \dots, \theta_{N-1}^\varepsilon, w_1, \dots, w_N) \tag{3.26}$$

where the function $S_{\varphi_0^\varepsilon}^N : \mathbf{X}^N \times \mathbb{R}^N \to \mathbf{X}$ is continuous (it is in fact affine in $\mathbf{w}$, and polynomial in θ.) Hence, for each $A \in \mathcal{B}(\mathbf{X})$ and $(\theta_0^\varepsilon, \dots, \theta_{N-1}^\varepsilon) \in \mathbf{X}^N$,

$$P(\varphi_0^\varepsilon; (\theta^\varepsilon)_0^{N-1}, A) = \int_{\mathbb{R}^N} \mathbf{1}_{\left\{S_{\varphi_0^\varepsilon}^N(\theta_0^\varepsilon, \dots, \theta_{N-1}^\varepsilon, w_1, \dots, w_N)\right\} \in A}\, \mu_w(dw_1) \dots \mu_w(dw_N).$$

The following lemma will be useful later in this section:

Lemma 3.3.1. *Suppose that the system (3.24) is under the control (3.21), and that assumption* **sys2** *is satisfied. Then there exists $N_1 \in \mathbb{Z}_+$ such that for every $\varphi_0^\varepsilon \in \mathbf{X}$, and a.e.-$[\mu^{Leb}]$ $(\theta_0^\varepsilon, \dots, \theta_{N_1-1}^\varepsilon) \in \mathbf{X}^{N_1}$ the matrix*

$$C_{\varphi_0^\varepsilon}^N \triangleq \left[\frac{\partial S_{\varphi_0^\varepsilon}^{N_1}}{\partial w_1} \Big| \cdots \Big| \frac{\partial S_{\varphi_0^\varepsilon}^{N_1}}{\partial w_{N_1}} \right] \tag{3.27}$$

is full rank.

Proof.

Since $S^N_{\varphi_0^\varepsilon}$ is linear in $\mathbf{w}$, the matrix (3.27) does not depend on $\mathbf{w}$. Furthermore, since for each $x \in \mathbf{X}$, $x^\top C^N_{\varphi_0^\varepsilon}$ is polynomial in θ, and a multi-variate polynomial is either zero everywhere or non-zero almost everywhere (see [Meyn, Caines, 1985]) we conclude that for each $N \in \mathbb{Z}_+$ either

$$\operatorname{rank} C^N_{\varphi_0^\varepsilon} < n1 + n2 + n3 - 1 \qquad \text{for all} \quad \theta_0^{N-1} \in \mathbf{X}^N,$$

or

$$\operatorname{rank} C^N_{\varphi_0^\varepsilon} = n1 + n2 + n3 - 1 \qquad \text{for } a.e\ [\mu^{Leb}] \quad \theta_0^{N-1} \in \mathbf{X}^N.$$

Hence, it is enough to show that there exists *one* sequence $\overline{\theta}_0^{N-1} = \{\overline{\theta}_0, \ldots, \overline{\theta}_{N-1}\}$ which makes $C^N_{\varphi_0^\varepsilon}$ full rank. We take $\overline{\theta}_0^{N-1} = \{\overline{\theta}, \ldots, \overline{\theta}\}$ where

$$\overline{\theta}^\top \triangleq (\ell_1, \ldots, \ell_{n_1}, b_2, \ldots, b_{n_2}, 1, 0, \ldots, 0),$$

and the polynomial $\ell(z) \triangleq 1 - \ell_1 z - \cdots - \ell_{n_1} z^{n1}$ is chosen so that the zeros of the polynomial

$$Q(\cdot) \triangleq \ell(\cdot) + c(\cdot) - a(\cdot)$$

lie outside the unit circle in $\mathbb{C}$, and $\deg Q(\cdot) \geq \deg a(\cdot)$.

To prove the lemma we will show that the linear Markovian system with state process $\overline{\varphi}^\top \triangleq (\overline{y}_k, \ldots, \overline{y}_{k-n_1+1}, \overline{u}_{k-1}, \ldots, \overline{u}_{k-n_2+1}, w_k, \ldots, w_{k-n_3+1})$ defined by

$$\overline{y}_{k+1} = \overline{\theta}^\top \overline{\varphi}_k + b_1 \overline{u}_k + w_{k+1} \tag{3.28}$$

with control (3.21) is locally stochastically controllable.

Observe that the system description (3.28) is equivalent to

$$Q(z)\overline{y}(z) = w(z), \tag{3.29}$$

and hence by the assumptions made on Q, the system (3.28) is stable in probability.

Therefore, a unique invariant probability $\overline{\pi}$ on $\mathcal{B}(\mathbf{X})$ exists for the Markov chain $\overline{\varphi}$ defined in equation (3.28). Consider the strictly stationary process $\overline{\varphi}$ on $\mathbf{X}^{\mathbb{Z}}$ for which $\overline{\varphi}_k$ has distribution $\overline{\pi}$ for each $k \in \mathbb{Z}$. In this case,

$$\overline{y}(z) = \frac{1}{Q(z)} w(z) \tag{3.30}$$

and

$$\overline{u}(z) = \frac{a(z) - c(z)}{zb(z)Q(z)} w(z). \tag{3.31}$$

If the system (3.28) is *not* locally stochastically controllable then for some $x \in \mathbf{X}$, $x \neq 0$,

$$x^{\top} \overline{\varphi}_k = 0 \qquad a.s.\ [\mathrm{P}_{\overline{\pi}}] \tag{3.32}$$

for every $k \in \mathbb{Z}$.

Rewriting (3.32) in the form

$$R(z)\overline{y}(y) = zS(z)\overline{u}(z) + T(z)w(\tilde{z}), \tag{3.33}$$

it follows that $\deg R(\cdot) \leq \deg a(\cdot) - 1$, $\deg S(\cdot) \leq \deg b(\cdot) - 1$, and we will now show that this violates the minimality condition in assumption **sys2**. If (3.33) does holds then by equations (3.30) and (3.31),

$$\frac{R(z)}{Q(z)} w(z) + \frac{zS(z)(a(z) - c(z))}{zb(z)Q(z)} w(z) = T(z)w(z) \tag{3.34}$$

which implies that the polynomial $b(\cdot)$ divides $S(\cdot)(a(\cdot) - c(\cdot))$. Since we have assumed that $b(\cdot)$ and $a(\cdot) - c(\cdot)$ are relatively prime and $\deg S(\cdot) < \deg b(\cdot)$, it follows that $S \equiv 0$. So, equation (3.34) becomes,

$$\frac{R(z)}{Q(z)} w(z) = T(z)w(z).$$

Since $\deg R(\cdot) < \deg a(\cdot) \leq \deg Q(\cdot)$, it follows that $R(\cdot)$ and $T(\cdot)$ are both zero. Hence there is no non-zero $x \in \mathbf{X}$ satisfying (3.32) and this proves the lemma. □

To complete the system description (3.24) we now propose a model for the parameter process θ^ε. Suppose that θ^ε is generated by the stable Markovian system

$$\theta^\varepsilon_{k+1} = G(\theta^\varepsilon_k, v^\varepsilon_{k+1}) \tag{3.35}$$

where $G: \mathbf{X} \times \mathbb{R} \to \mathbf{X}$ is continuous, $\mathbf{v}^\varepsilon \stackrel{\Delta}{=} \{v^\varepsilon_{k+1} : k \in \mathbb{Z}_+\}$ is an $\mathbb{R}^M$-valued i.i.d. processes with distribution μ^ε_v, the initial condition θ^ε_0 is independent of $\mathbf{v}^\varepsilon$, and $\mathbf{w}$ and $\mathbf{v}^\varepsilon$ are independent for all $\varepsilon \in [0,1]$. We also assume:

par1 The probabilities $\{\mu^\varepsilon_v : \varepsilon \in [0,1]\}$ satisfy

$$\mu^\varepsilon_v \xrightarrow{\text{weakly}} \mu^0_v \stackrel{\Delta}{=} \delta_0 \qquad \text{as} \quad \varepsilon \to 0.$$

par2 Condition **AS** is satisfied with $x^* = \theta^*$. Hence, by assumption **par1** when $\varepsilon = 0$,

$$\lim_{k\to\infty} \theta^0_k = \theta^*.$$

par3 For all $\varepsilon > 0$ the Markovian system generating θ^ε is locally stochastically controllable, and furthermore the probability

$$\mathrm{P}\left\{\theta^\varepsilon_1 \in \cdot \,|\theta^\varepsilon_0 = x\right\}$$

is equivalent to Lebesgue measure on an open set $O_{x,\varepsilon} \subset \mathbf{X}$ for every $x \in \mathbf{X}$ where $0 \in \bar{O}_{x,\varepsilon}$.

par4 The ε-parameterized family of systems described in equation (3.35) is uniformly (in ε) stable in probability. That is, for every initial condition $x \in \mathbf{X}$, and every $\delta > 0$ there exists a compact set $C \subset \mathbf{X}$ such that

$$\sup_{k\geq 0} \mathrm{P}\left\{\theta^\varepsilon_k \in C^c |\theta^\varepsilon_0 = x\right\} \leq \delta$$

for every $\varepsilon \in [0,1]$.

An example of a Markovian system satisfying these conditions is the $AR(1)$ model

$$\theta_{k+1}^{\varepsilon} = A(\theta_k^{\varepsilon} - \theta^*) + v_{k+1}^{\varepsilon} + \theta^*$$

where the matrix A is asymptotically stable, and for each $\varepsilon \in [0,1]$ the distribution of v_k^{ε} is Gaussian with zero mean and variance $\varepsilon^2 I$.

Observe that by conditions **par2** - **par4** and Proposition 2.3.2 it follows that for every $\varepsilon > 0$, θ^{ε} is an aperiodic Harris recurrent Markov chain with unique invariant probability $\pi_{\varepsilon}^{\theta}$.

If the feedback control (3.21) is applied then for fixed $\varepsilon \in [0,1]$, the closed loop system becomes

$$\begin{aligned} \theta_{k+1}^{\varepsilon} &= G(\theta_k^{\varepsilon}, v_{k+1}^{\varepsilon}) \\ \varphi_{k+1}^{\varepsilon} &= S_{\varphi_k^{\varepsilon}}^{1}(\theta_k^{\varepsilon}, w_{k+1}) \qquad k \in \mathbb{Z}_+, \end{aligned} \tag{3.36}$$

and the joint process $\mathbf{\Phi}^{\varepsilon} \triangleq \binom{\theta^{\varepsilon}}{\varphi^{\varepsilon}}$ is a Feller Markov chain on $\mathbf{X}^2$.

By assumptions **sys1** and **par2** the Markovian system (3.36) satisfies condition **AS** of Chapter II for all $\varepsilon \in [0,1]$. In fact, if the disturbance process $\binom{\mathbf{v}^{\varepsilon}}{\mathbf{w}}$ is set to zero in (3.36) then

$$\lim_{k\to\infty} \Phi_k^{\varepsilon} = \begin{pmatrix} \theta^* \\ 0 \end{pmatrix} \qquad \text{for every initial condition } \Phi_0^{\varepsilon} \in \mathbf{X}^2.$$

Furthermore, applying Lemma 3.3.1 together with assumption **par3** we may show $\mathbf{\Phi}^{\varepsilon}$ is locally stochastically controllable. The only important property that we cannot establish for $\mathbf{\Phi}^{\varepsilon}$ is stability, and to do this requires further assumptions; suppose that for some $p \in \mathbb{R}_+$ such that $p_0 > p > 2$, the closed loop system is uniformly (in ε) L^p stable:

sys6 For some constant $K_y > 0$ and all $x \in \mathbf{X}^2$, $\varepsilon \in [0,1]$,

$$\limsup_{k\to\infty} \mathrm{E}_x[|y_k^\varepsilon|^p] < K_y. \tag{3.37}$$

In this case it follows from **sys1** and **sys3** that for some constants K_u, $K_w > 0$,

$$\limsup_{k\to\infty} \mathrm{E}_x[|u_k^\varepsilon|^p] < K_u, \tag{3.38}$$

$$\limsup_{k\to\infty} \mathrm{E}_x[|w_k^\varepsilon|^p] < K_w, \tag{3.39}$$

for all $x \in \mathbf{X}^2$, and $0 \le \varepsilon \le 1$. Moreover, by assumption **par4** there exists a moment f on $\mathbf{X}$ such that for some $K_\theta > 0$,

$$\limsup_{k\to\infty} \mathrm{E}_x[f(\theta_k^\varepsilon)] < K_\theta \tag{3.40}$$

for every $\varepsilon \in [0,1]$, $x \in \mathbf{X}^2$. Hence, the closed loop system (3.24) is (uniformly in ε) stable in probability for each $0 \le \varepsilon \le 1$.

The following result shows that under general conditions a small stable perturbation of a linear stochastic system operating under feedback gives rise to a small perturbation of the infinite horizon cost.

Proposition 3.3.1. *Suppose that the Markovian system (3.24) satisfies conditions* **sys1** *-* **sys6** *and* **par1** *-* **par4**. *Then for every $\varepsilon \in [0,1]$ and $x \in \mathbf{X}$ the corresponding criterion functions J_∞^ε and L_∞^ε may be computed as follows:*

$$J_\infty^\varepsilon \triangleq \lim_{k\to\infty} \mathrm{E}_x[y_k^{\varepsilon\,2} + \rho u_k^{\varepsilon\,2}] = \int y^2(\lambda) + \rho u^2(\lambda)\,\pi_\varepsilon(d\lambda), \tag{3.41}$$

$$L_\infty^\varepsilon \triangleq \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} y_k^{\varepsilon\,2} + \rho u_k^{\varepsilon\,2} = \int y^2(\lambda) + \rho u^2(\lambda)\,\pi_\varepsilon(d\lambda). \tag{3.42}$$

Furthermore,

$$\lim_{\varepsilon\to 0} J_\infty^\varepsilon = J_\infty^0 = \sigma_w^2 + \rho\sigma_u^2.$$

Proof.

Since (3.24) is locally stochastically controllable for $0 < \varepsilon \leq 1$, and satisfies condition **AS** it follows that $\mathbf{\Phi}^{\varepsilon}$ is an aperiodic Harris recurrent Markov chain with unique invariant probability π_ε. Hence by Corollary 1.5.1, for every initial condition $\Phi_0^{\varepsilon} = x \in \mathbf{X}^2$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} y_k^{\varepsilon\,2} + \rho u_k^{\varepsilon\,2} = \int y^2(\lambda) + \rho u^2(\lambda)\,\pi_\varepsilon(d\lambda),$$

and this is (3.42).

By (3.37) - (3.40) there exists a constant $K_{\Phi} > 0$ such that for every $0 \leq \varepsilon \leq 1$,

$$\mathrm{E}_{\pi_\varepsilon}[\|\varphi_0^{\varepsilon}\|_2^2 + f(\theta_0^{\varepsilon})] < K_{\Phi}.$$

Since $g(x,y) \triangleq \|x\|_2^2 + f(y)$ is a moment on $\mathbf{X}^2$ it follows that the family of probabilities $\{\pi_\varepsilon : 0 \leq \varepsilon \leq 1\}$ is tight, and so conditions **R0** and **R1** are satisfied. Finally, by Proposition 3.2.1 and **par1**, condition **R2** is satisfied and the Markov transition operators $\{\mathbf{T}_\varepsilon\}$ corresponding to the ε-parameterized system (3.30) converge as $\varepsilon \to 0$.

Furthermore, since $\mathbf{\Phi}^{\varepsilon}$ is aperiodic, it follows that for every initial condition $x \in \mathbf{X}^2$, the resulting trajectory $\{\mu_k \triangleq \mathrm{U}_\varepsilon^k \delta_x\}$ converges weakly to π_ε as $k \to \infty$. By (3.37) and (3.38) it may be shown that the function $y^2(\cdot) + \rho u^2(\cdot)$ is uniformly integrable with respect to the probabilities $\{\mu_k : k \in \mathbb{Z}_+\}$. Applying Theorem 1.1.2 we see that

$$\lim_{k\to\infty} \mathrm{E}_x[y_k^{\varepsilon\,2} + \rho u_k^{\varepsilon\,2}] = \int y^2(\lambda) + \rho u^2(\lambda)\,\pi_\varepsilon(d\lambda) \tag{3.43}$$

for all $\varepsilon > 0$, and in the case $\varepsilon = 0$ equation (3.43) still holds with

$$\int y^2(\lambda) + \rho u^2(\lambda)\,\pi_0(d\lambda) = \sigma_w^2 + \rho\sigma_u^2$$

because in this case the parameter θ_k^0 converges to θ^* as $k \to \infty$.

To finish the proof of the Proposition we are left to show that

$$\lim_{\varepsilon\to 0} J^{\varepsilon}_{\infty} = J^{0}_{\infty} = \sigma^2_w + \rho\sigma^2_u. \tag{3.44}$$

This follows from Proposition 3.2.2: We have already established conditions **R0** - **R2** for this example and $\mathbf{T}_\varepsilon \to \mathbf{T}_0$ as $\varepsilon \to 0$. Hence,

$$\pi_\varepsilon \xrightarrow{\text{weakly}} \pi_0 \qquad \text{as } \varepsilon \to 0.$$

Using (3.37) and (3.38) once more it may be shown that the functions $y^2(\cdot) + \rho u^2(\cdot)$ are uniformly integrable with respect to the probabilities π_ε, $\varepsilon \in [0,1]$. By Theorem 1.1.2 it follows that (3.44) holds, and this completes the proof.

□

3.4 Future Applications

The example presented in the previous section suggests a number of applications. The main point of that example was to show that many of the asymptotic properties of a Markovian system vary continuously under perturbations of the structure of the system. It also illustrates how a Markovian system which is not locally stochastically controllable may be approximated by systems which do have this property. Hence, one possible approach to the stability analysis of general Markovian systems is to find a system model which is locally stochastically controllable and which approximates the system subject to analysis, and then use the results of Chapter II and this chapter to study the behavior of the approximate system model.

A simple way to construct a locally stochastically controllable approximate system is to inject an i.i.d. "dither" sequence $\mathbf{d}^\varepsilon$ into the system. For example, if the control law (3.21) is replaced by

$$zb(z) = (a(z) - c(z))y(z) + d^\varepsilon(z)$$

where $\mathbf{d}^{\varepsilon}$ is a Gaussian i.i.d stochastic process with zero mean and variance ε^2, then the resulting closed loop system will be locally stochastically controllable under general conditions. This technique has already been applied in parameter estimation algorithms to force a condition known as *persistent excitation* to hold

For example, suppose that $\mathbf{\Phi}$ is a Feller Markov chain with Markov transition operator $\mathbf{T}$ which is stable in probability and possesses exactly one invariant probability π. By Theorem 1.3.1 for every initial condition distribution μ_0, the resulting trajectory $\{\mu_k \stackrel{\Delta}{=} \mathbf{U}^k \mu_0 \, . \, k \in \mathbb{Z}_+\}$ satisfies

$$\frac{1}{N}\sum_{k=1}^{N} \mu_k \xrightarrow{\text{weakly}} \pi \qquad \text{as} \quad N \to \infty. \tag{3.45}$$

If the deterministic σ-algebra Σ_D of π is trivial then it seems plausible that (3.45) could be strengthened to simple convergence:

$$\mathbf{U}^k \mu_0 \xrightarrow{\text{weakly}} \pi \qquad \text{as} \quad k \to \infty. \tag{3.46}$$

One approach to establishing (3.46) would be to find a parameterized family $\{\mathbf{T}_\varepsilon : 0 \le \varepsilon \le 1\}$ of Markov transition operators such that **R1** and **R2** hold and $\mathbf{T}_\varepsilon \to \mathbf{T}$ as $\varepsilon \to 0$. If the Markov transition operators are chosen to be positive Harris recurrent with unique invariant probability π_ε then it follows that for every initial condition distribution μ_0,

$$\mathbf{U}_\varepsilon^k \mu_0 \xrightarrow{\text{weakly}} \pi_\varepsilon \qquad \text{as} \quad k \to \infty \tag{3.47}$$

Under certain conditions the existence of this limit for all $\varepsilon > 0$ will imply that (3.46) holds as well.

As an illustration suppose that the Markov chain $\mathbf{\Phi}$ is generated by a Markov system of the form (2.3) with state space $\mathbf{X} = \mathbb{R}^n$, and suppose that the following conditions are satisfied:

(1) $F(0,0) = 0$;

(2) For some $\lambda > 0$ the function F satisfies

$$\sup_{z \in \mathbb{R}^p} \sup_{\substack{x,y \in \mathbb{R}^n \\ x \neq y}} \frac{\|F(x,z) - F(y,z)\|_2}{\|x - y\|_2} < \lambda.$$

Since we have made no assumptions on the distribution of w_k, $k \in \mathbb{Z}_+$, we cannot apply any of the results of Chapter II to this example. The Markov chain $\mathbf{\Phi}$ is *not* Harris in general since for example, we may have $\mathbf{w} \equiv 0$ almost surely

However, it may be shown that $\mathbf{\Phi}$ satisfies condition **AS**, and that it is stable in probability and Feller Hence by Proposition 1.3.1, for every initial condition $x \in \mathbf{X}$ there exists an invariant probability π_x defined for $f \in \mathrm{C}$ by

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \mathbf{T}^k f\,(x) = \int f\, d\pi_x. \tag{3.48}$$

This result may be improved considerably by the approximation methods described above. The following result will be used in Chapter IV.

Proposition 3.4.1. *The Markov chain $\mathbf{\Phi}$ possesses exactly one invariant probability π, and for every initial condition distribution $\mu_0 \in \mathcal{M}$ and $f \in \mathbf{C}$,*

$$\lim_{k \to \infty} \mathrm{E}_{\mu_0}[f(\Phi_k)] = \int f\, d\pi. \tag{3.49}$$

Proof.

Consider the perturbed system

$$\Phi^{\varepsilon}_{k+1} \triangleq F(\Phi^{\varepsilon}_k, w_k) + d^{\varepsilon}_{k+1} \tag{3.50}$$

where $\mathbf{d}^{\varepsilon}$ is an $\mathbb{R}^n$-valued Gaussian i.i.d. stochastic process with zero mean and variance $\varepsilon^2 I$, and is independent of $\mathbf{w}$. For each $\varepsilon > 0$ the Markov system (3.50) satisfies condition

GA. is stable in probability, locally stochastically controllable, and hence by Proposition 2.3.2 it is aperiodic and positive Harris recurrent.

We will now show that Φ^ε is uniformly close to Φ in a probabilistic sense for $\varepsilon \cong 0$. For any initial condition $\Phi_0 = \Phi_0^\varepsilon = x \in \mathbf{X}$,

$$\begin{aligned}\|\Phi_{k+1}^\varepsilon - \Phi_{k+1}\|_2 &= \|F(\Phi_k^\varepsilon, w_{k+1}) - F(\Phi_k, w_{k+1}) + d_{k+1}^\varepsilon\|_2 \\ &\le \lambda\|\Phi_k^\varepsilon - \Phi_k\|_2 + \|d_{k+1}^\varepsilon\|_2\end{aligned}$$

by assumption 2. Hence for each $\delta > 0$,

$$\lim_{\varepsilon\to 0}\sup\ \sup_{k\in\mathbb{Z}_+}\ \sup_{x\in\mathbf{X}} \mathrm{P}_x\{\|\Phi_{k+1}^\varepsilon - \Phi_{k+1}\|_2 > \delta\} \le \lim_{\varepsilon\to 0} \mathrm{P}\Big\{\sum_{k=0}^{\infty} \lambda^k \|d_k^\varepsilon\|_2 > \delta\Big\} = 0. \tag{3.51}$$

This implies that for every $x \in \mathbf{X}$ and $f \in C_c$,

$$\lim_{\varepsilon\to 0}\ \sup_{k\in\mathbb{Z}_+} |\mathrm{E}_x[f(\Phi_k^\varepsilon)] - \mathrm{E}_x[f(\Phi_k)]| = 0. \tag{3.52}$$

In particular,

$$\begin{aligned}0 &= \lim_{\varepsilon\to 0}\ \limsup_{N\to\infty} \Big|\frac{1}{N}\sum_{k=1}^{N}\Big(\mathrm{E}_x[f(\Phi_k^\varepsilon)] - \mathrm{E}_x[f(\Phi_k)]\Big)\Big| \\ &= \lim_{\varepsilon\to 0} \Big|\int f\,d\pi_\varepsilon - \int f\,d\pi_x\Big|,\end{aligned}$$

where π_ε is the invariant probability for Φ^ε, and π_x is defined in equation (3.48). It follows that

$$\pi_\varepsilon \xrightarrow{\text{weakly}} \pi_x \quad \text{as} \quad \varepsilon \to 0, \qquad \text{and} \quad \pi \stackrel{\Delta}{=} \pi_x = \pi_y \quad \text{for all } x, y \in \mathbf{X}.$$

Furthermore, from this and equation (3.52) we have for each $x \in \mathbf{X}$ and $f \in C$,

$$\begin{aligned}\limsup_{k\to\infty} \Big|\mathrm{E}_x[f(\Phi_k)] - \int f\,d\pi\Big| &\le \limsup_{\varepsilon\to 0}\ \limsup_{k\to\infty} \Big(|\mathrm{E}_x[f(\Phi_k^\varepsilon)] - \mathrm{E}_x[f(\Phi_k)]| \\ &\quad + \Big|\mathrm{E}_x[f(\Phi_k^\varepsilon)] - \int f\,d\pi_\varepsilon\Big| \\ &\quad + \Big|\int f\,d\pi - \int f\,d\pi_\varepsilon\Big|\Big) \\ &= 0\end{aligned}$$

which proves the proposition. □

In the next chapter we investigate a very important application in what is now classical control theory; the robustness of linear stochastic systems operating under nonlinear feedback.

Chapter 4

NONLINEAR CONTROL and SECTOR CONDITIONS

In this chapter we investigate an important class of stochastic systems which satisfy condition **GA** of Chapter II. Consider the stochastic state space system

$$x_{k+1} = Ax_k + bu_k + G\xi_{k+1}, \tag{4.1)(i}$$

$$y_k = c^\top x_k + \varsigma_{k+1}, \qquad k \in \mathbb{Z}_+, \tag{ii}$$

where the processes $\mathbf{y}$ and $\mathbf{u}$ evolve on $\mathbb{R}$, $\mathbf{x}$ evolves on $\mathbb{R}^n$, the matrices c and b are $n \times 1$, and A is an $n \times n$ matrix. The disturbance process $\mathbf{w} \triangleq \{\binom{\xi_{k+1}}{\varsigma_{k+1}} : k \in \mathbb{Z}_+\}$ is i.i.d. and independent of (u_0, x_0).

Suppose that a nonlinear feedback control law is given of the form

$$u_k \triangleq -\varphi(y_k), \qquad \text{for all} \quad k \in \mathbb{Z}_+, \tag{4.2}$$

where the function φ is continuous. Then the resulting closed loop system

$$x_{k+1} = Ax_k - b\varphi(c^\top x_k + \varsigma_{k+1}) + G\xi_{k+1}, \tag{4.3)(i}$$

$$y_k = c^\top x_k + \varsigma_{k+1}, \qquad k \in \mathbb{Z}_+, \tag{ii}$$

is of the general form (2.3) introduced in Chapter II. In the next two sections we will use the results presented in Chapters I - II to establish a variety of results including the convergence of the underlying distribution of the input-state-output process, and the convergence of the mean square cost J_N for all initial conditions. In section 3 we will apply the results of Chapter III to show under extremely general assumptions that the value J_∞ of the limit of the mean square cost varies continuously under perturbations of the feedback control law φ.

4.1 Stability

In this section we will establish general stability results for the closed loop system (4.3). To determine whether or not the system (4.3) satisfies condition **GA** of Chapter II we must investigate the asymptotic behavior of the sequence **d** generated by the recursion

$$d_{k+1} = Ad_k - b\varphi(c^\top d_k). \tag{4.4}$$

The closed loop system (4.3) satisfies condition **GA** if and only if the sequence **d** converges to zero for every initial condition $d_0 \in \mathbb{R}^n$, and we will proceed by showing that the deterministic system (4.4) is globally asymptotically stable. A number of sufficient conditions are available to ensure that the system (4.4) is globally asymptotically stable. The conditions we present here are sufficient to ensure the asymptotic stability of (4.4), as well as the L^p stability of (4.3). Observe that by (4.3), the closed loop system is Markovianizable where the Markov state process $\mathbf{\Phi} \triangleq \binom{\mathbf{x}}{\mathbf{y}} = \{\binom{x_{k+1}}{y_k} : k \in \mathbb{Z}_+\}$ evolving on $\mathbf{X} \triangleq \mathbb{R}^{n+1}$ has the Feller property.

For a function $\gamma: \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ we define the *gain* of γ, $g(\gamma) \in [0, \infty]$, by

$$g(\gamma) \triangleq \sup_{x \in \mathbb{R}^{n_1}} \frac{\|\gamma(x)\|_2}{\|x\|_2}. \tag{4.5}$$

If γ is linear, so that it is realized by an $n_2 \times n_1$ matrix Λ then

$$\begin{aligned} g(\gamma) &= \text{greatest singular value of } \Lambda \\ &= \sqrt{\text{max e-value } (\Lambda^\top \Lambda)}. \end{aligned}$$

We say that the control φ defined in equation (4.2) lies in the sector (α, r) (see [Safonov, 1980]) if for all $x \in \mathbb{R}$,

$$|\varphi(x) - \alpha x| \leq r|x|.$$

Hence if the control φ lies in the sector (α, r) then it has the form

$$\varphi(x) = \alpha x + \ell(x)$$

where the gain of the function ℓ is less than r. Sector conditions such as this one have previously been used to establish the stability of deterministic continuous time systems operating under feedback (see [Zames, 1966] and [Popov, 1973].) Proposition 4.1.1 below generalizes these results to this stochastic control problem. We list here two assumptions that we will be referring to throughout this chapter:

nc1 For some $p \geq 1$, $E[||w_0||_p^p] < \infty$;

nc2 The control law φ lies in the sector (α, r) and

$$\lambda \triangleq g(A - \alpha bc^\top) + rg(b)g(c) < 1.$$

Proposition 4.1.1. *For the linear stochastic system (4.1) with control (4.2) suppose that assumptions* **nc1** *and* **nc2** *hold. Then the controlled system satisfies condition* **GA**, *and is stable in probability and* L^p*-stable.*

Proof.

Letting $\ell(x) = \varphi(x) - \alpha x$ for $x \in \mathbb{R}$, we may use (4.3) and assumptions **nc1** and **nc2** to estimate the norm of x_{k+1} as follows:

$$\begin{aligned}
\|x_{k+1}\|_2 &\le \|(A - \alpha bc^\top)x_k\|_2 + \|b\|_2|\ell(c^\top x_k + \varsigma_{k+1})| \\
&\quad + \alpha\|b\|_2|\varsigma_{k+1}| + \|G\xi_{k+1}\|_2 \\
&\le \mathrm{g}(A - \alpha bc^\top)\|x_k\|_2 + r\|b\|_2|(c^\top x_k + \varsigma_{k+1})| \\
&\quad + \alpha\|b\|_2|\varsigma_{k+1}| + \|G\xi_{k+1}\|_2 \\
&\le \lambda\|x_k\|_2 + C\|w_{k+1}\|_2 \qquad (4.6)
\end{aligned}$$

for some $C > 0$. Hence, replacing w_{k+1} by 0 in equation (4.6) shows that $d_k \to 0$ as $k \to \infty$, which establishes that (4.3) satisfies condition **GA**.

It follows from (4.6) and the triangle inequality that for every initial condition $\Phi_0 = x \in \mathbf{X}$,

$$\begin{aligned}
\limsup_{k\to\infty} \left(\mathrm{E}_x[\|x_{k+1}\|_2^p]\right)^{1/p} &\le \lambda \limsup_{k\to\infty} \left(\mathrm{E}_x[\|x_k\|_2^p]\right)^{1/p} \\
&\quad + C\left(\mathrm{E}[\|w_0\|_2^p]\right)^{1/p} \\
&\le C\frac{\left(\mathrm{E}[\|w_0\|_2^p]\right)^{1/p}}{1-\lambda} < \infty
\end{aligned}$$

Since all norms on $\mathbb{R}^n$ are equivalent, this bound implies that there exists a constant $C_x > 0$ such that

$$\limsup_{k\to\infty} \mathrm{E}_x[\|x_k\|_p^p] < C_x. \qquad (4.7)$$

Furthermore, we have for all $k \in \mathbb{Z}_+$,

$$\left(\mathrm{E}_x[|y_k|^p]\right)^{1/p} \le \left(\mathrm{g}(c)\mathrm{E}_x[\|x_k\|_p^p]\right)^{1/p} + \left(\mathrm{E}_x[\|w_{k+1}\|_p^p]\right)^{1/p}. \qquad (4.8)$$

Combining (4.7) and (4.8) we may find a constant $K > 0$ such that

$$\limsup_{k\to\infty} \mathrm{E}_x[\|x_k\|_p^p + |y_k|^p] \le K$$

for every $\Phi_0 = x \in \mathbf{X}$. We remark that the constant K does not depend on the specific structure of the feedback law φ, but only on the parameters A, b, c, r, α, and $\mathrm{E}[\|w_0\|_p^p]$.

This shows that (4.3) is L^p stable, and since $\|\cdot\|_p^p$ is a moment on $\mathbb{R}^{n+1}$ the state process $\mathbf{\Phi}$ is stable in probability.

□

4.2 Stochastic Controllability

We continue our investigation of the system (4.3) by establishing necessary and sufficient conditions for the Markov state process $\mathbf{\Phi} \triangleq \binom{\mathbf{x}}{\mathbf{y}}$ to be locally stochastically controllable. It is easily verified that in fact $\mathbf{x}$ is also a Markov process, and we will proceed by establishing necessary and sufficient conditions for the generalized controllability matrix C_T^x of the system (4.3) (i) generating $\mathbf{x}$ to be full rank. To do this we will occasionally need the following additional hypotheses:

nc3 The control φ is C^1;

nc4 The pair (A, G) is controllable;

nc5 $\mu_w \approx \mathbf{1}_{O_w}$ for some open set $O_w \subset \mathbb{R}^p$, and $0 \in \bar{O}_w$.

When condition **nc3** holds the Markov process $\mathbf{x}$ is of the form $x_{k+1} = F(x_k, w_{k+1})$ where F is C^1, and hence we may use (4.3) (i) to compute C_T^x by finding the derivatives of the function F:

$$\mathrm{C}_T^x = \left[\mathrm{A}_{T-1}^x \cdots \mathrm{A}_1^x \mathrm{B}_0^x | \mathrm{A}_{T-1}^x \cdots \mathrm{A}_2^x \mathrm{B}_1^x | \cdots\cdots | \mathrm{A}_{T-1}^x \mathrm{B}_{T-2}^x | B_{T-1}^x\right] \tag{4.9}$$

where, letting $\alpha_k \triangleq \frac{d\varphi}{dt}(y_k)$,

$$\mathrm{A}_k^x \triangleq \left[\frac{\partial F}{\partial x}\right]_{(x_k, w_{k+1})} = A - \alpha_k bc^\top,$$

and

$$\mathrm{B}_k^x \triangleq \left[\frac{\partial F}{\partial z}\right]_{(x_k, w_{k+1})} = [G| - \alpha_k b] \qquad \text{for all} \quad k \in \mathbb{Z}_+. \tag{4.10}$$

The following lemma greatly simplifies the computation of the rank of the matrix C_T^x. For an $n \times m$ matrix H let $\mathrm{CoKer}(H)$ denote the n-dimensional vector space

$$\mathrm{CoKer}(H) \triangleq \left\{x \in \mathbb{R}^n : x^\top H = 0\right\}.$$

Lemma 4.2.1. *The generalized controllability matrix C_T^x satisfies*

$$\mathrm{CoKer}(\mathrm{C}_T^x) = \mathrm{CoKer}\left(\left[A^{T-1}[G|\alpha_0 b]\middle|\cdots\middle|[A[G|\alpha_{T-2}b]\middle|[G|\alpha_{T-1}b]\right]\right) \tag{4.11}$$

Proof.

We will proceed by showing inductively that for $k = 0, \ldots T-1$ and $x \in \mathbb{R}^n$,

$$x^\top \left[A_{T-1}^x \cdots A_{T-k}^x B_{T-k-1}^x\middle|\cdots\middle|A_{T-1}^x B_{T-2}^x\middle|B_{T-1}^x\right] = 0$$

$$\text{if and only if} \qquad x^\top \left[A^k[G|\alpha_{T-k-1}b]\middle|\cdots|A[G|\alpha_{T-2}b]\middle|[G|\alpha_{T-1}b]\right] = 0. \tag{4.12}$$

For $k = 0$ equation (4.12) becomes

$$x^\top[G|\alpha_{T-1}b] = 0 \Longleftrightarrow x^\top[G| - \alpha_{T-1}b] = 0,$$

and this is obvious. Suppose now that (4.12) has been established for $k = n-1 \geq 0$. To establish the implication ($\Longrightarrow$) for $k = n$ observe that if $x \in \mathbb{R}^n$ satisfies

$$x^\top \left[A_{T-1}^x \cdots A_{T-n}^x B_{T-n-1}^x|\cdots|A_{T-1}^x B_{T-2}^x|B_{T-1}^x\right] = 0 \tag{4.13}$$

then by the induction hypothesis,

$$x^{\top}\left[A^{n-1}[G|\alpha_{T-n}b]\cdots\Big|A[G|\alpha_{T-2}b]\Big|[G|\alpha_{T-1}b]\right]=0. \tag{4.14}$$

Furthermore, by equations (4.13) and (4.14) it follows that

$$\begin{aligned}0 &= x^{\top}\left[A^{x}_{T-1}\cdots A^{x}_{T-n}B^{x}_{T-n-1}\right]\\ &= x^{\top}(A-\alpha_{T-1}bc^{\top})(A-\alpha_{T-2}bc^{\top})\cdots(A-\alpha_{T-n}bc^{\top})[G|-\alpha_{T-1-n}b]\\ &= x^{\top}A^{n}[G|-\alpha_{T-1-n}b].\end{aligned}$$

This and (4.14) establishes the implication ($\Longrightarrow$) in (4.12) when $k=n$.

To establish the reverse implication suppose that $x\in\mathbb{R}^n$ satisfies

$$x^{\top}A^{i}[G|\alpha_{T-1-i}b]=0,\qquad \text{for all}\quad 0\le i\le n, \tag{4.15}$$

so that by the induction hypothesis

$$x^{\top}\left[A^{x}_{T-1}\cdots A^{x}_{T-n+1}B^{x}_{T-n}|\cdots|B^{x}_{T-1}\right]=0. \tag{4.16}$$

To complete the proof of the lemma we are left to show that

$$x^{\top}A^{x}_{T-1}\cdots A^{x}_{T-n}B^{x}_{T-n-1}=0,$$

and this follows from equations (4.10) and (4.15):

$$\begin{aligned}x^{\top}A^{x}_{T-1}\cdots A^{x}_{T-n+1}B^{x}_{T-n} &= x^{\top}(A-\alpha_{T-1}bc^{\top})(A-\alpha_{T-2}bc^{\top})\cdots(A-\alpha_{T-n}bc^{\top})[G|-\alpha_{T-1-n}b]\\ &= x^{\top}A^{n}[G|-\alpha_{T-1-n}b]=0.\end{aligned}$$

□

Using Lemma 4.2.1 we may now give the following sufficient condition for the generalized controllability matrix C_T for (4.3) to be full rank.

Proposition 4.2.1. *If conditions* **nc3** *and* **nc4** *hold then the generalized controllability matrix* C_n *for (4.3) is full rank.*

Proof.

The generalized controllability matrix C_n is of the form

$$C_n = \begin{bmatrix} A^x_{n-1} \cdots A^x_1 B^x_0 \; A^x_{n-1} \cdots A^x_2 B^x_1 | \cdots | A^x_{n-1} B^x_{n-2} | B^x_{n-1} \\ \# \; \cdot \; \cdot \; \cdots \; \cdot \; \cdot \; \# \; 1 \end{bmatrix} \tag{4.17}$$

where # denotes a variable which does not concern us. Hence by Lemma 4.2.1, the rank of C_n is greater or equal to the rank of the matrix

$$\begin{bmatrix} A^{n-1}G \cdots | AG | G | \mathbf{0} \\ 0 \; \cdot \; \cdots \; \cdot \; \mathbf{0} \; 1 \end{bmatrix}.$$

Since (A, G) is controllable, the rank of this matrix is $n + 1$ and this completes the proof.

□

We will now consider the complete observations case where $\varsigma \equiv 0$. In this case the system equations become

$$\begin{aligned} x_{k+1} &= Ax_k + bu_k + G\xi_{k+1}, \\ y_k &= c^\top x_k, \qquad k \in \mathbb{Z}_+, \end{aligned} \tag{4.18}$$

where the distribution μ_ξ of ξ_0 satisfies condition **A4**. When the control (4.2) is applied the closed loop system is

$$\begin{aligned} x_{k+1} &= Ax_k - b\varphi(c^\top x_k) + G\xi_k, \\ y_k &= c^\top x_k, \qquad k \in \mathbb{Z}_+. \end{aligned} \tag{4.19}$$

As before, this system is Markovianizable but in this simpler case we may take the Markov state process $\mathbf{\Phi} = \mathbf{x}$.

Letting $A_k = A - \alpha_k bc^\top$, and $B_k = G$ for $k \in \mathbb{Z}_+$, the generalized controllability matrix for the closed loop system becomes

$$\begin{aligned} \mathrm{C}_T &= [\mathrm{A}_{T-1} \cdots \mathrm{A}_1\mathrm{B}_0 | \mathrm{A}_{T-1} \cdots \mathrm{A}_2\mathrm{B}_1 | \cdots\cdots | \mathrm{A}_{T-1}\mathrm{B}_{T-2} | B_{T-1}] \\ &= \left[(A - \alpha_{T-1}bc^\top) \cdots (A - \alpha_1 bc^\top)G | \cdots |(A - \alpha_{T-1}bc^\top)G|G\right] \end{aligned}$$

In the complete observations case the hypothesis that (A, G) is controllable is not enough to ensure that the matrix C_T is full rank for some T. For a counter example, take $\varphi = -id.$, and $A = bc^\top$. Then the state process $\mathbf{x}$ has the form

$$x_{k+1} = Gw_{k+1}, \qquad k \in \mathbb{Z}_+,$$

and the generalized controllability matrix for this system is full rank if and only if G is full rank.

On the other hand, if G is full rank and φ is continuous then $\mathbf{x}$ is locally stochastically controllable.

Let us now return to the stability analysis of (4.3) with conditions **nc1** - **nc5** in force. Since the Markov state process $\mathbf{\Phi}$ is stable in probability, locally stochastically controllable, and satisfies condition **GA** we conclude from Proposition 2.3.1 that $\mathbf{\Phi}$ is an aperiodic, Harris recurrent Markov chain with unique invariant probability π. Hence, the performance criterion L_∞ defined in Chapter I may be computed (with a possibly infinite value) using the invariant probability π:

$$
\begin{aligned}
L_\infty &\triangleq \limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} y_k^2 + \rho u_k^2 \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} y_k^2 + \rho u_k^2 \\
&= \int y^2 + \rho u^2 \, d\pi
\end{aligned}
$$

We summarize these facts in the following proposition:

Proposition 4.2.2. *Suppose that the closed loop system satisfies conditions* **nc1** - **nc5**. *Then the state process* $\mathbf{\Phi}$ *is Harris recurrent and aperiodic. Furthermore, the s.m.s. performance criterion* L_∞ *is a.s. constant and independent of initial conditions.* □

We have the following corollary to Proposition 4.2.2:

Corollary 4.2.2. *Suppose that the closed loop system satisfies the conditions* **nc1** - **nc5** *and that the constant* p *used in condition* **nc1** *strictly greater than 2. Then the closed loop system is s.m.s. stable, mean square stable, and the performance criterion* J_∞ *is independent of inititial conditions and satisfies*

$$
\begin{aligned}
J_\infty &\triangleq \limsup_{k\to\infty} \mathrm{E}_x[y_k^2 + \rho u_k^2] \\
&= \lim_{k\to\infty} \mathrm{E}_x[y_k^2 + \rho u_k^2] \\
&= \int y^2 + \rho u^2 \, d\pi \\
&\le K_\infty \triangleq ((1 + \rho(r + \alpha)) \int y^2 \, d\pi \\
&< \infty
\end{aligned}
$$

where the constant K_∞ *depends only on* r, α, A, b, c, *and* $\mathrm{E}[\|w_0\|_p^p]$.

Proof.

Since $\mathbf{\Phi}$ is Harris recurrent, for every initial condition $\Phi_0 = x \in \mathbf{X}$,

$$\mu_k \xrightarrow{\text{weakly}} \pi \qquad \text{as } k \to \infty, \tag{4.20}$$

where μ_k is the distribution of Φ_k. Furthermore, by Proposition 4.1.1 there exists a constant $K_p > 0$ such that for every initial condition $\Phi_0 = x \in \mathbf{X}$,

$$\begin{aligned}\limsup_{k\to\infty} \mathrm{E}_x[|y_k|^p + \rho|u_k|^p] &= \limsup_{k\to\infty} \int |y|^p + \rho|u|^p \, d\mu_k \\ &\le K_p \end{aligned} \tag{4.21}$$

From equation (4.21) it follows that the function $y^2(\cdot) + \rho u^2(\cdot)$ is uniformly integrable with respect to the probabilities $\{\mu_k : k \in \mathbb{Z}_+\}$. Applying Theorem 1.1.2 shows that

$$\begin{aligned} J_\infty &\triangleq \limsup_{k\to\infty} \mathrm{E}_x[y_k^2 + \rho u_k^2] \\ &= \lim_{k\to\infty} \mathrm{E}_x[y_k^2 + \rho u_k^2] \\ &= \int y^2 + \rho u^2 \, d\pi, \end{aligned}$$

and the last term is less then $((1 + \rho(r+\alpha)) \int y^2 \, d\pi$ by the sector condition on φ. □

4.3 Structural Robustness

The example introduced in this chapter is ideal for illustrating the results of Chapter III. Sector conditions of the type described in Section 1 were originally devised to establish the stability of a linear system for an entire class of feedback control laws. We will now extend these results to show that if the control laws

$$u_k^\varepsilon \triangleq -\varphi^\varepsilon(y_k^\varepsilon), \qquad 0 \le \varepsilon \le 1, \tag{4.22}$$

all lie in the fixed sector (α, r), then under very general conditions (including the convergence of φ_ε to φ_0 as $\varepsilon \to 0$) we may conclude that the invariant probability π_ε corresponding to the control law φ_ε converges to π_0 as $\varepsilon \to 0$. Furthermore, we will show that the performance criterion J^ε_∞ defined in the previous section also converges:

$$\lim_{\varepsilon \to 0} J^\varepsilon_\infty = J^0_\infty.$$

We remark that we do not know of any way of establishing these results, or the results presented in Section 4.2 without the use of the methods introduced in this thesis.

Proposition 4.3.1. *For the linear stochastic system (4.1) with control (4.22) suppose that for each $0 \le \varepsilon \le 1$, conditions* **nc1** - **nc5** *hold, and that the constant p used in condition* **nc1** *is strictly greater than 2. Suppose further that for every $N \in \mathbb{Z}_+$,*

$$\lim_{\varepsilon \to 0} \sup_{|x| \le N} |\varphi_\varepsilon(x) - \varphi_0(x)| = 0.$$

That is, φ_ε converges to φ_0 uniformly on compact sets. Then $\pi_\varepsilon \xrightarrow{\text{weakly}} \pi_0$, and $J^\varepsilon_\infty \to J^0_\infty$ as $\varepsilon \to 0$

Proof.

By the conditions of the proposition it is easily verified that the Markov transition functions $\{\mathbf{T}_\varepsilon : 0 \le \varepsilon \le 1\}$ satisfy conditions **R0** - **R2** and that $\mathbf{T}_\varepsilon \to \mathbf{T}_0$ as $\varepsilon \to 0$. Hence $\pi_\varepsilon \to \pi_0$, and by Proposition 4.1.1 (and the remarks made in it's proof) there is a constant $K > 0$ such that

$$\int y^p + \rho u^p \, d\pi_\varepsilon < K \qquad \text{for all} \quad 0 \le \varepsilon \le 1.$$

Hence the function $y^2 + \rho u^2$ is uniformly integrable with respect to the invariant probabilities $\{\pi_\varepsilon : 0 \le \varepsilon \le 1\}$, and applying Theorem 1.1.2 we find that

$$\lim_{\varepsilon \to 0} J^\varepsilon_\infty = J^0_\infty.$$

□

We now turn to the problem of removing the smoothness condition on φ and the noise controllability condition on the closed loop system (4.3). Although our results are not complete as yet, we have a partial solution to this problem and the methods used to establish these partial results are interesting on their own. It is likely that these methods will in useful in providing further extensions to the results of this chapter. Using the following strengthening of condition **nc2**, we will proceed by showing that the closed loop system (4.3) satisfies the conditions of Proposition 3.4.1

nc2′ The control law φ lies satisfies $\varphi(0) = 0$, and for some $\alpha, r \in \mathbb{R}_+$,

$$\|b\|_2\|c\|_2 \sup_{\substack{s,t \in \mathbb{R}^n \\ s \neq t}} \left| \frac{\varphi(s) - \varphi(t) + \alpha(t - s)}{s - t} \right| \leq r,$$

and

$$\lambda \triangleq \mathbf{g}(A - \alpha b c^{\top}) + r\mathbf{g}(b)\mathbf{g}(c) < 1.$$

Proposition 4.3.2. *Suppose that conditions* **nc1** *and* **nc2′** *hold for the Markov state process* Φ *described by equation (4.3). Then there exists a unique invariant probability* π, *and for every initial condition* $\Phi_0 = x \in \mathbf{X}$, *and every* $f \in C$ *we have*

$$\lim_{k \to \infty} \mathrm{E}_x f(\Phi_k) = \int f \, d\pi \tag{4.23}$$

Observe that since we make no restrictions on the distribution of the disturbance process, Proposition 4.3.2 holds for the complete observations case by setting $\varsigma \equiv 0$.

Proof.

We will show that the Markov chain $\mathbf{x}$ defined in (4.3)(i) satisfies the conditions of Proposition 3.4.1. Condition 1 is obvious and using **nc2′** we will now show that condition 2 holds: For $x \neq y$,

$$\begin{aligned}\frac{\|A(x-y) - b(\varphi(c^\top x + r) - \varphi(c^\top y + r))\|_2}{\|x-y\|_2} &\leq \mathrm{g}(A - \alpha b c^\top) \\ &\quad + \|b\|_2 \frac{|\ell(c^\top x + r) - \ell(c^\top y + r)|}{\|x-y\|_2} \\ &\leq \mathrm{g}(A - \alpha b c^\top) + r\|b\|_2 \frac{\|c^\top (x-y)\|_2}{\|x-y\|_2} \\ &\leq \lambda. \end{aligned} \tag{4.24}$$

Hence all of the conditions of Proposition 3.4.1 are satisfied and we conclude that there exists a unique invariant probability ϖ for the Markov chain $\mathbf{x}$ and that for every initial condition $x_0 \in \mathbb{R}^n$, and every continuous and bounded function $g\colon \mathbb{R}^n \to \mathbb{R}$

$$\lim_{k\to\infty} \mathrm{E}_x[g(x_k)] = \int g\, d\varpi.$$

Since $y_k = c^\top x_k + \varsigma_{k+1}$ it follows that for every $f \in \mathbf{C}$ and $\Phi_0 = x \in \mathbf{X}$,

$$\lim_{k\to\infty} \mathrm{E}_x[f(\Phi_k)] = \int f\, d\pi$$

where $\pi \in \mathcal{M}$ is defined for $f \in \mathbf{C}$ by $\int f(x,y)\,\pi(dx,dy) \triangleq$

$$\int_{\mathbb{R}^{n+1}} \int_{\mathbf{X}} f(Ax - b\varphi(c^\top x + \lambda_1) + G\lambda_2, c^\top x + \lambda_1)\varpi(dx)\mu_\varsigma(d\lambda_1)\mu_\xi(d\lambda_2)$$

and this completes the proof. □

Chapter 5 APPLICATIONS TO ADAPTIVE CONTROL

To illustrate the results in stochastic adaptive control obtainable from the theory of Markov chains described in Chapters I - III we will present here a detailed analysis of a class of random parameter $AR(p)$ stochastic systems under optimal control. We will find that under reasonable conditions, the closed loop system equations give rise to a Markov chain Φ which has all of the desirable properties described in Chapters I and II. In particular, when these conditions are satisfied the state process Φ satisfies condition **GA**, is stable in probability and locally stochastically controllable. These facts will be used to give a complete description of the asymptotic properities of the output process of the closed loop system.

Consider the following $AR(1)$ (= state space) random parameter model:

$$y_{k+1} = \theta_{k+1} y_k + u_k + w_{k+1} \tag{5.1}$$

$$\theta_{k+1} = \alpha\theta_k + e_{k+1}, \qquad |\alpha| < 1, \ k \in \mathbb{Z}_+, \tag{5.2}$$

where the disturbance process $\binom{e}{w}$ is Gaussian and satisfies

$$\mathrm{E}\begin{pmatrix} e_n \\ w_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5.3}$$

$$\mathrm{E}\left[\begin{pmatrix} e_n \\ w_n \end{pmatrix}(e_k, w_k)\right] = \begin{pmatrix} \sigma_e^2 & 0 \\ 0 & \sigma_w^2 \end{pmatrix} \delta_{n-k}, \qquad n, k \geq 1. \tag{5.4}$$

The time varying parameter process θ is not observed directly but is partially observed through the input and output processes $\mathbf{u}$ and $\mathbf{y}$. We assume that for $k \geq 1$ the input process satisfies $u_k \in \mathcal{F}_k$, where $\mathcal{F}_k \triangleq \sigma\{y_1, \ldots, y_k\}$.

Equations (5.1) and (5.2) define a controlled Markov transition function P with state space $\mathbb{R}^2$ where for θ, u, $y \in \mathbb{R}$ and $A \in \mathcal{B}(\mathbb{R}^2)$,

$$P\left(\begin{pmatrix} y \\ \theta \end{pmatrix}; u, A\right) \triangleq \mathrm{P}\left\{\begin{pmatrix} (\alpha\theta + e_1)y + u + w_1 \\ \alpha\theta + e_1 \end{pmatrix} \in A\right\},$$

where e_1 and w_1 are defined above. With the map $\Psi: \mathbb{R}^2 \to \mathbb{R}^1$ defined by $\Psi\binom{y}{\theta} \triangleq y$, the pair (P, Ψ) is a stochastic state space system as defined in Chapter I.

Our goal is to find a control law which (s.m.s) stabilizes this system and minimizes the expected s.m.s. criterion function defined in Section 7 of Chapter I. Since this model is in (linear) state space form with state θ, and because of the assumptions made on $(\mathbf{w}, \mathbf{e})$, the conditional expectation $\mathrm{E}[\theta_k \mid \mathcal{F}_{k-1}]$ is computable using the Kalman filter (see [Mayne, 1963]) whenever the initial condition (u_0, y_0, θ_1) for (5.1), (5.2) is Gaussian. We may use the resulting algorithm to compute the "certainty equivalence" adaptive minimum variance control $u_k = y^*_{k+1} - \widehat{\theta}_{k+1} y_k$, $k \in \mathbb{Z}_+$. For simplicity we henceforth assume that the reference signal $y^* \equiv 0$. Then for an arbitrary initial condition $\Phi_0 = (V_1, \widehat{\theta}_1, y_0)^\top$ this control may be computed recursively via the equations

$$\widehat{\theta}_{k+1} = \alpha\widehat{\theta}_k - \alpha V_k y_k y_{k-1}(V_k y_{k-1}^2 + \sigma_w^2)^{-1} \tag{5.5}$$

$$V_{k+1} = \sigma_e^2 + \alpha^2\sigma_w^2 V_k (V_k y_{k-1}^2 + \sigma_w^2)^{-1}, \quad k \geq 1,$$

and letting $\widetilde{\theta}_k \triangleq \theta_k - \widehat{\theta}_k$, the closed loop system becomes

$$\widetilde{\theta}_{k+1} = \alpha\widetilde{\theta}_k - \alpha V_k y_k y_{k-1}(V_k y_{k-1}^2 + \sigma_w^2)^{-1} + e_k \tag{5.6}$$

$$y_k = \widetilde{\theta}_k y_{k-1} + w_k \tag{5.7}$$

$$V_{k+1} = \sigma_e^2 + \alpha^2\sigma_w^2 V_k (V_k y_{k-1}^2 + \sigma_w^2)^{-1}, \quad k \geq 1. \tag{5.8}$$

We note that $\tilde{\theta}_{k+1} = \theta_{k+1} - \mathrm{E}[\theta_{k+1} \mid \mathcal{F}_k]$ and $V_{k+1} = \mathrm{E}[\tilde{\theta}^2_{k+1} \mid \mathcal{F}_k]$ whenever $\tilde{\theta}_1$ is distributed $N(0, V_1)$ and y_0 and V_1 are constant.

As one would expect, by the recursive nature of the feedback control law the closed loop system is Markovianizable. In fact, the triple

$$\mathbf{\Phi} = \{\Phi_k\}_{k=0}^{\infty} \triangleq \left\{ \begin{pmatrix} V_{k+1} \\ \tilde{\theta}_{k+1} \\ y_k \end{pmatrix} \right\}_{k=0}^{\infty}$$

is a Feller Markov process with state space $\mathbf{X} \triangleq \mathbb{R}^+ \times \mathbb{R}^2$. We will use these facts and Proposition 5.1.1 below to establish asymptotic properties of the criterion functions described in Chapter I.

5.1 Stochastic Controllability

We are fortunate enough to have the following extremely useful result:

Proposition 5.1.1. *The system described by equations (5.6) - (5.8) is locally stochastically controllable.*

Proof.

By Proposition 2.2.1. of Chapter II it is enough to show that for some T the controllability matrix for this system is full rank for almost every sequence

$$\left\{ \begin{pmatrix} e_1 \\ w_1 \end{pmatrix}, \ldots, \begin{pmatrix} e_T \\ w_T \end{pmatrix} \right\} \in (\mathbb{R}^2)^{(T)}. \tag{5.9}$$

We will show that this is the case for $T = 2$. To construct the second order controllability matrix we use the notation of Chapter II: Let $F: \mathbf{X} \times \mathbb{R}^2 \to \mathbf{X}$, $(x, z) \to F(x, z)$, denote the C^∞ function defined by

$$F\left(\begin{pmatrix} V_2 \\ \tilde{\theta}_2 \\ y_1 \end{pmatrix}, \begin{pmatrix} e_2 \\ w_2 \end{pmatrix} \right) = \begin{pmatrix} \sigma_e^2 + \alpha^2 \sigma_w^2 V_2 (V_2 y_1^2 + \sigma_w^2)^{-1} \\ \alpha\tilde{\theta}_2 - \alpha V_2 y_1 (\tilde{\theta}_2 y_1 + w_2)(V_2 y_1^2 + \sigma_w^2)^{-1} + e_2 \\ \tilde{\theta}_2 y_1 + w_2 \end{pmatrix}.$$

With $x \triangleq (V_2, \tilde{\theta}_2, y_1)$ and $z \triangleq (e_2, w_2)$, the partial derivatives of F are given by

$$\left[\frac{\partial F}{\partial z}\right]_{(x,z)} = \begin{bmatrix} 0, & 0 \\ 1, & \frac{-\alpha V_2 y_1}{V_2 y_1^2 + \sigma_w^2} \\ 0, & 1 \end{bmatrix}$$

$$\left[\frac{\partial F}{\partial x}\right]_{(x,z)} = \begin{bmatrix} \#, & 0, & \frac{-2\alpha^2 \sigma_w^2 V_2^2 y_1}{(V_2 y_1^2 + \sigma_w^2)^2} \\ \#, & \#, & \#, \\ \#, & \#, & \#, \end{bmatrix},$$

where we use # to replace a function of $\left\{ \begin{pmatrix} V_i \\ \tilde{\theta}_i \\ y_{i-1} \end{pmatrix}, \begin{pmatrix} e_i \\ w_i \end{pmatrix} \right\}_{i=1,2}$ which is irrelevant to the present discussion.

Hence C_2, the second order controllability matrix, has the form

$$C_2 = \begin{bmatrix} 0, & \frac{-2\alpha^2 \sigma_w^2 V_2^2 y_1}{(V_2 y_1^2 + \sigma_w^2)^2}, & 0, & 0 \\ \#, & \#, & 1, & \# \\ \#, & \#, & 0, & 1 \end{bmatrix}$$

which is full rank whenever $y_1 = \tilde{\theta}_1 y_0 + w_1$ is non-zero. This shows that for each $\begin{pmatrix} V_1 \\ \tilde{\theta}_1 \\ y_0 \end{pmatrix} \in \mathbb{R}^+ \times \mathbb{R}^2$, the matrix C_2 is full rank for a.e. $[\mu^{Leb}]$ $\left(\begin{pmatrix} e_1 \\ w_1 \end{pmatrix}, \begin{pmatrix} e_2 \\ w_2 \end{pmatrix} \right) \in \mathbb{R}^2 \times \mathbb{R}^2$ and so by Proposition 2.2.1 the closed loop system is locally stochastically controllable.

□

We now show that the Markov state process $\mathbf{\Phi}$ satisfies condition **GA**. In Section 2 below we will find that for $\sigma_e^2 < 1$ it is stable in probability, and hence positive Harris recurrent and aperiodic.

To see that $\mathbf{\Phi}$ satisfies condition **GA**, observe that for any $k \in \mathbb{Z}_+$ and $x \in \mathbf{X}$, the asymptotic behavior of the state readout map $S_x^k(\cdot)$ evaluated at 0 may be analyzed by "turning off" the noise in equations (5.6) - (5.8). Hence, with $(\widehat{V}_{k+1}, \widehat{\widehat{\theta}}_{k+1}, \widehat{y}_k) \triangleq S_x^k(0,\ldots,0)$,

$$\begin{pmatrix} \widehat{V}_{k+1} \\ \widehat{\widehat{\theta}}_{k+1} \\ \widehat{y}_k \end{pmatrix} = \begin{pmatrix} \sigma_e^2 + \alpha^2\sigma_w^2 \widehat{V}_k(\widehat{V}_k\widehat{y}_{k-1}^2 + \sigma_w^2)^{-1} \\ \alpha\widehat{\widehat{\theta}}_k - \alpha\widehat{V}_k\widehat{\widehat{\theta}}_k\widehat{y}_{k-1}^2(\widehat{V}_k\widehat{y}_{k-1}^2 + \sigma_w^2)^{-1} \\ \widehat{\widehat{\theta}}_k\widehat{y}_{k-1} \end{pmatrix}.$$

This implies that $|\widehat{\widehat{\theta}}_{k+1}| < \alpha|\widehat{\widehat{\theta}}_k|$, from which it follows that for every $x \in \mathbf{X}$,

$$\lim_{k\to\infty} S_x^k(0,\ldots,0) = \begin{pmatrix} \frac{\sigma_e^2}{1-\alpha^2} \\ 0 \\ 0 \end{pmatrix}.$$

5.2 Stability

In this section we prove that Φ is stable in probability and hence positive Harris recurrent when $\sigma_e^2 < 1$. By Theorem 1.3.1 of Chapter 1, an invariant probability will exist if $\mathrm{E}_{\mu_0}\|\Phi_k\|^2 = \mathrm{E}_{\mu_0}(V_{k+1}^2 + \widetilde{\theta}_{k+1}^2 + y_k^2)$ is uniformly bounded for some initial distribution μ_0. In the proposition below we establish the boundedness of this quantity whenever $\sigma_e^2 < 1$, and the initial distribution is chosen to ensure that the equations above are truly generating $\mathrm{E}_{\mu_0}[\theta_k | \mathcal{F}_{k-1}]$. An example of such an initial distribution is that where $(V_1, \widetilde{\theta}_1, y_0) \sim \delta_0$.

Proposition 5.2.2. *For the system described by equations (5.6) - (5.8) with initial distribution δ_0 (i.e. the unit probability mass at zero)*

$$\limsup_{k\to\infty} \mathrm{E}_{\delta_0}\left[\|\Phi_k\|^2\right] < \infty \text{ for } \sigma_e^2 < 1$$
$$= \infty \text{ for } \sigma_e^2 \geq 1.$$

Proof.

If $\sigma_e^2 \geq 1$ then $V_k \geq 1$ for all $k > 0$. Recall that $V_k = \mathrm{E}_{\delta_0}[\widetilde{\theta}_k^2 \mid \mathcal{F}_{k-1}]$ so

$$\begin{aligned} \mathrm{E}_{\delta_0}\left[y_k^2 \mid \mathcal{F}_{k-1}\right] &= V_k y_{k-1}^2 + \sigma_w^2 \\ &\geq y_{k-1}^2 + \sigma_w^2. \end{aligned} \tag{5.10}$$

Thus $\mathrm{E}_{\delta_0}\left[y_k^2\right] \geq k\sigma_w^2$ and

$$\limsup_{k\to\infty} \mathrm{E}_{\delta_0}\left[\|\Phi_k\|^2\right] = \infty.$$

Suppose that $\sigma_e^2 < 1$. Then $V_k \leq \frac{\sigma_e^2}{1-\alpha^2}$ for all $k \geq 0$ and it follows that

$$\mathrm{E}_{\delta_0}\left[\widetilde{\theta}_k^2\right] = \mathrm{E}_{\delta_0}\left[V_k\right] \leq \frac{\sigma_e^2}{1-\alpha^2}$$

and

$$\mathrm{E}_{\delta_0}\left[V_k^2\right] \leq \left(\frac{\sigma_e^2}{1-\alpha^2}\right)^2.$$

To establish the proposition it is enough to show that

$$\limsup_{k\to\infty} \mathrm{E}_{\delta_0}\left[y_k^2\right] < \infty.$$

Fix $\sigma_e^2 < \rho < 1$. It is easy to show that

$$V_{k+1} < \rho \text{ if and only if } y_{k-1}^2 > Q, \tag{5.11}$$

where $Q \triangleq \sigma_w^2\left(\frac{\alpha^2}{\rho-\sigma_e^2} - \frac{1}{V_k}\right)$. From (5.10) we see that

$$\begin{aligned} \mathrm{E}_{\delta_0}[y_{k+1}^2 \mid \mathcal{F}_k] &= \left(\mathbf{1}_{\{V_{k+1}<\rho\}} + \mathbf{1}_{\{V_{k+1}\geq\rho\}}\right) V_{k+1} y_k^2 + \sigma_w^2 \\ &\leq \rho y_k^2 + \mathbf{1}_{\{V_{k+1}\geq\rho\}} V_{k+1} y_k^2 + \sigma_w^2. \end{aligned} \tag{5.12}$$

So, since

$$\mathrm{E}_{\delta_0}\left[\mathbf{1}_{\{V_{k+1}\geq\rho\}} V_{k+1} y_k^2\right] \leq \frac{\sigma_e^2}{(1-\alpha^2)} \mathrm{E}_{\delta_0}\left[\mathbf{1}_{\{y_{k-1}^2 \leq Q\}} y_k^2\right],$$

we may compute the following estimate of $E_{\delta_0}[y_{k+1}^2]$ by smoothing and the inequality in (5.12):

$$\begin{aligned} E_{\delta_0}[y_{k+1}^2] &\le \rho E_{\delta_0}[y_k^2] + \frac{\sigma_e^2}{1-\alpha^2} E_{\delta_0}[y_k^2 \mid y_{k-1}^2 \le Q] + \sigma_w^2 \\ &\le \rho E_{\delta_0}[y_k^2] + \sigma_w^2 \left[\frac{\alpha^2}{\rho - \sigma_e^2} \left(\frac{\sigma_e^2}{1-\alpha^2} \right)^2 + \frac{\sigma_e^2}{1-\alpha^2} + 1 \right] \end{aligned}$$

which establishes that $\limsup_{k\to\infty} E_{\delta_0}[y_k^2] < \infty$.

□

Applying Theorem 1.3.1 to the case where $0 < \sigma_e^2 < 1$ we see there is an invariant probability π which is *unique* since Φ is locally stochastically controllable and satisfies condition **GA**. Furthermore, it is easily verified that

$$\pi \approx \mathbf{1}_{O}\mu^{Leb} \qquad \text{where} \quad O \triangleq (\sigma_w^2, \frac{\sigma_w^2}{1-\alpha^2}) \times \mathbf{R}^2$$

and that $P\{\Phi \text{ enters } O\} = 1$. Hence by Proposition 2.2.4, Φ is stable in probability. Applying Corollary 2.2.4 and Proposition 2.3.2 it follows that Φ is aperiodic and positive Harris recurrent

Before applying the results of Theorem 1.4.5 to this example we will establish a few properties of the unique invariant probability π. First, because $\pi \approx \mathbf{1}_{O}\mu^{Leb}$,

$$\pi \left\{ \sigma_e^2 < V_1 < \sigma_e^2/(1-\alpha^2) \right\} = 1. \tag{5.13}$$

Furthermore, by Proposition 5.2.2 and the fact that $\frac{1}{N}\sum_{k=1}^{N} U^k \delta_0 \xrightarrow{\text{weakly}} \pi$, we find that

$$\begin{aligned} E_\pi[y_0^2] &\triangleq \int_{\mathcal{B}(\mathbf{X}^{z,\cdot})} y_0^2 P_\pi(d\omega) \\ &= \lim_{n\to\infty} E_\pi[n \wedge y_0^2] \\ &= \lim_{n\to\infty} \left\{ \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} E_{\delta_0}[n \wedge y_k^2] \right\} \\ &\le \limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} E_{\delta_0}[y_k^2] < \infty. \end{aligned}$$

In Proposition 5.2.3 below this result is improved, giving upper and lower bounds on $E_\pi[y_0^2]$. First we must prove the following lemma:

Lemma 5.2.1. *For the stationary Markov process on $\mathbb{Z}$ described by equations (5.6) - (5.8) with distribution* P_π *(P_π can be extended from $\mathcal{B}(\mathbf{X}^{\mathbb{Z}_+})$ to $\mathcal{B}(\mathbf{X}^{\mathbb{Z}})$ since π is invariant), we have*

(i) $E_\pi[\widetilde{\theta}_k \mid \mathcal{F}^*_{k-1}] = 0$ *a.s.* $[P_\pi]$, *and*

(ii) $E_\pi[\widetilde{\theta}_k^2 \mid \mathcal{F}^*_{k-1}] = V_k$ *a.s.* $[P_\pi]$,

where $\mathcal{F}^*_k \triangleq \sigma\{y_i : -\infty < i \leq k\}$.

Proof.

Suppose Φ_0 has distribution δ_0 so that

$$\frac{1}{N}\sum_{k=1}^{N} \mu_k \xrightarrow{\text{weakly}} \pi \qquad \text{as } N \to \infty. \tag{5.14}$$

From (5.6) and (5.7) we have

$$\begin{aligned}\widetilde{\theta}_{k+1} = \alpha &\left[1 - V_k y_{k-1}^2 (V_k y_{k-1}^2 + \sigma_w^2)^{-1}\right] \widetilde{\theta}_k \\ &- \alpha V_k w_k y_{k-1} (V_k y_{k-1}^2 + \sigma_w^2)^{-1} + e_k.\end{aligned} \tag{5.15}$$

So for all $p \geq 1$,

$$\|\widetilde{\theta}_{k+1}\|_p \leq \alpha \|\widetilde{\theta}_k\|_p + c \tag{5.16}$$

where $c > 0$ depends only on p, σ_e^2, and σ_w^2. Hence using an argument similar to the one used to show that $E_\pi[y_0] < \infty$ we may show that

$$E_\pi[|\widetilde{\theta}|^p] < \infty.$$

By Theorem 1.5.2 of Chapter I,

$$\infty > \mathrm{E}_{\pi}\left[\sup_{N} \frac{1}{N}\sum_{k=1}^{N} \widetilde{\theta}_k^2\right]$$
$$= \int_{\mathbf{X}} \mathrm{E}\left[\sup_{N} \frac{1}{N}\sum_{k=1}^{N} \widetilde{\theta}_k^2 \mid \Phi_0 = \begin{pmatrix} V_1 \\ \widetilde{\theta}_1 \\ y_0 \end{pmatrix}\right] p_{\pi}(V_1, \widetilde{\theta}_1, y_0) dV_1\, d\widetilde{\theta}_1\, dy_0$$

where p_π is the density of π. Hence for *a.a.* $[\mu^{Leb}]$ $(V_1, y_0) \in \left[\sigma_e^2, \frac{\sigma_e^2}{(1-\alpha^2)}\right] \times \mathbb{R}$

$$\int_{\mathbb{R}} \mathrm{E}\left[\sup_{N} \frac{1}{N}\sum_{k=1}^{N} \widetilde{\theta}_k^2 \mid \Phi_0 = \begin{pmatrix} V_1 \\ \widetilde{\theta}_1 \\ y_0 \end{pmatrix}\right] p_{\pi}(V_1, \widetilde{\theta}_1, y_0) d\widetilde{\theta}_1 < \infty. \tag{5.17}$$

Choose $(\bar{V}_1, \bar{y}_0) \in \left[\sigma_e^2, \frac{\sigma_e^2}{(1-\alpha^2)}\right] \times \mathbb{R}$ such that (5.17) holds and

$$\bar{V}_1 < \rho^2 \triangleq \alpha^2 \frac{\bar{V}_1^2 \sigma_w^2 \bar{y}_0^2}{(\bar{V}_1 \bar{y}_0^2 + \sigma_w^2)} + \sigma_e^2. \tag{5.18}$$

This is possible because (5.18) defines a non-empty open subset of $\left[\sigma_e^2, \sigma_e^2/(1-\alpha^2)\right] \times \mathbb{R}$ (non-empty because it contains $(\sigma_e^2 + \varepsilon, \bar{y}_0)$ for $\bar{y}_0 \neq 0$ and ε sufficiently small.) Observe that $p_\pi(\bar{V}_1, \widetilde{\theta}_1, \bar{y}_0)$ is a Gaussian density with variance ρ^2 which by (5.18) is greater than $\bar{V}_1$. It follows that for some constant $K > 0$ and all $\widetilde{\theta}_1 \in \mathbb{R}$

$$\frac{1}{\sqrt{2\pi \bar{V}_1}} \exp\left\{-\frac{\widetilde{\theta}_1^2}{2\bar{V}_1}\right\} \leq K p(\bar{V}_1, \widetilde{\theta}_1, \bar{y}_0).$$

Define μ_0 to be the distribution on $\mathbf{X}$ given by $\mu_0 = \delta_{\bar{y}_0} \delta_{\bar{V}_1} N(0, \bar{V}_1)$. That is, under μ_0, $y_0 \equiv \bar{y}_0$, $V_1 \equiv \bar{V}_1$, and $\widetilde{\theta}_1 \sim N(0, \bar{V}_1)$; the Gaussian distribution with mean zero and variance $\bar{V}_1$. By (5.17)

$$\mathrm{E}_{\mu_0}\left[\sup_{N} \frac{1}{N}\sum_{k=1}^{N} \widetilde{\theta}_k^2\right] \leq K \int_{\mathbb{R}} \mathrm{E}\left[\sup_{N} \frac{1}{N}\sum_{k=1}^{N} \widetilde{\theta}_k^2 \mid \Phi_0 = \begin{pmatrix} \bar{V}_1 \\ \widetilde{\theta}_1 \\ \bar{y}_0 \end{pmatrix}\right] p_{\pi}(\bar{V}_1, \widetilde{\theta}_1, \bar{y}_0) d\widetilde{\theta}_1$$
$$< \infty, \tag{5.19}$$

and furthermore, for all $M \in \mathbb{Z}_+$ and all Borel $f : \mathbb{R}^M \to \mathbb{R}$,

$$\mathrm{E}_{\mu_0}\left[\tilde{\theta}_k f(y_{k-1},\cdots,y_{k-M})\right] = 0 \tag{5.20}$$

and

$$\mathrm{E}_{\mu_0}\left[\tilde{\theta}_k^2 f(y_{k-1},\cdots,y_{k-M})\right] = \mathrm{E}_{\mu_0}\left[V_k f(y_{k-1},\cdots,y_{k-M})\right], \qquad k \in \mathbb{Z}_+. \tag{5.21}$$

By the corollary to Theorem 1.5.1 we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\tilde{\theta}_k f(y_{k-1},\cdots,y_{k-M}) = \mathrm{E}_\pi\left[\tilde{\theta}_{M+1} f(y_M,\cdots,y_1)\right] \qquad a.s.\ [\mathrm{P}_{\mu_0}].$$

Taking expectations of both sides of this equation and using (5.19), (5.20), and the dominated convergence theorem gives

$$\begin{aligned}
\mathrm{E}_\pi\left[\tilde{\theta}_{M+1} f(y_M,\cdots,y_1)\right] &= \mathrm{E}_{\mu_0}\left[\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\tilde{\theta}_k f(y_{k-1},\cdots,y_{k-M})\right] \\
&= \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\mathrm{E}_{\mu_0}\left[\tilde{\theta}_k f(y_{k-1},\cdots,y_{k-M})\right] \\
&= 0
\end{aligned}$$

which establishes part (i) of the lemma. Part (ii) is proved using the same argument. □

Proposition 5.2.3. *We have the following upper and lower bounds on the variance of y_0 under the probability π:*

$$\frac{\sigma_w^2}{1-\sigma_e^2}(1+\alpha^2\sigma_e^2) \le \mathrm{E}_\pi[y_0^2] \le \frac{\sigma_w^2}{1-\sigma_e^2}\left(1+\frac{\alpha^2\sigma_e^2}{1-\alpha^2}\right) \tag{5.22}$$

and we remark that we know of no way to obtain these inequalities without the use of these methods.

Proof.

Taking expectations in (5.7) and (5.8) under the probability P_π and using Lemma 5.2.1 we obtain

$$E_\pi\left[y_0^2\right] = E_\pi\left[V_1 y_0^2\right] + \sigma_w^2 \tag{5.23}$$

$$\sigma_e^2 E_\pi\left[y_0^2\right] = E_\pi\left[V_1 y_0^2\right] - \alpha^2\sigma_w^2 E_\pi\left[V_1\right]. \tag{5.24}$$

And hence,

$$(1 - \sigma_e^2)E_\pi\left[y_0^2\right] = \sigma_w^2 + \alpha^2\sigma_w^2 E_\pi\left[V_1\right].$$

which, together with (5.13), proves Proposition 5.2.3.

□

5.3 Summary

We now establish a number of asymptotic properties of the controlled system (5.6) - (5.8). First suppose that $\sigma_e^2 < 1$. By Theorems 1.4.5 and 1.5.1 of Chapter I for every initial distribution $\mu_0 \in \mathcal{M}$,

$$\lim_{k\to\infty} P_{\mu_0}\{|y_k| > \varepsilon\} = \pi\{y_0^2 > \varepsilon^2\} \le \frac{1}{\varepsilon^2}E_{\mu_0}[y_0^2]; \tag{5.25}$$

and

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} y_k^2 = E_\pi\left[y_0^2\right] \quad a.s.\ [P_{\mu_0}]. \tag{5.26}$$

Furthermore, by Theorem 1.4.5,

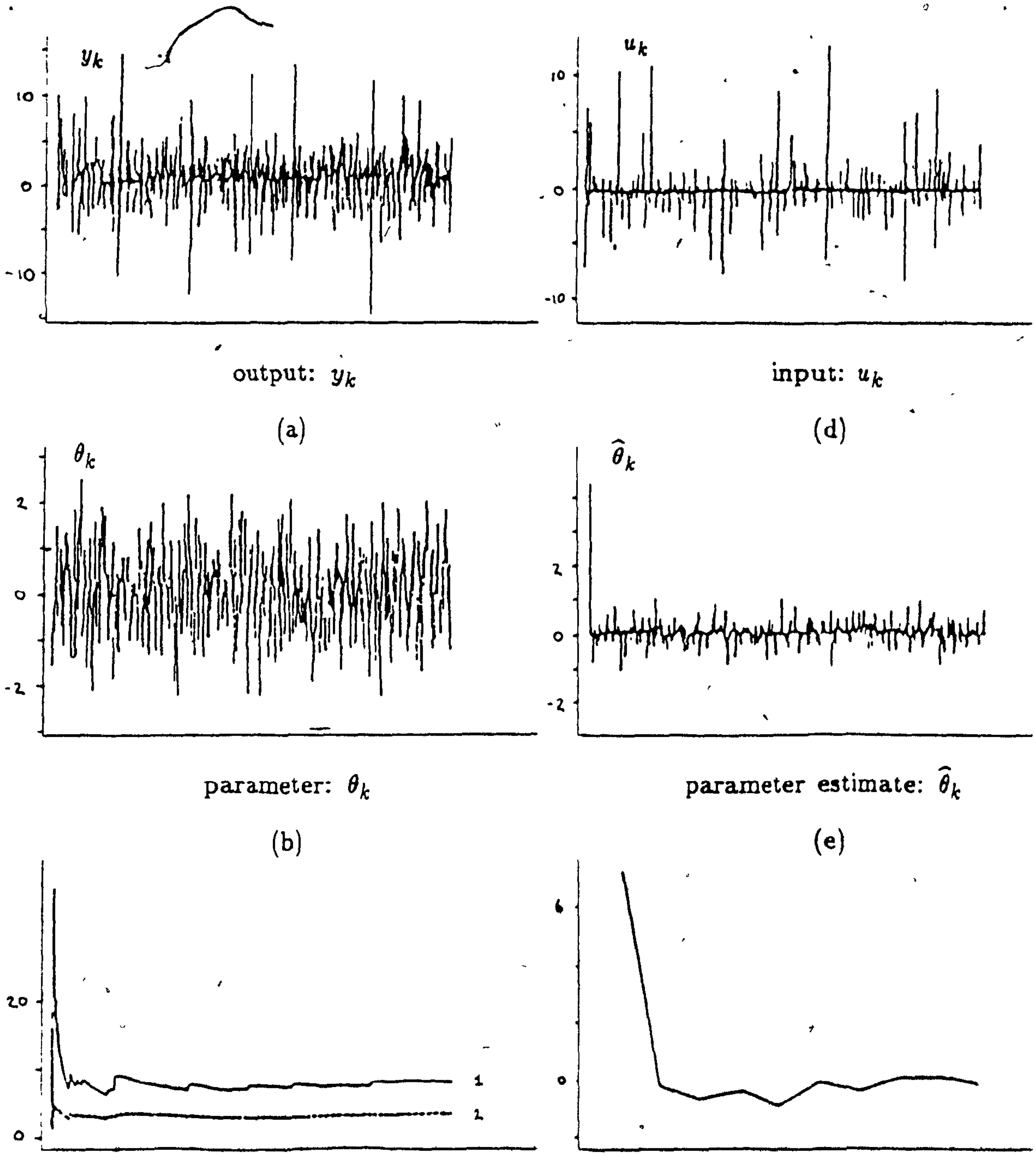
$$\lim_{k\to\infty} E_x[y_k^2] = E_\pi[y_0^2] \qquad \text{for a.a. } x \in \mathcal{O}\ [\mu^{Leb}] \tag{5.27}$$

where $\mathcal{O} = [\sigma_e^2, \sigma_e^2/(1-\alpha^2)] \times \mathbf{R}^2$.

We stress that these resusits hold even though the estimation algorithm (5 6) - (5.8) is not necessarily generating conditional expectations with these initial conditions. The computer simulation below is taken from [Aloneftis, 1987]. It illustrates the asymptotic properties of the controlled system. In this simulation $\sigma_e^2 = \frac{1}{2}$, $\sigma_w^2 = 3$, and consequently $6.75 \leq E_\pi[y_0^2] \leq 7$.

Figure 5.1 Experimental results for the example

$\alpha = 1/2$, $\sigma_e^2 = 1/2$, and $\sigma_w^2 = 3$.



output: y_k

(a)

input: u_k

(d)

parameter: θ_k

(b)

parameter estimate: $\widehat{\theta}_k$

(e)

s.m.s. output and disturbance:

1) $\frac{1}{N}\sum_{k=1}^{N} y_k^2$ 2) $\frac{1}{N}\sum_{k=1}^{N} w_k^2$

(c)

Sample mean correlation function

$\frac{1}{700}\sum_{k=300}^{1000} y_k y_{k-\tau};\ \tau = 0, \ldots, 9$

(f)

Note that it is not known how to obtain (5.25) - (5.27) by stochastic Lyapunov methods even in the constant unknown parameter case.

Finally, we may apply the results of Chapter III to this example. Observe that we have shown that for each $(\alpha, \sigma_w^2, \sigma_e^2)^\top \in (-1,1) \times \mathbb{R}_+ \times \mathbb{R}_+$ there is an invariant probability $\pi \triangleq \pi(\alpha, \sigma_w^2, \sigma_e^2)$. For any compact subset $C \subset (-1,1) \times \mathbb{R}_+ \times \mathbb{R}_+$ the corresponding invariant probabilities $\{\pi(\alpha, \sigma_w^2, \sigma_e^2) : (\alpha, \sigma_w^2, \sigma_e^2) \in C\}$ possess uniformly bounded second moments, and hence are tight. Applying Proposition 3.2.3 it follows that the *map* $\pi : (-1,1) \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathcal{M}$ taking $(\alpha, \sigma_w^2, \sigma_e^2) \longrightarrow \pi(\alpha, \sigma_w^2, \sigma_e^2)$ is continuous. This is an interesting result, but not very useful: Idealy we would like to establish the robustness of $\mathbf{\Phi}$ with respect to perturbations of the distribution of the processes **w** and **e**, or under stable perturbations of the Markov transition operator **T** but the solution to this problem eludes us at the present.

We conclude by observing that a similar but more restricted result is obtainable in the $AR(p), p > 1$, case.

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DEPARTMENT OF ELECTRICAL ENGINEERING

Course 304-688A **VLSI TESTING** September 1987

Instructor: Professor J. Rajski (398-7123)

The course covers various important aspects of the crucial area of VLSI testing. As it can be seen from the enclosed contents, it will span a wide range of topics in both the practical and theoretical framework. In addition, a hands-on use of testing equipment will ensure that the concepts learned in the course are exercised in practical terms as well.

The main emphasis of the course is to orient designers of VLSI chips and boards to think about testing problems in parallel with the design process. With the growing complexity of VLSI systems, their testing is becoming even more complex and almost impossible in many cases. Thus, the course will consider structured design-for-testability as a necessary requirement for designing complex systems. The emerging concept of built-in self-test (BIST) will also be considered in detail.

The lectures will be held on Wednesdays at 2-3p.m. in room 284 (MacDonald Engineering Building).

The first organizational meeting is on September 9.

TENTATIVE LECTURE SCHEDULE

1. **Introduction**
   - aims and objectives of testing,
   - cost of testing and diagnosis,
   - economics of testing (yield and defect level),
   - physical failures and fault models,
   - transistor-level, gate-level and functional-level fault models.

2. **Component testing**
   - automatic test equipment,
   - characterization testing,
   - dc and ac parametric testing.

3. **Test generation for combinational circuits**
   - the stuck-at fault model,
   - the sensitized path,
   - algorithmic methods (d-algorithm, podem, fan),
   - complexity of test generation.

4. **Fault simulation**
   - testdetect,
   - parallel fault simulation,
   - deductive fault simulation,
   - concurrent fault simulation,
   - critical path tracing,
   - region analysis.

5. **Automatic test pattern generation**
   - manual, random vs. algorithmic test pattern generation (Hitest),
   - heuristic methods and artificial intelligence,
   - fault dictionaries, fault dropping,
   - test pattern languages.

6. **Test generation for switch-level**
   - stuck-open and stuck-on faults,
   - CMOS complex gates,
   - networks of complex gates,
   - transition fault testing.

7. **Test generation for PLA's**
   - test generation for two level circuits (complexity),
   - cross-point fault model and redundancy,
   - generation of input vectors,
   - pruning algorithm.

8. **Memory testing**
   - memory faults,
   - memory patterns.

9. **Microprocessor testing**
   - structural, functional and behavioral testing,
   - functional level fault models,
   - testing bus oriented architectures,
   - testing flow of control.

10 **Structured design for testability – random logic**
   - ad hoc methods,
   - scan path techniques

11. **Design for testability – regular structures**
   - easily testable networks,
   - function-independent testing,
   - easily testable PLA's.

12. **Problems with structured DFT, and BIST**
   - test application time and cost,
   - embedded modules.

13. **Built-in-self-test (BIST)**
   - hardware for stimuli generation, random patterns,
   - data compression techniques, signature analysis,
   - BIST for random logic and regular structures.

14. **BIST at chip level and board level**
   - boundary scan.

Laboratory Experiments and Projects:

The VLSI Design Lab has the following Hewlett Packard Testing Equipment to run various experiments for this course:

HP 8180A Data Generator
HP 8182 Data Analyzer
HP 4145A Semiconductor Parameter Analyzer

In addition, a Wentworth Prober is also available.

Each group of two students will perform only two experiments. The first one is designed to familiarize the students with the equipment and some basic concepts in testing. The second will be a part of a project that will require other software tools to generate test vectors, analyze fault coverage, and perform postprocessing of test results. The following projects will be offered:

a) CMOS testing,
b) PLA testing,
c) Memory testing,
d) Testing of random logic,
e) Testing of a multiplier,
f) Testing of a scannable circuit,
g) Diagnosis of scannable circuit,.

The final evaluation will be based on assignments (40%), the final report (30%), and the demonstration of the experiment (30%).